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Decomposition of λK_v into 6-Circuits with Two Chords

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Abstract In this paper, we discuss the G-decomposition of λK_v into 6-circuits with two chords. We construct some holey G-designs using sharply 2-transitive group, and present the recursive structure by PBD. We also give a unified method to construct G-designs when the index equals the edge number of the discussed graph. Finally, the existence of G-$GD_\lambda(v)$ is given.

Keywords graph design; holey graph design; sharply 2-transitive group.

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1. Introduction

Let K_v be a complete graph with v vertices, and G=$(V(G),E(G))$ be a finite simple graph. A G-decomposition (or G-design) is a pair $(X,\mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any edge of K_v occurs in exactly λ blocks of $\mathcal{B}$. For simplicity, such a G-design is denoted by G-$GD_\lambda(v)$. Obviously, the necessary conditions for the existence of a G-$GD_\lambda(v)$ are

$$v \geq |V(G)|,\ \lambda v(v-1)\equiv 0 \mod 2|E(G)|,\ \lambda(v-1)\equiv 0 \mod d, \tag{*}$$

where d is the greatest common divisor of the degrees of the vertices in $V(G)$.

Let $K_{n_1,n_2,\ldots,n_t}$ be a complete multipartite graph with vertex set $\bigcup_{i=1}^t X_i$, where these X_i are disjoint and $|X_i|=n_i$, $1\leq i\leq t$. For a given graph G, a holey G-design, denoted by G-$HD_\lambda(n_1n_2\cdots n_t)$, is a pair $(X,\mathcal{B})$, where X is the vertex set of $K_{n_1,n_2,\ldots,n_t}$ and $\mathcal{B}$ is a collection of subgraphs of $K_{n_1,n_2,\ldots,n_t}$ called blocks, such that each block is isomorphic to G and any edge of $K_{n_1,n_2,\ldots,n_t}$ occurs in exactly λ blocks of $\mathcal{B}$. When the multipartite graph has a_i partite of size g_i $1\leq i\leq r$, the holey G-design is denoted by G-$HD_\lambda({g_1}^{a_1}{g_2}^{a_2}\cdots {g_r}^{a_r})$. For $\lambda=1$, the index 1 is often omitted. A G-$HD_\lambda(1^v w^1)$ is called an incomplete G-design, denoted by G-$ID_\lambda(v+w,w)$. Obviously, a G-$GD_\lambda(v)$ can be regarded as a G-$HD_\lambda(1^v)$, a G-$ID_\lambda(v+0,0)$ or a G-$ID_\lambda((v-1)+1,1)$.

From [2], there are 6 graphs–6-circuit with two chords, which are listed below:

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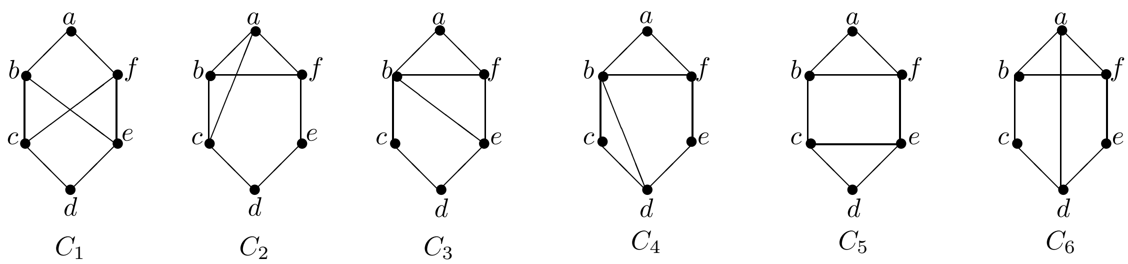

Figure 1 Graphs—6-circuit with two chords

For convenience, all graphs above are denoted by (a, b, c, d, e, f).

For $\lambda = 1$, we have solved the existence of graph designs for these graphs.

Lemma 1.1[4] *For graph C_k, $1 \le k \le 6$, there exists a G-$GD(v)$ if and only if $v \equiv 0, 1 \pmod{16}$ and $v \ge 16$.*

The graph design C_1-$GD_\lambda(v)$ for $\lambda > 1$ has been completed in [5]. In this paper, we shall focus on the left five graphs for $\lambda > 1$, i.e., $C_k, 2 \le k \le 6$.

By $(*)$, we need discuss the following v and λ:

$$\lambda = 2,\ v \equiv 0, 1 \ (\text{mod}\, 8); \quad \lambda = 4,\ v \equiv 0, 1 \ (\text{mod}\, 4); \quad \lambda = 8,\ v \ge 6. \tag{**}$$

Our main conclusions will be:

Theorem 1.2 *The necessary conditions for the existence of C_k-$GD_\lambda(v)$, $2 \le k \le 6$, are also sufficient.*

The following definition and lemmas are important for our constructing methods in this paper.

