Journal of Mathematical Research & Exposition
July, 2009, Vol. 29, No. 4, pp. 745–752
DOI:10.3770/j.issn:1000-341X.2009.04.022
Http://jmre.dlut.edu.cn

The Killing Forms of Lie Triple Systems

ZHANG Zhi Xue[1], *GAO Rui*[2]
(1. Institute of Mathematics and Computers, Hebei University, Hebei 071002, China;
2. Department of Mathematics, Cangzhou Teacher's College, Hebei 061000, China)
(E-mail: zzx_hb@yahoo.com.cn)

Abstract For Lie triple systems in the characteristic zero setting, we obtain by means of the Killing forms two criterions for semisimplicity and for solvability respectively, and then investigate the relationship among the Killing forms of a real Lie triple system T_0, the complexification T of T_0, and the realification of T.

Keywords Lie triple system; Killing form; complexification.

Document code A
MR(2000) Subject Classification 17B05; 17C55
Chinese Library Classification O152.5

1. Preliminaries

Throughout the article, we shall be concerned with Lie triple systems and Lie algebras which are finite dimensional over a field F of characteristic zero.

First, we recall some definitions, notations and facts which can be found in [1] or [2].

A Lie triple system(L.t.s) is a vector space T over a field F with a ternary composition $[\ ,\ ,\ ]$ which is trilinear and satisfies the following identities:

$$[xyz] = -[yxz], \tag{1.1}$$

$$[xyz] + [yzx] + [zxy] = 0, \tag{1.2}$$

and

$$[uv[xyz]] = [[uvx]yz] + [x[uvy]z] + [xy[uvz]] \tag{1.3}$$

for all $u, v, x, y, z \in T$.

Define $L(x, y)$, $R(x, y) \in \mathrm{End}_F T$ by $L(x, y)z := [xyz]$, $R(x, y)z := [zyx]$. We see that (1.1), (1.2) and (1.3) are equivalent to

$$L(x, y) = -L(y, x), \tag*{(1.1)$'$}$$

$$L(x, y) = R(x, y) - R(y, x), \tag*{(1.2)$'$}$$

and

$$[L(x, y), L(u, v)] = L([xyu], v) + L(u, [xyv]), \tag*{(1.3)$'$}$$

Received date: 2007-05-16; **Accepted date**: 2008-04-16
Foundation item: the Natural Science Foundation of Hebei Province (Nos. A200500008; A2007000138).

for all $u, v, x, y \in T$.

A subspace I of a L.t.s T is called an ideal of T if $[ITT] \subseteq I$.

An ideal I of T is called solvable if there is a positive integer k for which $I^{(k)} \neq 0$, $I^{(k+1)} = 0$, where $I^{(1)} = [TII]$, $I^{(k)} = [TI^{(k-1)}I^{(k-1)}]$. T is called semisimple if the radical $R(T)$ (the unique maximal solvable ideal) of T is zero.

A derivation of a L.t.s T is a linear transformation D of T into T such that

$$D[xyz] = [Dxyz] + [xDyz] + [xyDz].$$

Identity (1.3) shows that all $L(x,y)$ are derivations. A derivation D of the form $D = \Sigma L(x_i, y_i)$, $\forall x_i, y_i \in T$, is said to be inner. The set $D(T)$ of all derivations of T is a Lie algebra of linear transformation acting in T, the so-called derivation algebra of T.

Let H be the subalgebra of $D(T)$ generated by all $L(x,y)$, x, $y \in T$. By $(1.3)'$, H is the linear span of the $L(x,y)'s$, which is called the inner derivation algebra of T. We consider the vector space direct sum $L(T) = H \oplus T$.

Define for elements $X_i = h_i \oplus x_i$, $h_i \in H$, $x_i \in T$, $i = 1, 2$, a product

$$[X_1, X_2] = ([h_1, h_2] + L(x_1, x_2)) \oplus (h_1 x_2 - h_2 x_1).$$

Note that, in particular, we have $[x,y] = L(x,y)$, $[h,x] = hx$, for $x, y \in T$, and $h \in H$. It is easy to prove that $L(T)$ together with the above product is a Lie algebra which is called the standard imbedding Lie algebra of T[1].

