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On Stabilization for Linear Time-Varying Systems

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Abstract This paper deals with the stabilization problem for linear time-varying systems within the framework of nest algebras. We give a necessary and sufficient condition for a class of plants to be stabilizable, and we also study the simultaneous and strong stabilization problems.

Keywords nest algebra; causal; linear system; coprime factorization; stabilization.

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1. Introduction

The stabilization problem for linear systems has been studied within various frameworks. For a discussion of the origins of the problem, we refer to [1]. In the course of discussion of stabilization, the coprime factorization is always used. It was shown in [1, 2] that every internally stabilizable causal linear time-varying system admits doubly coprime factorizations and there is a Youla-kučera-like parametrization of all stabilizing controllers which is conceptually similar to the classical result for LTI systems. However, it is shown that internal stabilizability is generally not equivalent to the existence of doubly coprime factorizations. Many results on stabilization problems were presented in a number of literatures, for example, see [3–8].

Within the framework of nest algebras (see [9]), the purpose of this paper is to study the stabilization problem for linear time-varying systems. The several authors have studied the strong stabilization problems within the framework of nest algebras[10,11]. In this paper we give a necessary and sufficient condition for the stabilization of the standard feedback system which does not admit coprime factorizations. We also study the simultaneous and strong stabilization problems.

This paper is organized as follows. In Section 2, we give some notations and definitions and introduce the linear system within the framework of nest algebras. In Section 3, we give a necessary and sufficient condition for the stabilization of the standard feedback system which does not admit coprime factorizations. Sections 4 and 5 are devoted to simultaneous stabilization and strong stabilization problems, respectively. The paper ends with cited references.

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2. Preliminaries

We recall some basic concepts[9] that will be useful in this paper. First, we introduce the definitions about complete nest and nest algebra.

Let $\mathcal{H}$ be a separable complex Hilbert space. $\mathcal{L}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, the range $R(T)$ of T is $\{Tx : x \in \mathcal{H}\}$. T is an orthogonal projection if T is idempotent ($T = T^2$) and self-adjoint.

Definition 2.1 *A family $\mathcal{N}$ of closed subspaces of the Hilbert space $\mathcal{H}$ is a complete nest if*

(1) $\{0\}, \mathcal{H} \in \mathcal{N}$.

(2) For $N_1, N_2 \in \mathcal{N}$, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

(3) If N_α is a subfamily in $\mathcal{N}$, then $\cap_\alpha N_\alpha$ and $\vee_\alpha N_\alpha$ are also in $\mathcal{N}$.

Every closed subspace N of $\mathcal{H}$ is identifiable with the orthogonal projection P_N with range N and therefore we can think of a nest as a family of projections.

Every set $\mathcal{P}$ of projections in $\mathcal{L}(\mathcal{H})$ determines an algebra Alg$\mathcal{P}$ of operators,

$$\mathrm{Alg}\mathcal{P} = \{T \in \mathcal{L}(\mathcal{H}) : (I - P)TP = 0, P \in \mathcal{P}\}.$$

If $\mathcal{N}$ is a nest, and $\mathcal{P}$ is its associated family of orthogonal projections, Alg$\mathcal{P}$ is called a nest algebra.

Suppose $\mathcal{H}$ is a separable complex Hilbert space, and $\mathcal{P}$ is a complete nest on $\mathcal{H}$. We parametrize $\mathcal{P}$ and write $\mathcal{P} = \{P_t : t \in \Gamma\}$. Let $Q_t = I - P_t$, $\mathcal{R} = \{Q_t : t \in \Gamma\}$. We assume $P_{t_1} \le P_{t_2}$ for $t_1 \le t_2$, and for each $t \in \Gamma$ we define a seminorm on $\mathcal{H}$ by $\|x\|_t = \|P_t x\|$, $x \in \mathcal{H}$, $P_t \ne I$. The family $\{\| \|_t : t \in \Gamma\}$ of seminorms defines a topology on $\mathcal{H}$, called the resolution topology. Convergence in this topology is described as follows: a sequence $\{x_n\}$ converges to $x \in \mathcal{H}$ if, for all seminorms, $\|x_n - x\|_t \longrightarrow 0$. The resolution topology is a metric topology[9]. Let $\mathcal{H}_e$ denote the completion of the metric space $\mathcal{H}$.

Definition 2.2 *A linear transformation T on $\mathcal{H}_e$ is causal if for each $t \in \Gamma$, $P_t T = P_t T P_t$. A linear system on $\mathcal{H}_e$ is a causal linear transformation on $\mathcal{H}_e$, which is continuous with respect to the resolution topology.*

It is clear that the set of linear systems on $\mathcal{H}_e$ is an algebra. We denote this algebra by $\mathcal{L}$.

Definition 2.3 *A linear transformation $T : \mathcal{H}_e \longrightarrow \mathcal{H}_e$ is stable if there exists $M > 0$ such that for each $x \in \mathcal{H}_e$ and $t \in \Gamma$, $\|Tx\|_t \le M\|x\|_t$.*

We denote the collection of stable linear transformations on $\mathcal{H}_e$ by $\mathcal{S}$. Then $\mathcal{S}$ is a weakly closed algebra containing the identity[12].