A pairwise balanced design $B[K, 1; v]$ is a pair $(V, \mathcal{B})$, where V is a v-set (point set) and $\mathcal{B}$ is a family of subsets (blocks) of V with block sizes from K such that every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$. When $K = \{k\}$, $B[K, 1; v] = B[k, 1; v]$ is just a balanced incomplete block design.

Lemma 1.3[3] *Let G be a simple graph, K be a set of positive integers, and m, u, v, λ, μ be positive integers.*

(1) If there exist a $B[K, 1; v]$ and a G-$HD_\lambda(m^k)$ for each $k \in K$, then there exists a G-$HD_\lambda(m^v)$.

(2) If there exists a G-$HD_\lambda(m^u)$, then there exists a G-$HD_{\lambda\mu}(m^u)$.

Lemma 1.4[5] *Let G be a simple graph, and h, m, n, λ be positive integers, $w \ge 0$.*

(1) If there exist a G-$HD_\lambda(m^h)$, a G-$ID_\lambda(m + w, w)$ and a G-$GD_\lambda(m + w)$ (or G-$GD_\lambda(w)$), then there exists a G-$GD_\lambda(mh + w)$.

(2) If there exist a G-$HD_\lambda(m^h n^1)$, a G-$ID_\lambda(m + w, w)$ and a G-$GD_\lambda(n + w)$, then there exists a G-$GD_\lambda(mh + n + w)$.

Lemma 1.5 *Let m be a positive integer, $q = 3, 4, 5$, $w = 0, 1$ and $i = 1, 2$. If there exist a*

G-$HD_2(m^q)$ and a G-$GD_2(im+w)$, then there exists a G-$GD_2(v)$ for $v \equiv 0, 1 \pmod m$ and $v \geq m$.

Note. The above lemma is just the modified version of Theorem 2.2.7 in [4].

Lemma 1.6 *Let positive integer $w < 8$, q=3, 4, 5 and $t \in \{1, 2, 6, 8\}$. If there exist a G-$HD_\lambda(8^q)$, a G-$ID_\lambda(8+w, w)$ and a G-$GD_\lambda(8t+w)$, then there exists a G-$GD_\lambda(v)$ for $v \equiv w \pmod 8$ and $v \geq 8+w$.*

Proof Let $v = 8t + w$, $t \geq 1$. From [1], there exists a $B[\{3, 4, 5\}, 1; t]$ for any $t \geq 3$, $t \neq 6, 8$. Hence, by Lemma 1.3(1), there exists a G-$HD_\lambda(8^t)$ for any $t \geq 3$, $t \neq 6, 8$, from the existence of G-$HD_\lambda(8^q)$ for $q = 3, 4, 5$. Furthermore, by Lemma 1.4(1), there exists a G-$GD_\lambda(8t+w)$ for any $t \geq 3$, $t \neq 6, 8$, from the known G-$ID_\lambda(8+w, w)$ and G-$GD_\lambda(8+w)$. Adding the given G-$GD_\lambda(8t+w)$ for $t = 1, 2, 6, 8$, we obtain the conclusion. □

2. Construction of HD via sharply 2-transitive group

Let H be a transformation group acting on the n-set N. For any two ordered 2-subsets (x, y) and (x', y') from N, if there exists unique $\xi \in H$ satisfying $(\xi x, \xi y)$=(x', y'), then H is called a sharply 2-transitive group on N.

Lemma 2.1[4] *Let F_q be a finite field, where q is a prime power. Then, for the multiplication of transformations, all linear transformations on F_q*

$$f_{c,d} : x \longmapsto cx + d \quad \forall x \in F_q$$

form a sharply 2-transitive group on F_q: $L_q = \{f_{c,d} : c \in F_q^*,\ d \in F_q\}$.

Lemma 2.2 *Let G be a graph with $2e$ edges. If*

(1) There exists a mapping f (i.e., vertex labeling) from its vertex set $V(G)$ to the set Z_{2e} such that the induced mapping on its edge set (i.e., edge labeling)

$$f^* :\ \{x, y\} \longmapsto |f(x) - f(y)| \quad \forall \{x, y\} \in E(G)$$

satisfies $\{f^*\{x, y\} : \{x, y\} \in E(G)\} = \{1, 1, 2, 2, \ldots, e-1, e-1\} \bigcup \{0, e\}$, where $|f(x) - f(y)| = f(y) - f(x)$ if $0 \leq f(y) - f(x) \leq e$ and $|f(x) - f(y)| = f(x) - f(y)$ if $e < f(y) - f(x) < 2e$;

(2) G is q-vertex-colorable (the coloring set is Q);

(3) There exists a sharply 2-transitive group on Q,

then there exists a G-$HD_2((2e)^q)$, where q is a prime power.