Clearly, T is the (-1) eigenspace and H is the $(+1)$ eigenspace of $\theta \in \mathrm{Aut}L(T)$ defined by $\theta(a+h) := -a + h$, for $a \in T$, $h \in H$. The automorphism θ of order 2 is called the main involution automorphism of $L(T)$.

To conclude this section, we record an important result which will be needed later on.

Theorem 1.1[2] *T is solvable if and only if the standard imbedding algebra $L(T)$ is solvable.*

2. The Killing forms of Lie triple systems

Definition 2.1[1] *The Killing form of a Lie triple system T is the bilinear form*

$$\rho(x,y) := \mathrm{tr}(R(x,y) + R(y,x)), \quad x, y \in T \ (\mathrm{tr} = \mathrm{trace}).$$

Clearly, the Killing form $\rho(x,y)$ is symmetric.

Definition 2.2 *The Killing form ρ of a Lie triple system T is called non-degenerate if $T^{\perp} = 0$, where $T^{\perp} = \{x \in T \mid \rho(x,y) = 0$, for $y \in T\}$.*

Let ρ, λ and λ_H be the Killing forms of the Lie triple system T, of the standard imbedding Lie algebra $L(T)$ and of the inner derivation algebra H of T, respectively. The following theorem describes the relationship among them.

Theorem 2.1 *Suppose ρ, λ and λ_H are given as the above. Then*

$$\lambda(H, T) = 0, \tag{2.1}$$

$$\lambda(h, h') = \lambda_H(h, h') + \mathrm{tr}(h \cdot h'), \quad \forall h, h' \in H, \tag{2.2}$$

$$\lambda(a,a')=\rho(a,a'),\quad \forall a,a'\in T,\ i.e.\ \rho=\lambda\mid_{T\times T}. \tag{2.3}$$

Proof Since $\theta: a+h\longmapsto -a+h$, $\forall h\in H$, $a\in T$, is an automorphism of $L(T)$, $-\lambda(h,a)=\lambda(h,-a)=\lambda(h,\theta a)=\lambda(\theta h,a)=\lambda(h,a)$, we have $\lambda(h,a)=0$, which proves (2.1).

Since H and T are invariant under $adh\cdot adh'$ and $L(T)=H\oplus T$, we obtain

$$\mathrm{tr}_{L(T)}(adh\cdot adh')=\mathrm{tr}_H(adh\cdot adh')+\mathrm{tr}_T(adh\cdot adh'),\quad \forall h,h'\in H.$$

But $adh\cdot adh'(a)=[h,[h'a]]=h\cdot h'(a)$, for $a\in T$, therefore $\mathrm{tr}_T(adh\cdot adh')=\mathrm{tr}(h\cdot h')$ and hence (2.2) holds.

Let $\{h_1,h_2,\ldots,h_m,a_1,a_2,\ldots,a_n\}$ be a homogeneous basis of $L(T)$ formed from bases of H and T. We assume that the elements ada_i $(i=1,2,\ldots,n)$ are expressed by means of these bases as follows:

$$[a_i,a_j]=\Sigma_p K^p_{i,j}h_p,\ \ [a_i,h_j]=\Sigma_l S^l_{i,j}a_l.$$

Then

$$\begin{aligned}
&[a_i,[a_j,h_k]]=[a_i,\Sigma_l S^l_{j,k}a_l]=\Sigma_l S^l_{j,k}\Sigma_p K^p_{i,l}h_p,\\
&\mathrm{tr}_H(ada_i\cdot ada_j)=\Sigma_k\Sigma_l S^l_{j,k}K^k_{i,l},\\
&[a_i,[a_j,a_k]]=[a_i,\Sigma_p K^p_{j,k}h_p]=\Sigma_p K^p_{j,k}\Sigma_l S^l_{i,p}a_l,\\
&\mathrm{tr}_T(ada_i\cdot ada_j)=\Sigma_p K^p_{j,k}\Sigma_k S^k_{i,p}=\Sigma_p\Sigma_k K^p_{j,k}S^k_{i,p}.
\end{aligned}$$