The following proposition is Theorem 5.4.2 of [9].

Proposition 2.1 *The following are equivalent:*

(1) T on $\mathcal{H}_e$ is stable.

(2) T is causal and $T|\mathcal{H}$ is a bounded operator.

(3) $T \in \mathcal{L}$ and is the extension to $\mathcal{H}_e$ of an operator in Alg$\mathcal{R}$.

This proposition allows us to identify the algebra $\mathcal{S}$ of stable operators on $\mathcal{H}_e$ with the nest algebra Alg$\mathcal{R}$. The restriction of $T \in \mathcal{S}$ to $\mathcal{H}$ is in Alg$\mathcal{R}$ and the extension of $S \in \text{Alg}\mathcal{R}$ to $\mathcal{H}_e$ is in $\mathcal{S}$.

From now, let $\mathcal{H} = \ell^2$ be the usual Hilbert space of square summable sequences with the standard norm $\|x\|_2^2 = \sum_{j=0}^{\infty} |x_j|^2 < \infty$, $x = \langle x_0, x_1, x_2, \ldots \rangle \in \ell^2$. Then $\mathcal{H}$ is a Hilbert space with an inner product

$$(x, y) = \sum_{n=1}^{\infty} x_n \bar{y}_n.$$

It is easy to check that $\mathcal{H}_e = \{\langle x_0, x_1, x_2, \ldots \rangle : x_i \in \mathbf{C}\}$.

For each $n \geq 0$, let P_n denote the standard truncation projection defined on $\mathcal{H}$ and $\mathcal{H}_e$ by

$$P_n \langle x_0, x_1, \ldots, x_n, x_{n+1}, \ldots \rangle = \langle x_0, x_1, \ldots, x_n, 0, 0, \ldots \rangle,$$

and $Q_n = I - P_n$. Let $P_{-1} = 0$ and $P_\infty = I$. Then $\mathcal{P} = \{P_n : n = -1, 0, 1, \ldots\}$ is a complete nest, as is $\mathcal{R} = \{Q_n : n = -1, 0, 1, \ldots\}$. We will be concerned with the following nest algebra

$$\text{Alg}\mathcal{R} = \{T \in \mathcal{B}(\mathcal{H}) : (I - Q_n) T Q_n = 0\}.$$

We also need the notions of right and left strong representations for a linear system. Recall that the graph of a linear transformation L with domain $\mathcal{D}(L)$ in $\mathcal{H}$ is

$$G(L) = \left\{ \begin{pmatrix} x \\ Lx \end{pmatrix} : x \in \mathcal{D}(L) \right\}.$$

The following definitions are from [9].

Definition 2.4 *A plant L has a strong right representation $\begin{pmatrix} M \\ N \end{pmatrix}$ with M and N stable if*

(1) $G(L) = \text{Ran} \begin{pmatrix} M \\ N \end{pmatrix}$,

(2) $\begin{pmatrix} M \\ N \end{pmatrix}$ *has a stable left inverse, i.e., there exist X, Y stable such that*

$$(Y, X) \begin{pmatrix} M \\ N \end{pmatrix} = I.$$

A plant L has a strong left representation $(-\hat{N}, \hat{M})$ with $\hat{M}, \hat{N}$ stable if

(1) $G(L) = \text{Ker}(-\hat{N}, \hat{M})$,

(2) $(-\hat{N}, \hat{M})$ *has a stable right inverse, i.e., there exist $\hat{X}, \hat{Y}$ stable such that*

$$(-\hat{N}, \hat{M}) \begin{pmatrix} -\hat{X} \\ \hat{Y} \end{pmatrix} = I.$$

The following result on strong representation was proved in [9] .

Theorem 2.1 *Suppose $M, N \in \mathcal{S}$. Then $\begin{pmatrix} M \\ N \end{pmatrix}$ is a strong right representation of $L \in \mathcal{L}$ if and only if*

(1) There exist $X, Y \in \mathcal{S}$ such that $(Y, X)\begin{pmatrix} M \\ N \end{pmatrix} = I$.

(2) M is invertible in $\mathcal{L}$.

Remark We have seen that a right representation $\begin{pmatrix} M \\ N \end{pmatrix}$ for L gives a right fractional representation $L = NM^{-1}$ for L. If the representation is strong, this representation is called a right coprime factorization for L. The dual representation gives a left fractional representation $L = \hat{M}^{-1}\hat{N}$ and if it is strong, we have a left coprime factorization for L.

3. Stabilization problem

In this section, we consider the standard feedback system and study its stabilization problem.