Proof We will construct a holey-design G-$HD_2((2e)^q)$ on $Z_{2e} \times Q$, where the set of partites is $\{Z_{2e} \times \{i\} :\ i \in Q\}$ and Q is just the q-vertex-coloring set. Denote the q-vertex-coloring of G by C, and the graph is labeled according to condition (1) by B. Let L_q be the sharply 2-transitive group on Q. Then $(B, C) = \{(f(x), C(x)) : x \in V(G)\} \bmod (Z_{2e}, L_q)$ forms the block set of G-$HD_2((2e)^q)$.

In fact, since C is a q-vertex-coloring graph, the differences in the base blocks are all mixed differences between distinct holes (not pure difference in the same hole).

The mixed differences between two distinct holes are $0, \pm 1, \pm 2, \ldots, \pm(e-1)$ and e. In the block B, each edge valuation of $\{1, 2, \ldots, (e-1)\}$ appears exactly two times and each edge valuation of $\{0, e\}$ appears exactly once. Under the acting of the sharply-2 transitive group L_q, each edge (x, y) of C takes each ordered pair from Q exactly once. Therefore, in the base blocks each mixed difference in $\{0, e, \pm 1, \pm 2, \ldots, \pm(e-1)\}$ between any two distinct holes appears exactly two times. This completes the proof. □

Lemma 2.3 *For graph $G \in \{C_k : 2 \le k \le 6\}$, there exists a G-$HD_2(8^q)$ for $q = 3, 4, 5$.*

Proof For each graph $G(a, b, c, d, e, f)$, we will construct the desired G-$HD_2(8^q)$ on $X{=}Z_8 \times Z_q$ with partites $Z_8 \times \{x\}, x \in F_q$. By Lemma 2.2, we need only to construct the corresponding vertex labeling and vertex coloring, which are listed below.

C_2: $B = (0, 1, 4, 5, 3, 3),\ C = (0, 1, 2, 1, 0, 2)$; C_3: $B = (2, 4, 4, 0, 1, 3),\ C = (0, 1, 2, 1, 0, 2)$;
C_4: $B = (0, 1, 4, 6, 6, 2),\ C = (0, 1, 0, 2, 1, 2)$; C_5: $B = (0, 1, 5, 2, 3, 3),\ C = (0, 1, 2, 1, 0, 2)$;
C_6: $B = (0, 1, 5, 2, 3, 3),\ C = (0, 1, 2, 1, 0, 2)$. □

3. $\lambda = 2$

In this section, by $(**)$, the scope of order v for the existence of G-$GD_2(v)$ is $v \equiv 0, 1 \pmod 8$. By the known holey designs and recursive constructions in Sections 1 and 2, it is enough to construct a few GDs with index 2 for some small orders.

Lemma 3.1 *For graph $G \in \{C_k : 2 \le k \le 6\}$, there exists a G-$GD_2(v)$ for $v \in \{8, 9, 16, 17\}$.*

Proof For $v \in \{8, 9\}$, we list vertex set and blocks below.

$\underline{v = 8}$: $X{=}Z_7 \bigcup \{\infty\}$, mod 7.

$C_2 : (0, 1, 2, 6, \infty, 5),\ C_3 : (1, 0, \infty, 6, 2, 3),\ C_4 :\ (1, 0, 6, 2, \infty, 3),\ C_5 : (\infty, 0, 2, 6, 3, 1),$
$C_6 : (2, 6, \infty, 0, 1, 3).$

$\underline{v = 9}$: $X{=}Z_9$, mod 9.

$C_2 :\ (0, 1, 2, 8, 3, 7),\ C_3 : (1, 0, 4, 8, 2, 3),\ \ C_4 : (1, 0, 8, 2, 7, 3),\ \ C_5 : (6, 0, 2, 8, 3, 1),$
$C_6 :\ (2, 8, 5, 0, 1, 3).$

$\underline{v = 16, 17}$: The designs can be obtained by Lemmas 1.1 and 1.3(2). □

Theorem A *For graph $G \in \{C_k : 2 \le k \le 6\}$, there exists a G-$GD_2(v) \iff v \equiv 0, 1 \pmod 8$ and $v \ge 8$.*

Proof The conclusion holds by Lemmas 1.5, 2.3 and 3.1. □

4. $\lambda = 4$

In this section, by $(**)$, the scope of order v for the existence of G-$GD_4(v)$ is $v \equiv 0, 1 \pmod 4$ and $v \ge 8$. By the known G-designs, holey designs and recursive constructions in Section 1–3, it is enough to construct a few GDs and IDs with index 4 for some small orders.