Obviously, $\mathrm{tr}_T(ada_i\cdot ada_j)=\mathrm{tr}_H(ada_j\cdot ada_i)$. Since H and T are invariant under $ada_i\cdot ada_j$, we obtain

$$\begin{aligned}
\mathrm{tr}_{L(T)}(ada_i\cdot ada_j)&=\mathrm{tr}_H(ada_i\cdot ada_j)+\mathrm{tr}_T(ada_i\cdot ada_j)\\
&=\mathrm{tr}_T(ada_j\cdot ada_i)+\mathrm{tr}_T(ada_i\cdot ada_j).
\end{aligned}\tag{2.4}$$

Moreover,

$$\begin{aligned}
[a_i,[a_j,a]]&=-[a_i,[a,a_j]]=[[a,a_j],a_i]=L(a,a_j)a_i=[aa_ja_i]\\
&=R(a_i,a_j)a,\quad \forall a\in T.
\end{aligned}$$

Therefore,

$$\mathrm{tr}_T(ada_i\cdot ada_j)=\mathrm{tr}R(a_i,a_j). \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$\lambda(a_i,a_j)=\mathrm{tr}(R(a_i,a_j)+R(a_j,a_i))=\rho(a_i,a_j),$$

which is the third assertion of the theorem.

Theorem 2.2 *Suppose $\rho(x,y)$ is the Killing form of a Lie triple system T. Then*

(1) $\rho(Ax,Ay)=\rho(x,y)$ *for* $A\in \mathrm{Aut}T$*;*

(2) $\rho(Dx,y)+\rho(x,Dy)=0$ *for* $D\in \mathrm{Der}T$*;*

(3) $\rho(x,y)$ *is right invariant, and left invariant, i.e.,*

$$\rho(R(a,b)x,y)=\rho(x,R(b,a)y),\quad \rho(L(a,b)x,y)=\rho(x,L(b,a)y);$$

(4) If $\rho(x,y)$ is non-degenerate, then $\rho(x,y) = 2\mathrm{tr}R(x,y)$.

Proof (1) $\forall A \in \mathrm{Aut}T$, we have $A[xyz] = [AxAyAz]$, or $AR(z,y)x = R(Az,Ay)Ax$,

$$AR(z,y) = R(Az,Ay)A,\ R(Az,Ay) = AR(z,y)A^{-1},$$

then

$$\begin{aligned}\rho(Ax,Ay) &= \mathrm{tr}(R(Ax,Ay) + R(Ay,Ax)) \\ &= \mathrm{tr}(AR(x,y)A^{-1} + AR(y,x)A^{-1}) \\ &= \mathrm{tr}(A\cdot(R(x,y)+R(y,x))A^{-1}) \\ &= \mathrm{tr}(R(x,y)+R(y,x)) \\ &= \rho(x,y).\end{aligned}$$

(2) For $D \in \mathrm{Der}T$, we have $D[xyz] = [Dxyz] + [xDyz] + [xyDz]$, i.e.,

$$DR(z,y) = R(z,y)D + R(z,Dy) + R(Dz,y),$$

then

$$\mathrm{tr}(R(z,Dy) + R(Dz,y)) = \mathrm{tr}(DR(z,y) - R(z,y)D) = 0.$$

So,

$$\rho(Dx,y) + \rho(x,Dy) = \mathrm{tr}(R(Dx,y) + R(y,Dx) + R(x,Dy) + R(Dy,x)) = 0.$$

(3) Notice that the Killing form λ of Lie algebra $L(T)$ is invariant and $\rho = \lambda\mid_{T\times T}$. For $a,b,x,y \in T$, we have

$$\begin{aligned}\rho(R(a,b)x,y) &= \lambda(R(a,b)x,y) = \lambda([xba],y) = \lambda([[x,b],a],y) \\ &= \lambda([x,b],[a,y]) = \lambda(x,[b,[a,y]]) \\ &= \lambda(x,[[y,a],b]) = \lambda(x,[yab]) \\ &= \lambda(x,R(b,a)y) = \rho(x,R(b,a)y).\end{aligned}$$

As for the proof of left invariance of ρ, we can employ identity $L(x,y) = R(x,y) - R(y,x)$, or argue by using the invariance of λ.