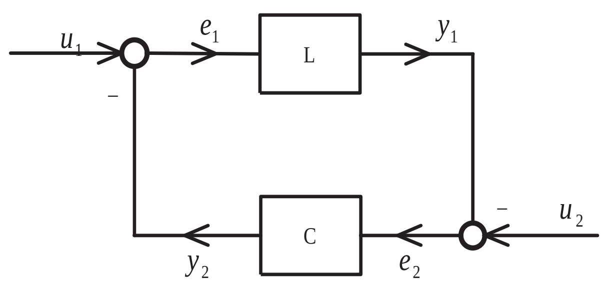

Figure 1 Standard feedback configuration

In Figure 1, L represents a given system (plant) and C a compensator or controller. u_1, u_2 denote the externally applied inputs; e_1, e_2 denote the inputs to the plant and compensator, respectively; and y_1, y_2 denote the outputs of the compensator and plants, respectively. The closed loop system equation is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I & C \\ L & -I \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

The system is well posed if the internal input $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ can be expressed as a causal function of the external input $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. This is equivalent to requiring that $\begin{pmatrix} I & C \\ L & -I \end{pmatrix}$ be invertible [9, Chapter 6]. This inverse is easily computed formally and is given by the transfer matrix

$$H(L, C) = \begin{pmatrix} (I + CL)^{-1} & C(I + LC)^{-1} \\ L(I + CL)^{-1} & -(I + LC)^{-1} \end{pmatrix}.$$

L and C may not be stable. This means that there may be an input u in $\mathcal{H}$ such that Lu or Cu may not be in $\mathcal{H}$. Let $\mathcal{D}(L) = \{u \in \mathcal{H} : Lu \in \mathcal{H}\}$ and $\mathcal{D}(C) = \{u \in \mathcal{H} : Cu \in \mathcal{H}\}$. Then $\begin{pmatrix} I & C \\ L & -I \end{pmatrix}$ can be seen as a linear transformation from $\mathcal{D}(L) \bigoplus \mathcal{D}(C)$ into $\mathcal{H} \bigoplus \mathcal{H}$.

Definition 3.1 *The closed loop system determined by the plant L and the compensator C is stable if all the entries of $H(L, C)$ are stable systems on $\mathcal{H}$. The plant L is stabilizable if there exists a linear system C such that the closed loop system determined by L and C is stable.*

It is well known that a given plant is stabilizable if and only if it has a doubly coprime factorization[9, Chapter 6]. Now we give a necessary and sufficient condition for a class of plants to be stabilizable.

Theorem 3.1 *Suppose $L = NM^{-1}$, $M, N \in \mathcal{S}$. Then L is stabilizable if and only if there exist $X, Y \in \mathcal{L}$ such that Y is invertible and*

(1) $(Y, X)\begin{pmatrix} M \\ N \end{pmatrix} = YM + XN = I.$

(2) Every entry of $\begin{pmatrix} M \\ N \end{pmatrix}(Y, X) = \begin{pmatrix} MY & MX \\ NY & NX \end{pmatrix}$ *is in $\mathcal{S}$. If these conditions are satisfied, then the controller $C = Y^{-1}X$ stabilizes L.*

Proof Suppose that the compensator C stabilizes L, $L = NM^{-1}$. Then we have

$$A_1 = (I + CL)^{-1} \in \mathcal{S}, \quad A_2 = C(I + LC)^{-1} = (I + CL)^{-1}C \in \mathcal{S},$$

$$A_3 = L(I + CL)^{-1} \in \mathcal{S}, \quad A_4 = (I + LC)^{-1} \in \mathcal{S}.$$

Let $X = M^{-1}A_2, Y = M^{-1}A_1, X, Y \in \mathcal{L}$, and $C = A_1^{-1}A_2 = Y^{-1}X$. Then

$$(Y, X)\begin{pmatrix} M \\ N \end{pmatrix} = YM + XN = M^{-1}(A_1M + A_2N) = M^{-1}(I + CL)^{-1}(M + CN)$$

$$= M^{-1}(I + CL)^{-1}(I + CL)M = I.$$

This implies that L satisfies the condition (1).

Since

$$A_1 = MY, \quad A_2 = (I + CL)^{-1}C = MYY^{-1}X = MX,$$

$$A_3 = LA_1 = NM^{-1}MY = NY,$$

$$A_4 = (I + LC)^{-1} = I - LC(I + LC)^{-1} = I - NM^{-1}MX = I - NX,$$

and

$$\begin{pmatrix} M \\ N \end{pmatrix}(Y, X) = \begin{pmatrix} MY & MX \\ NY & NX \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & I - A_4 \end{pmatrix},$$

every entry of $\begin{pmatrix} M \\ N \end{pmatrix}(Y, X)$ is in $\mathcal{S}$. This implies that L satisfies the condition (2).

Conversely, suppose that the plant L satisfies the conditions of the theorem. Let $C = Y^{-1}X$. By the condition (1),

$$(I + CL) = I + Y^{-1}XNM^{-1} = Y^{-1}(YM + XN)M^{-1} = (MY)^{-1},$$

we have

$$(I + CL)^{-1} = MY, \quad (I + CL)^{-1}C = MYY^{-1}X = MX,$$

$$L(I + CL)^{-1} = NM^{-1}MY = NY,$$

and

$$(I + LC)^{-1} = I - LC(I + LC)^{-1} = I - NM^{-1}MX = I - NX.$$

Since

$$H(L, C) = \begin{pmatrix} (I + CL)^{-1} & C(I + LC)^{-1} \\ L(I + CL)^{-1} & -(I + LC)^{-1} \end{pmatrix} = \begin{pmatrix} MY & MX \\ NY & NX - I \end{pmatrix}$$

and by the condition (2), we obtain that every entry of $H(L, C)$ is in $\mathcal{S}$, and thus $C = Y^{-1}X$ stabilizes L. The proof is completed. $\square$

The following theorem is just a dual result of Theorem 3.1.