Lemma 4.1 *There exists a C_2-$ID_2(8+w,w)$. Further there exists a C_2-$ID_4(8+w,w)$ for $w=4,5$, too.*

Proof For $w\in\{4,5\}$, we list vertex set and blocks below.

$\underline{w=4}$: $X=Z_8\bigcup\{A,B,C,D\}$.

$(6,5,B,0,7,D)$, $(5,4,A,1,0,D)$, $(5,3,2,1,D,7)$, $(6,4,3,C,5,0)$, $(3,0,A,7,6,C)$, $(7,2,A,3,4,C)$, $(7,4,B,2,6,C)$, $(7,5,4,B,6,1)$, $(0,2,C,1,4,D)$, $(3,2,B,1,6,D)$, $(6,5,A,1,B,0)$, $(7,3,B,5,C,1)$, $(2,0,7,6,A,4)$, $(4,2,1,D,3,6)$, $(3,1,0,A,2,5)$.

$\underline{w=5}$: $X=Z_8\bigcup\{A,B,C,D,E\}$.

$(6,A,7,1,B,0)$, $(5,A,1,4,E,3)$, $(3,A,7,4,C,2)$, $(4,A,6,3,D,1)$, $(4,D,7,5,E,2)$, $(6,C,7,2,D,3)$, $(1,C,0,3,E,7)$, $(6,D,5,0,A,4)$, $(1,E,6,2,A,5)$, $(2,B,7,0,E,1)$, $(5,C,3,2,E,6)$, $(2,B,6,1,C,5)$, $(2,C,4,5,D,0)$, $(2,0,5,4,3,1)$, $(0,E,7,5,B,4)$, $(0,B,3,1,D,6)$, $(3,B,4,0,D,7)$. □

Lemma 4.2 *For graph $G\in\{C_k:\ 3\le k\le 6\}$, there exists a G-$ID_4(8+w,w)$ for $w=4,5$.*

Proof For $w\in\{4,5\}$, we list vertex set and blocks below.

$\underline{w=4}$: $X=Z_8\bigcup\{A,B,C,D\}$.

$C_3:(A,4,0,B,1,5)$, $(C,2,1,D,3,5)$, $(A,0,6,A,2,7)\mod 8$;
$(0,3,6,D,4,C)$, $(1,3,5,C,7,D)$, $(C,2,5,7,D,0)$, $(0,6,1,C,3,D)$, $(C,1,4,D,2,7)$, $(6,4,2,D,5,C)$.

$C_4:(A,4,B,0,1,5)$, $(C,2,D,0,3,5)$, $(A,0,B,3,4,7)\mod 8$;
$(3,1,D,0,2,C)$, $(0,6,C,7,1,D)$, $(C,0,6,4,D,2)$, $(2,4,C,5,3,D)$, $(C,1,7,5,D,3)$, $(4,6,D,7,5,C)$.

$C_5:(A,4,0,B,1,5)$, $(C,2,0,D,3,5)$, $(A,0,3,B,4,7)\mod 8$;
$(3,C,2,0,D,1)$, $(6,C,7,1,D,0)$, $(C,1,3,D,5,7)$, $(5,C,4,2,D,3)$, $(C,2,4,D,6,0)$, $(4,C,5,7,D,6)$.

$C_6:(4,1,A,0,B,5)$, $(5,2,C,7,D,3)$, $(2,0,A,3,B,7)\mod 8$;
$(0,C,1,D,4,2)$, $(5,C,3,D,2,7)$, $(5,0,C,2,D,3)$, $(1,C,5,D,6,3)$, $(1,4,C,6,D,7)$, $(4,C,7,D,0,6)$.

$\underline{w=5}$: $X=Z_8\bigcup\{A,B,C,D,E\}$.

$C_3:(0,A,4,E,3,2)$, $(0,D,5,2,6,3)$, $(B,0,1,C,4,2)\mod 8$;
$(4,E,0,7,6,5)$, $(7,E,3,4,5,6)$, $(0,D,3,C,1,6)$, $(6,C,5,D,2,4)$, $(3,E,7,0,1,2)$, $(1,D,7,C,5,3)$, $(C,0,D,7,5,2)$, $(C,7,2,D,4,1)$, $(0,C,4,D,6,3)$, $(0,E,4,3,2,1)$.