(4) Suppose $\rho(x,y)$ is non-degenerate. If $R(x,y)^*$ denotes the adjoint endomorphism of $R(x,y)$ in T with respect to ρ, then we get $R(x,y)^* = R(y,x)$ by the right invariancy of ρ, which implies $\rho(x,y) = 2\mathrm{tr}R(x,y)$, because $\mathrm{tr}R(x,y)^* = \mathrm{tr}R(x,y)$.

It is well known that a Lie algebra L is solvable if and only if $\lambda(x,y) = 0$, for $x \in L$, $y \in L^{(1)} = [L,L]$, where λ is the Killing form of L (Cartan's Criterion). Now we will see that there is an analogous theorem in the case of Lie triple systems. For this purpose, we first prove the following theorem:

Theorem 2.3 *Let λ and ρ be as in Theorem 2.1. The followings are equivalent:*

(a) $\lambda(x,y) = 0$, for all $x \in L(T)$, $y \in L(T)^{(1)}$;

(b) $\rho(x,y) = 0$, for all $x \in T$, $y \in T^{(1)}$.

Proof If we assume (a), then we obtain (b) by using

$$T^{(1)} = [TTT] = [H, T] \subseteq [L(T), L(T)] = L(T)^{(1)}.$$

Conversely, suppose (b) holds. Then for $a_i, b_i \in T$, $i = 1, 2$, we have by the invariance of λ,

$$\lambda(L(a_1, b_1), L(a_2, b_2)) = \lambda([a_1, b_1], [a_2, b_2]) = \lambda(a_1, [b_1a_2b_2]) = \rho(a_1, [b_1a_2b_2]) = 0,$$

which implies $\lambda(H, H) = 0$. Moreover, we have $\lambda(H, T) = 0$ by Theorem 2.1.

Next, we see that

$$L(T)^{(1)} = [T + [T, T], T + [T, T]] \subseteq T^{(1)} + [T, T] = T^{(1)} + H.$$

Let $x \in L(T)$, $x = a_1 + h_1$, $a_1 \in T$, $h_1 \in H$, $y \in L(T)^{(1)}$, $y = a_2 + h_2$, $a_2 \in T^{(1)}$, $h_2 \in H$. Then

$$\lambda(x, y) = \lambda(a_1, a_2) = \rho(a_1, a_2) = 0.$$

Now we are in a position to state the Cartan's Criterion for solvability in the case of Lie triple systems as follows:

Corollary 2.4 *A Lie triple system T is solvable if and only if its Killing form $\rho(x, y) = 0$, for all $x \in T$, $y \in T^{(1)}$.*

Proof This follows at once from Theorems 1.1, 2.3 and the Cartan's Criterion for Lie algebras.

The following two Lemmas are easy to verify.

Lemma 2.5 *Let $I \subset T$ be an ideal of T. Then $I^{\perp}$ is also an ideal of T, where*

$$I^{\perp} = \{x \in T \mid \rho(x, y) = 0, \quad \forall y \in I\}.$$

Lemma 2.6 *Let I be an ideal of T. Denote by $\rho_I(x, y)$ and $\rho(x, y)$ the Killing forms of I and T respectively. Then $\rho_I(x, y) = \rho(x, y)$, $\forall x, y \in I$.*

Theorem 2.7 *Let T be a finite dimensional Lie triple system over a field of characteristic 0. Then T is semisimple if and only if its Killing form $\rho(x, y)$ is non-degenerate.*

Proof Suppose first that the radical $R(T) = 0$. Clearly, $\rho(T^{\perp}, T) = 0$, in particular, $\rho(T^{\perp}, T^{\perp(1)}) = 0$. Considering Lemma 2.6 and applying Corollary 2.4 to the ideal $T^{\perp}$, we conclude that $T^{\perp}$ is solvable, and $T^{\perp} \subseteq R(T) = 0$, i.e., ρ is non-degenerate.