Theorem 3.2 *Suppose $L = \hat{M}^{-1}\hat{N}, \hat{M}, \hat{N} \in S$. Then L is stabilizable if and only if there exist $\hat{X}, \hat{Y} \in \mathcal{L}$ such that $\hat{Y}$ is invertible and*

(1) $(-\hat{N}, \hat{M}) \begin{pmatrix} -\hat{X} \\ \hat{Y} \end{pmatrix} = \hat{N}\hat{X} + \hat{M}\hat{Y} = I.$

(2) Every entry of $\begin{pmatrix} -\hat{X} \\ \hat{Y} \end{pmatrix} (-\hat{N}, \hat{M}) = \begin{pmatrix} \hat{X}\hat{N} & -\hat{X}\hat{M} \\ -\hat{Y}\hat{N} & \hat{N}\hat{N} \end{pmatrix}$ *is in $\mathcal{S}$.*

If these conditions are satisfied, then the controller $C = \hat{X}\hat{Y}^{-1}$ stabilizes L.

Corollary 3.1 *(1) Suppose that $L = NM^{-1} \in \mathcal{L}$ and there exist $X, Y \in \mathcal{L}$ such that $(Y, X) \begin{pmatrix} M \\ N \end{pmatrix} = I$. If $MY, MX, NX, NY \in \mathcal{S}$ and Y is invertible in $\mathcal{L}$, then L is stabilizable and $C = Y^{-1}X$ stabilizes L.*

(2) Suppose that $L = \hat{M}^{-1}\hat{N} \in \mathcal{L}$ and there exist $\hat{X}, \hat{Y} \in \mathcal{L}$ such that $(-\hat{N}, \hat{M}) \begin{pmatrix} -\hat{X} \\ \hat{Y} \end{pmatrix} = I$. If $\hat{X}\hat{N}, \hat{X}\hat{M}, \hat{Y}\hat{N}, \hat{Y}\hat{M} \in \mathcal{S}$ and $\hat{Y}$ is invertible in $\mathcal{L}$, then L is stabilizable and $C = \hat{X}\hat{Y}^{-1}$ stabilizes L.

The following corollary is immediate from Theorems 3.1 and 3.2.

Corollary 3.2 *(1) Suppose $L = NM^{-1} \in \mathcal{L}$ and $\begin{pmatrix} M \\ N \end{pmatrix}$ is a strong right representation of L, i.e., there exist $X, Y \in \mathcal{S}$ such that $(Y, X) \begin{pmatrix} M \\ N \end{pmatrix} = I$. If Y is invertible in $\mathcal{L}$, then L is stabilizable and $C = Y^{-1}X$ stabilizes L.*

(2) Suppose $L = \hat{M}^{-1}\hat{N} \in \mathcal{L}$ and $(-\hat{N}, \hat{M})$ is a strong left representation of L, i.e., there exist $\hat{X}, \hat{Y} \in \mathcal{S}$ such that $(-\hat{N}, \hat{M})\begin{pmatrix} -\hat{X} \\ \hat{Y} \end{pmatrix} = I$. If $\hat{Y}$ is invertible, then L is stabilizable and $C = \hat{X}\hat{Y}^{-1}$ stabilizes L.

(3) Suppose that $L \in \mathcal{L}$ has a strong left representation $(-\hat{N}, \hat{M})$ and a strong right representation $\begin{pmatrix} M \\ N \end{pmatrix}$. If there exist $X, Y, \hat{X}, \hat{Y} \in \mathcal{S}$ such that $Y, \hat{Y}$ are invertible in $\mathcal{L}$ and satisfy the double Bezout identity

$$\begin{pmatrix} Y & X \\ -\hat{N} & \hat{M} \end{pmatrix}\begin{pmatrix} M & -\hat{X} \\ N & \hat{Y} \end{pmatrix} = \begin{pmatrix} M & -\hat{X} \\ N & \hat{Y} \end{pmatrix}\begin{pmatrix} Y & X \\ -\hat{N} & \hat{M} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then L is stabilizable and $C = Y^{-1}X = \hat{X}\hat{Y}^{-1}$ stabilizes L.

4. Simultaneous Stabilization

We now turn to the problem of simultaneous stabilization. Given L_1, L_2, when does there exist $C \in \mathcal{L}$ for which $\{L_1, C\}, \{L_2, C\}$ are both stable?

The following Lemma is Theorem 6.4.8 in [9].

Lemma 4.1 *Suppose $L \in \mathcal{L}$ and there exist $M, N, X, Y, \hat{M}, \hat{N}, \hat{X}, \hat{Y} \in \mathcal{S}$ such that $\begin{pmatrix} M \\ N \end{pmatrix}$ and $(-\hat{N}, \hat{M})$ are, respectively, strong right and left representation for L that satisfy the double Bezout identity*

$$\begin{pmatrix} Y & X \\ -\hat{N} & \hat{M} \end{pmatrix}\begin{pmatrix} M & -\hat{X} \\ N & \hat{Y} \end{pmatrix} = \begin{pmatrix} M & -\hat{X} \\ N & \hat{Y} \end{pmatrix}\begin{pmatrix} Y & X \\ -\hat{N} & \hat{M} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then

(1) L is stabilizable.

(2) $C \in \mathcal{L}$ stabilizes L if and only if it has a strong right representation $\begin{pmatrix} \hat{Y} - NQ \\ \hat{X} + MQ \end{pmatrix}$ and a strong left representation $(-(X + Q\hat{M}), Y - Q\hat{N})$ for some $Q \in \mathcal{S}$.

We denote by $\mathcal{S}(L)$ the set of controllers $C \in \mathcal{L}$ for which $\{L, C\}$ is stable. Lemma 4.1 gives us a parametric representation for every element of $\mathcal{S}(L)$.

The following Lemma is Theorem 6.3.5 in [9].

Lemma 4.2 *Suppose $L \in \mathcal{L}$ has a strong right representation $\begin{pmatrix} M \\ N \end{pmatrix}$. Then any strong right representation $\begin{pmatrix} M_1 \\ N_1 \end{pmatrix}$ of L is of the form $\begin{pmatrix} M \\ N \end{pmatrix} S$ with S invertible in $\mathcal{S}$.*

Theorem 4.1 *Suppose $L \in \mathcal{L}$ satisfies the condition of Lemma 4.1. Then any $C \in \mathcal{S}(L)$ is stabilizable and L stabilizes C.*

Proof If $C \in \mathcal{S}(L)$, then there exists a $Q \in \mathcal{S}$ such that $\begin{pmatrix} \hat{Y} - NQ \\ \hat{X} + MQ \end{pmatrix}$ and $(-(X + Q\hat{M}), Y - Q\hat{N})$ are strong right representation and strong left representation of C, respectively. Since

$$\hat{M}\hat{Y} + \hat{N}\hat{X} = I, \hat{N}M - \hat{M}N = 0,$$

we have

$$(\hat{M}, \hat{N}) \begin{pmatrix} \hat{Y} - NQ \\ \hat{X} + MQ \end{pmatrix} = \hat{M}\hat{Y} + \hat{N}\hat{X} + (\hat{N}M - \hat{M}N)Q = I.$$

Since $\hat{M}$ is invertible, by Corollary 3.2 (1), we obtain that C is stabilizable and $L = \hat{M}^{-1}\hat{N}$ stabilizes C.

Now we are in a position to state the main result of this section.

Theorem 4.2 *Suppose $L \in \mathcal{L}$ satisfies the conditions of Lemma 4.1 with M, N, X, Y, $\hat{M}$, $\hat{N}$, $\hat{X}$, $\hat{Y} \in \mathcal{S}$ and $L_1 \in \mathcal{L}$ also satisfies the conditions of Lemma 4.1. Then there exists $C \in \mathcal{L}$ which simultaneously stabilizes L, L_1 if and only if there exist Q, $R \in \mathcal{S}$ such that $\begin{pmatrix} M - (\hat{X} + MQ)R \\ N + (\hat{Y} - NQ)R \end{pmatrix}$ is a strong right representation and $(-(\hat{N}+R(Y-Q\hat{N})), \hat{M}-R(X+Q\hat{M}))$ is a strong left representation of L_1, respectively.*

Proof If L, L_1 are simultaneously stabilized by C, then by Lemma 4.1 there exists a $Q \in \mathcal{S}$ such that $C = (\hat{X} + MQ)(\hat{Y} - NQ)^{-1} \in \mathcal{S}(L)$ and $C \in \mathcal{S}(L_1)$. By Theorem 4.1, we know that L_1 must stabilize C, namely, $L_1 \in \mathcal{S}(C)$. Since

$$(\hat{M}, \hat{N}) \begin{pmatrix} \hat{Y} - NQ \\ \hat{X} + MQ \end{pmatrix} = I$$

and

$$(-(X + Q\hat{M}), Y - Q\hat{N}) \begin{pmatrix} -N \\ M \end{pmatrix} = XN + YM = I,$$

the double Bezout identity

$$\begin{aligned} &\begin{pmatrix} \hat{M} & \hat{N} \\ -(X + Q\hat{M}) & Y - Q\hat{N} \end{pmatrix} \begin{pmatrix} \hat{Y} - NQ & -N \\ \hat{X} + MQ & M \end{pmatrix} \\ &= \begin{pmatrix} \hat{Y} - NQ & -N \\ \hat{X} + MQ & M \end{pmatrix} \begin{pmatrix} \hat{M} & \hat{N} \\ -(X + Q\hat{M}) & Y - Q\hat{N} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

holds.