Conversely, suppose that the Killing form $\rho(x, y)$ is non-degenerate, i.e., $T^{\perp} = \{0\}$. To prove that T is semisimple it suffices to prove that every solvable ideal of T is included in $T^{\perp}$.

Let I be a solvable ideal of T, such that $I^{(r+1)} = 0$, $I^{(r)} \neq 0$ $(r \geq 0)$. Since $I^{(r+1)} = (I^{(r)})^{(1)}$, we may assume that there exists an ideal $I \neq 0$, such that $I^{(1)} = 0$. Choose a basis $\{e_1, e_2, \ldots, e_r, e_{r+1}, \ldots, e_n\}$ for T, $r \leq n$, such that $\{e_1, e_2, \ldots, e_r\}$ is a basis for I. Notice that

$$[TIT] + [TTI] + [ITT] \subset I, \ [TII] = [ITI] = [IIT] = 0. \tag{2.6}$$

Set

$$R(e_{i_3}, e_{i_2})e_{i_1} = \Sigma_{p=1}^{n} a_{i_1,i_2,i_3}^{p} e_p, \ a_{i_1,i_2,i_3}^{p} \in F, \ 1 \leq i_1, i_2, i_3 \leq n.$$

From (2.6) we have

$$a^p_{i_1,i_2,i_3} = 0, \tag{2.7}$$

when $p > r$ and at least one $i_j \le r$, or at least two $i_j \le r$, among i_1, i_2, i_3.

Let $1 \le l, i_2 \le n$, $1 \le i_1 \le r$. Then from (2.7) we obtain

$$R(e_{i_1}, e_{i_2})e_l = \Sigma^n_{p=1} a^p_{l,i_2,i_1} e_p = \Sigma^r_{p=1} a^p_{l,i_2,i_1} e_p.$$

Therefore,

$$\mathrm{tr}R(e_{i_1}, e_{i_2}) = \Sigma^r_{l=1} a^l_{l,i_2,i_1} = 0.$$

In other words, $\mathrm{tr}R(I,T) = 0$. Similarly, $\mathrm{tr}R(T,I) = 0$, hence $\rho(I,T) = 0$, which yields $I \subseteq T^{\perp} = 0$, as desired.

Remark Using the standard imbedding Lie algebras, Meyberg[1] has already proved this theorem. Here, we present a direct proof by using the definition of the Killing forms of Lie triple systems.

3. The complexification of real Lie triple systems

In this section we assume $F = R$ or C, i.e., all Lie triple systems are assumed to be real or complex. First we introduce a definition for a Lie triple system, which is an analogue in the case of Lie algebras[3].

Definition 3.1 *Let T be a Lie triple system over R. An R-linear endomorphism J of T is called a compatible complex structure, if J satisfies:*

(1) J is a complex structure on the vector space T, that is, $J^2 = -\mathrm{id}$.

(2) $J[xyz] = [(Jx)yz]$, for $x, y, z \in T$.

From condition (2) and the definition of L.t.s, we have

$$J[xyz] = [(Jx)yz] = [x(Jy)z] = [xy(Jz)].$$

It is easy to verify that a real Lie triple system T_0 with a compatible complex structure J can be turned into a complex Lie triple system, denoted by $\overline{T_0}$, by putting

$$(a + ib)x = ax + bJx,\ i = \sqrt{-1},\ x \in T_0,\ a, b \in R,$$

and with the ternary operation inherited from T_0.

Let T_0 be an arbitrary real Lie triple system. We form the tensor product $T_0^C := C \otimes_R T_0$ and regard it as a vector space over $C : \alpha(\beta \otimes x) := \alpha\beta \otimes x$, for any $\alpha, \beta \in C, x \in T_0$. Obviously, T_0^C is a Lie triple system with $[\alpha \otimes x, \beta \otimes y, \gamma \otimes z] = \alpha\beta\gamma \otimes [xyz]$.