By Lemma 4.1, C is stabilizable and a controller stabilizes C if and only if it has a strong right representation $\begin{pmatrix} M - (\hat{X} + MQ)R \\ N + (\hat{Y} - NQ)R \end{pmatrix}$ and a strong left representation $(-(\hat{N} + R(Y - Q\hat{N})), \hat{M} - R(X + Q\hat{M}))$ for some $R \in \mathcal{S}$. So L_1 has the strong left representation and the strong right representation in the theorem.

On the other hand, suppose L_1 has the representations in the theorem. Since $\hat{M}N - \hat{N}M = 0$ and

$$((Y - Q\hat{N}), X + Q\hat{M}) \begin{pmatrix} -(\hat{X} + MQ) \\ \hat{Y} - NQ \end{pmatrix} = 0,$$

we have

$$((Y - Q\hat{N}), X + Q\hat{M}) \begin{pmatrix} M - (\hat{X} + MQ)R \\ N + (\hat{Y} - NQ)R \end{pmatrix} = I.$$

By Corollary 3.2, $C = (Y - Q\hat{N})^{-1}(X + Q\hat{M})$ stabilizes L_1 and by Lemma 4.1, this C also stabilizes L. So L and L_1 can be simultaneously stabilized by C. The proof is completed. □

If L is given and satisfies the condition of Lemma 4.1, the theorem above gives a parametric representation of L_1 which can be simultaneously stabilized with L. In other words, any linear system can be simultaneously stabilized with L by a $C = (\hat{X} + MQ)(\hat{Y} - NQ)^{-1} = (Y - Q\hat{N})^{-1}(X + Q\hat{M})$ if it has the strong left and strong right representations in Theorem 4.2 for some $Q, R \in \mathcal{S}$.

5. Strong stabilization

Practicing control engineers are reluctant to use unstable compensators for the purpose of stabilization. This motivates us to consider whether there exists a stable compensator for a given plant L.

Definition 5.1 *$L \in \mathcal{L}$ is strongly stabilizable if L can be stabilized by a $C \in \mathcal{S}$.*

The following lemma is Theorem 6.6.4 in [9].

Lemma 5.1 *Suppose $C \in \mathcal{L}$ stabilizes L. Then $(-(I + LC)^{-1}L, (I + LC)^{-1})$ is a strong left representation for L and $\begin{pmatrix} (I + CL)^{-1} \\ L(I + CL)^{-1} \end{pmatrix}$ is a strong right representation for L if and only if $C \in \mathcal{S}$.*

Theorem 5.1 *Suppose $L \in \mathcal{L}$ is stabilizable and $C_0 \in \mathcal{S}$ is a known controller of L. Then $C \in \mathcal{L}$ stabilizes L if and only if it has a strong right representation $\begin{pmatrix} I - LQ \\ C_0 + Q \end{pmatrix}$ and a strong left representation $(-(C_0 + Q), I - QL)$ for some $Q \in \mathcal{S}$.*

Proof Since $C_0 \in \mathcal{S}$ stabilizes L, by Lemma 5.1, $\begin{pmatrix} (I + C_0L)^{-1} \\ L(I + C_0L)^{-1} \end{pmatrix}$ is a strong right representation for L and $(-(I + LC_0)^{-1}L, (I + LC_0)^{-1})$ is a strong left representation for L. Clearly, L satisfies the following double Bezout identity:

$$\begin{pmatrix} I & C_0 \\ -(I + LC_0)^{-1}L & (I + LC_0)^{-1} \end{pmatrix} \begin{pmatrix} (I + C_0L)^{-1} & -C_0 \\ L(I + C_0L)^{-1} & I \end{pmatrix}$$
$$= \begin{pmatrix} (I + C_0L)^{-1} & -C_0 \\ L(I + C_0L)^{-1} & I \end{pmatrix} \begin{pmatrix} I & C_0 \\ -(I + LC_0)^{-1}L & (I + LC_0)^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then by Lemma 4.1, if $C \in \mathcal{L}$ stabilizes L, then it has a strong right representation

$$\begin{pmatrix} I - L(I + C_0L)^{-1}R \\ C_0 + (I + C_0L)^{-1}R \end{pmatrix}$$

and a strong left representation $(-(C_0 + R(I + LC_0)^{-1}, I - R(I + LC_0)^{-1}L)$ for some $R \in \mathcal{S}$. Let

$$Q = (I + C_0L)^{-1}R$$

or

$$Q = R(I + LC_0)^{-1}.$$

Then C admits a strong right representation $\begin{pmatrix} I - LQ \\ C_0 + Q \end{pmatrix}$ and a strong left representation $(-(C_0 + Q), I - QL)$ for some $Q \in \mathcal{S}$.

If C has a strong right representation $\begin{pmatrix} I - LQ \\ C_0 + Q \end{pmatrix}$ and a strong left representation $(-(C_0 + Q), I - QL)$ for some $Q \in \mathcal{S}$, by Corollary 3.2 we verify directly that $C \in \mathcal{L}$ stabilizes L. The proof is completed. □

The theorem above gives a parametric representation for C without giving the known coprime factorization of L first, instead, a known $C_0 \in \mathcal{S}$ stabilizes L.

We now turn to the problem on simultaneous stabilization. Given $L_0, L_1 \in \mathcal{L}$, we consider their simultaneous stabilization problem.

Theorem 5.2 *Given $L_0, L_1 \in \mathcal{L}$ stabilizable and suppose $C_0 \in \mathcal{S}$ stabilizes L_0 and $C_1 \in \mathcal{S}$ stabilizes L_1. Then there exists a $C \in \mathcal{L}$ which simultaneously stabilizes L_0, L_1 if and only if there exists $T \in \mathcal{S}$ such that $(I + C_0L_1)(I + C_1L_1)^{-1} + T(L_1 - L_0)(I + C_1L_1)^{-1}$ is invertible in $\mathcal{S}$.*

Proof By Theorem 5.1, the controllers that stabilize L_0, L_1, respectively, have the strong left representations $(-(C_0 + Q), I - QL_0)$ and $(-(C_1 + R), I - RL_1)$ for some $Q, R \in \mathcal{S}$. Thus by Theorem 6.3.5 in [9], L_0 and L_1 can be simultaneously stabilized if and only if there exist $Q_0, R_1 \in \mathcal{S}$ such that

$$(-(C_0 + Q_0), I - Q_0L_0) = Z(-(C_1 + R_1), I - R_1L_1)$$

for some invertible Z in $\mathcal{S}$. This is equivalent to

$$(C_0 + Q_0) = Z(C_1 + R_1),\ I - Q_0L_0 = Z(I - R_1L_1).$$

Rewrite this as

$$(I, Q_0)\begin{pmatrix} C_0 & I \\ I & -L_0 \end{pmatrix} = Z(I, R_1)\begin{pmatrix} C_1 & I \\ I & -L_1 \end{pmatrix}$$

or

$$(I, Q_0)\begin{pmatrix} C_0 & I \\ I & -L_0 \end{pmatrix}\begin{pmatrix} C_1 & I \\ I & -L_1 \end{pmatrix}^{-1} = Z(I, R_1).$$

Since

$$\begin{pmatrix} C_1 & I \\ I & -L_1 \end{pmatrix}^{-1} = \begin{pmatrix} L_1(I+C_1L)^{-1} & (I+L_1C_1)^{-1} \\ (I+C_1L_1)^{-1} & -C_1(I+L_1C_1)^{-1} \end{pmatrix},$$

we have

$$(I, Q_0) \begin{pmatrix} (C_0L_1+I)(I+C_1L_1)^{-1} & (C_0-C_1)(I+L_1C_1)^{-1} \\ (L_1-L_0)(I+C_1L_1)^{-1} & (I+L_0C_1)(I+L_1C_1)^{-1} \end{pmatrix} = Z(I, R_1).$$

Thus $(C_0L_1+I)(I+C_1L_1)^{-1} + Q_0(L_1-L_0)(I+C_1L_1)^{-1} = Z$ is invertible in $\mathcal{S}$.

Conversely, if there exists $T \in \mathcal{S}$ such that

$$(I+C_0L_1)(I+C_1L_1)^{-1} + T(L_1-L_0)(I+C_1L_1)^{-1}$$

is invertible in $\mathcal{S}$, then taking

$$Z = (C_0L_1+I)(I+C_1L_1)^{-1} + Q_0(L_1-L_0)(I+C_1L_1)^{-1}$$

$$Q_0 = T,$$

and

$$R_1 = Z^{-1}((C_0-C_1)(I+L_1C_1)^{-1} + Q_0(I+L_0C_1)(I+L_1C_1)^{-1})$$

leads to the following equation

$$(-(C_0+Q_0), I-Q_0L_0) = Z(-(C_1+R_1), I-R_1L_1).$$

This implies that L_0 and L_1 can be simultaneously stabilized. The proof is completed. □

Similarly to Theorem 5.2, we immediately get the following theorem for the simultaneous stabilization problem of $L_0, L_1, \ldots, L_n \in \mathcal{L}$.

Theorem 5.3 *Suppose that $L_i \in \mathcal{L}$ and C_i is a known stable controller that stabilizes L_i for $i = 0, 1, \ldots, n$. Then there exists a $C \in \mathcal{L}$ which simultaneously stabilizes $L_0, L_1, \ldots, L_n$ if and only if there exists $T_i \in \mathcal{S}$ such that $(I+C_0L_i)(I+C_iL_i)^{-1} + T_i(L_i-L_0)(I+C_iL_i)^{-1}$ is invertible in $\mathcal{S}$ for $i = 0, 1, \ldots, n$.*

Proof If there exists a C which stabilizes $L_0, L_1, \ldots, L_n$, then there exist $Q_0, R_i \in \mathcal{S}$ such that

$$(-(C_0+Q_0), I-Q_0L_0) = Z_1(-(C_1+R_1), I-R_1L_1) = \cdots = Z_n(-(C_n+R_n), I-R_nL_n)$$

for some invertible Z_i $(i = 1, 2, \ldots, n)$ in $\mathcal{S}$. Rewrite this as

$$(I, Q_0) \begin{pmatrix} C_0 & I \\ I & -L_0 \end{pmatrix} = Z_i(I, R_i) \begin{pmatrix} C_i & I \\ I & -L_i \end{pmatrix},$$

or

$$(I, Q_0) \begin{pmatrix} C_0 & I \\ I & -L_0 \end{pmatrix} \begin{pmatrix} C_i & I \\ I & -L_i \end{pmatrix}^{-1} = Z_i(I, R_i).$$