This complex Lie triple system is called the complexification of T_0. We can formally think of T_0^C as

$$T_0^C = \{x + iy \mid x, y \in T_0 \ \text{ and } \ i = \sqrt{-1}\}.$$

A real Lie triple system T_0 is called a real form of a complex Lie triple system T if its complexification is isomorphic to T.

On the other hand, given a complex Lie triple system T, then by restricting the ground field to the real field, we obtain a real Lie triple system denoted by T^R, which will be called the realification of T.

Let $J : x \to ix$, for any $x \in T^R$. Obviously, J is a compatible complex structure of T^R, which is called the regular complex structure on T^R. Clearly, $\overline{T^R} = T$.

Let T_0 be an arbitrary real Lie triple system, T the complexification of T_0, T^R the realification of T. Notice that $T = T_0 \dot{+} iT_0$ and $T^R = T_0 \dot{+} JT_0$.

Theorem 3.1 *Let ρ_0, ρ and ρ^R denote the Killing forms of Lie triple systems T_0, T and T^R, respectively. Then:*

(1) $\rho_0(x, y) = \rho(x, y)$, for $x, y \in T_0$;

(2) $\rho^R(x, y) = 2\mathrm{Re}\rho(x, y)$, for $x, y \in T^R$ (Re=real part).

Proof Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of real Lie triple system T_0. Obviously, it is also a basis of T, and $\{x_1, x_2, \ldots, x_n, Jx_1, Jx_2, \ldots, Jx_n\}$ is a basis of T^R, where J is the regular complex structure of T^R.

Let $x, y \in T_0$, the endomorphism $R(x, y)$ viewed as acting on T_0 or as acting on T, has the same matrix expression with respect to the basis $\{x_1, x_2, \ldots, x_n\}$. Thus, the first relation is proved.

For the second, $\forall x, y \in T^R (= T)$, let $B + \sqrt{-1}C$ denote the matrix of $R(x, y)$ with respect to the basis $\{x_1, x_2, \ldots, x_n\}$ of T, B and C being real. In other words,

$$R(x, y)x_i = \Sigma_{j=1}^n b_{ji}x_j + \sqrt{-1}\Sigma_{j=1}^n c_{ji}x_j, \quad i = 1, \ldots, n,$$

$$B = (b_{ji})_{n\times n}, \quad C = (c_{ji})_{n\times n}.$$

Then,

$$\begin{aligned} R(x, y)(Jx_i) &= [(Jx_i)yx] = J[x_iyx] = JR(x, y)x_i \\ &= J(\Sigma_{j=1}^n b_{ji}x_j + \sqrt{-1}\Sigma_{j=1}^n c_{ji}x_j) \\ &= -\Sigma_{j=1}^n c_{ji}x_j + \Sigma_{j=1}^n b_{ji}Jx_j. \end{aligned}$$

This shows that as an endomorphism on T^R, the matrix of $R(x, y)$ with respect to the basis $\{x_1, x_2, \ldots, x_n, Jx_1, Jx_2, \ldots, Jx_n\}$ is

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

Then, $\mathrm{tr}_{T^R} R(x, y) = 2\mathrm{Re}\,\mathrm{tr}_T R(x, y)$, and $\rho^R(x, y) = 2\mathrm{Re}\rho(x, y)$.

Theorem 3.2 *The Killing forms ρ_0, ρ and ρ^R are all non-degenerate if and only if one of them is.*

Proof (1) Assume that ρ_0 is non-degenerate. Suppose $x + iy \in T$, where $x, y \in T_0$, such that $\rho(x + iy, T) = 0$. Then

$$\rho(x, T_0) + i\rho(y, T_0) = 0.$$

From Theorem 3.1(1), $\rho_0(x, T_0) = \rho_0(y, T_0) = 0$. Since ρ_0 is non-degenerate, we have $x = y = 0$, then $x + iy = 0$. Therefore, ρ is non-degenerate.