The remainder proof is just similar to that of Theorem 5.2. The proof is completed. □

Theorem 5.4 *Suppose $L_0, L_1 \in \mathcal{L}$ are stabilizable and $C_0 \in \mathcal{S}$ stabilizes L_0. Then L_0 and L_1 can be simultaneously stabilized if and only if there exist $Q, R \in \mathcal{S}$ such that*

$$\begin{pmatrix} (I + C_0L_0)^{-1} - (C_0 + (I + C_0L_0)^{-1}Q)R \\ L_0(I + C_0L_0)^{-1} + (I - L_0(I + C_0L_0)^{-1}Q)R \end{pmatrix}$$

is a strong right representation and

$$(-((I + L_0C_0)^{-1}L_0 + R(I - Q(I + L_0C_0)^{-1}L_0), (I + L_0C_0)^{-1} - R(C_0 + Q(I + L_0C_0)^{-1})))$$

is a strong let representation for L_1.

Proof Since $L_0 \in \mathcal{L}$ is stabilized by a known $C_0 \in \mathcal{S}$, by Theorem 5.1 and its proof, $C \in \mathcal{L}$ stabilizes L_0 if and only if it has a strong right representation

$$\begin{pmatrix} I - L_0(I + C_0L_0)^{-1}Q \\ C_0 + (I + C_0L_0)^{-1}Q \end{pmatrix}$$

and a strong left representation

$$(-(C_0 + Q(I + L_0C_0)^{-1}), I - Q(I + L_0C_0)^{-1}L_0)$$

for some $Q \in \mathcal{S}$. If $C \in \mathcal{S}(L_0)$ stabilizes L_1, then there is a $Q \in \mathcal{S}$ such that

$$C = (C_0 + (I + C_0L_0)^{-1}Q)(I - L_0(I + C_0L_0)^{-1}Q)^{-1}$$

stabilizes L_1. By Theorem 4.1, L_1 stabilizes C. Since the following double Bezout equation is satisfied:

$$\begin{pmatrix} (I + L_0C_0)^{-1} & (I + L_0C_0)^{-1}L_0 \\ -(C_0 + Q(I + L_0C_0)^{-1}) & I - Q(I + L_0C_0)^{-1}L_0 \end{pmatrix} \begin{pmatrix} I - L_0(I + C_0L_0)^{-1}Q & -L_0(I + C_0L_0)^{-1} \\ C_0 + (I + C_0L_0)^{-1}Q & (I + C_0L_0)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} I - L_0(I + C_0L_0)^{-1}Q & -L_0(I + C_0L_0)^{-1} \\ C_0 + (I + C_0L_0)^{-1}Q & (I + C_0L_0)^{-1} \end{pmatrix} \begin{pmatrix} (I + L_0C_0)^{-1} & (I + L_0C_0)^{-1} \\ -(C_0 + Q(I + L_0C_0)^{-1}) & I - Q(I + L_0C_0)^{-1}L_0 \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

by Lemma 4.1, L_1 stabilizes C if and only if it has a strong right representation

$$\begin{pmatrix} (I + L_0C_0)^{-1} - (C_0 + (I + C_0L_0)^{-1}Q)R \\ L_0(I + C_0L_0)^{-1} + (I - L_0(I + C_0L_0)^{-1}Q)R \end{pmatrix}$$

and a strong left representation

$$(-((I + L_0C_0)^{-1}L_0 + R(I - Q(I + L_0C_0)^{-1}L_0)), (I + L_0C_0)^{-1} - R(C_0 + Q(I + L_0C_0)^{-1}))$$

for some $R \in \mathcal{S}$.

Conversely, suppose L_1 has the strong right representation

$$\begin{pmatrix} (I + L_0C_0)^{-1} - (C_0 + (I + C_0L_0)^{-1}Q)R \\ L_0(I + C_0L_0)^{-1} + (I - L_0(I + C_0L_0)^{-1}Q)R \end{pmatrix}.$$

Since

$$(I-Q(I+L_0C_0)^{-1}L_0, C_0+Q(I+L_0C_0)^{-1}) \begin{pmatrix} (I+L_0C_0)^{-1}-(C_0+(I+C_0L_0)^{-1}Q)R \\ L_0(I+C_0L_0)^{-1}+(I-L_0(I+C_0L_0)^{-1}Q)R \end{pmatrix} = I,$$

by Corollary 3.1,

$$C = (I - Q(I + L_0C_0)^{-1}L_0)^{-1}(C_0 + Q(I + L_0C_0)^{-1})$$

stabilizes L_1 and by the proof of Theorem 5.1, C also stabilizes L_0. The proof is completed. □

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