(2) Assume ρ is non-degenerate. For any $x, y \in T^R$, by Theorem 3.1(2),

$$\rho^R(Jx, y) = 2\mathrm{Re}\rho(ix, y) = -2\mathrm{Im}\rho(x, y).$$

Let $y \in T^R$, such that, $\rho^R(y, x) = 0$, for all $x \in T^R$. Then by Theorem 3.1(2), $\mathrm{Re}\rho(x, y) = 0$. Moreover, $\mathrm{Im}\rho(x, y) = -\frac{1}{2}\rho^R(Jx, y) = -\frac{1}{2}\rho^R(y, Jx) = 0$, so $\rho(x, y) = 0$, $\forall x \in T^R$, then $\rho(y, T^R) = 0$. That is, $\rho(y, T) = 0$, hence $y = 0$. Therefore, ρ^R is non-degenerate.

(3) Assume ρ^R is non-degenerate. Let $x \in T_0$, such that $\rho_0(x, T_0) = 0$. From Theorem 3.1(1), $\rho(x, T_0) = 0$, hence, $\rho(x, T) = 0$. i.e.,

$$\rho(x, y) = 0, \quad \forall y \in T.$$

From Theorem 3.1(2), $\rho^R(x, y) = 2\mathrm{Re}\rho(x, y) = 0$, $\forall y \in T^R$. Since ρ^R is non-degenerate, $x = 0$. Therefore, $\rho_0(x, y)$ is non-degenerate.

Corollary 3.3 *T_0, T and T^R are all semi-simple if and only if one of them is.*

This Corollary follows from Theorems 2.7 and 3.2.

Theorem 3.4 *Suppose ρ_0, ρ and ρ^R are given as the above. Then the following are equivalent:*

(a) $\rho_0(x, y) = 0$, for all $x \in T_0$, $y \in T_0^{(1)}$;

(b) $\rho(x, y) = 0$, for all $x \in T$, $y \in T^{(1)}$;

(c) $\rho^R(x, y) = 0$, for all $x \in T^R$, $y \in (T^R)^{(1)}$.

Proof (a)$\Rightarrow$(b). This follows from the identities: $T = T_0 \dot{+} iT_0$, $T^{(1)} = T_0^{(1)} \dot{+} iT_0^{(1)}$ and Theorem 3.1(1).

(b)$\Rightarrow$(c). This follows from Theorem 3.1(2) and the fact that T and T^R agree set theoretically.

(c)$\Rightarrow$(a). If (c) holds, then $\rho^R(x, y) = 0$, for all $x \in T_0$, $y \in T_0^{(1)}$, because $T^R = T_0 \dot{+} JT_0$ and $(T^R)^{(1)} = T_0^{(1)} \dot{+} JT_0^{(1)}$. Therefore, by Theorem 3.1,

$$\rho_0(x, y) = \rho(x, y) = \mathrm{Re}\rho(x, y) = \frac{1}{2}\rho^R(x, y) = 0.$$

Corollary 3.5 *T_0, T and T^R are all solvable if and only if one of them is.*

References

[1] MEYBERG K. *Lectures on Algebras and Triple Systems* [M]. The University of Virginia, Charlottesville, Va., 1972.

[2] LISTER W G. *A structure theory of Lie triple systems* [J]. Trans. Amer. Math. Soc., 1952, **72**: 217–242.

[3] HELGASON S. *Differential Geometry, Lie Groups, and Symmetric Spaces* [M]. Academic Press, Inc., New York-London, 1978.

[4] ZHANG Zhixue, SHI Yiqian, ZHAO Lina. *Invariant symmetric bilinear forms on Lie triple systems* [J]. Comm. Algebra, 2002, **30**(11): 5563–5573.

[5] ZHANG Zhixue, LI Huajun, DONG Lei. *Invariant bilinear forms on anti-Lie triple systems* [J]. Chinese Ann. Math. Ser. A, 2004, **25**(4): 429–436. (in Chinese)

[6] ZHAO Li'na, LI Xuewen, ZHANG Zhixue. *Non-degenerate invariant bilinear forms on nonassociative triple systems* [J]. Chinese Ann. Math. Ser. B, 2005, **26**(2): 275–290.