$C_4:(0,A,2,3,C,1)$, $(0,B,1,4,D,3)$, $(0,E,2,5,1,4)\mod 8$;
$(0,D,1,7,C,6)$, $(C,2,D,4,6,0)$, $(2,D,3,5,C,4)$, $(7,6,5,4,2,0)$, $(C,3,D,5,7,1)$, $(3,C,2,0,D,1)$, $(4,C,5,7,D,6)$, $(6,7,0,1,3,5)$, $(4,3,2,1,7,5)$, $(1,2,3,4,6,0)$.

$C_5:(4,A,1,5,B,0)$, $(E,0,3,D,4,7)$, $(C,0,3,E,4,7)\mod 8$;
$(6,C,7,1,D,0)$, $(C,1,3,D,5,7)$, $(5,C,4,2,D,3)$, $(4,C,5,7,D,6)$, $(A,2,4,B,6,0)$, $(3,C,2,0,D,1)$, $(B,2,4,A,6,0)$, $(C,2,4,D,6,0)$, $(B,1,7,A,5,3)$, $(A,1,7,B,5,3)$.

$C_6:(A,0,D,2,E,1)$, $(B,0,D,4,E,3)$, $(C,0,3,5,6,4)\mod 8$;

$(4, C, 1, 3, 2, 7)$, $(5, C, 6, 2, 1, 0)$, $(6, 3, C, 2, 4, 7)$, $(1, B, 6, 3, 5, 7)$, $(0, 3, B, 4, 1, 7)$, $(2, B, 5, 4, 6, 0)$, $(3, A, 6, 5, 7, 2)$, $(6, 5, A, 7, 0, 1)$, $(0, A, 1, 2, 5, 4)$, $(6, 4, 3, 0, 5, 1)$. □

In what follows, for a block B, $B \times m$ means m times of the block B for $m > 0$.

Lemma 4.3 *For graph $G \in \{C_k : 2 \le k \le 6\}$, there exists G-$GD_4(v)$ for $v \in \{12, 13, 20, 21, 52, 53, 68, 69\}$.*

Proof For $v \in \{12, 13, 20, 21, 52, 53, 68, 69\}$, we list vertex set and blocks below.

$\underline{v = 12}$: $X = Z_{11} \bigcup \{\infty\}$, mod 11.

$C_2 : (0, 3, 10, 8, \infty, 9) \times 2$, $(1, 0, 5, 8, 3, 4)$; $C_3 : (10, 1, \infty, 2, 0, 4) \times 2$, $(4, 0, 3, 6, 1, 5)$;
$C_4 : (\infty, 0, 9, 3, 5, 1) \times 2$, $(5, 0, 4, 3, 9, 1)$; $C_5 : (\infty, 0, 4, 2, 10, 1) \times 2$, $(4, 0, 6, 10, 7, 1)$;
$C_6 : (10, 2, \infty, 1, 0, 4) \times 2$, $(6, 10, 3, 0, 1, 7)$.

$\underline{v = 13}$: $X = Z_{13}$, mod 13.

$C_2 : (0, 1, 5, 8, 2, 6) \times 2$, $(0, 12, 10, 7, 9, 11)$; $C_3 : (12, 1, 8, 2, 0, 4) \times 2$, $(4, 0, 3, 6, 1, 5)$;
$C_4 : (7, 0, 11, 3, 5, 1) \times 2$, $(5, 0, 4, 3, 11, 1)$; $C_5 : (7, 0, 4, 2, 12, 1) \times 2$, $(4, 0, 8, 12, 9, 1)$;
$C_6 : (12, 2, 8, 1, 0, 4) \times 2$, $(8, 12, 3, 0, 1, 9)$.

$\underline{v = 20}$: $X = Z_{19} \bigcup \{\infty\}$, mod 19.

$C_2 : (4, 0, 2, 9, 16, 8)$, $(2, 0, 8, 9, 14, 5) \times 2$, $(5, 2, 11, \infty, 10, 9) \times 2$;
$C_3 : (2, 0, 1, 9, 8, 4)$, $(\infty, 10, 5, 14, 8, 11) \times 2$, $(4, 0, 7, 14, 6, 9) \times 2$;
$C_4 : (2, 0, 3, 6, 10, 4)$, $(\infty, 10, 1, 8, 3, 11) \times 2$, $(9, 0, 3, 7, 13, 8) \times 2$;
$C_5 : (0, 5, 9, 16, 11, 7)$, $(\infty, 10, 1, 7, 3, 11) \times 2$, $(0, 3, 12, 11, 5, 8) \times 2$;
$C_6 : (0, 4, 12, 6, 7, 8)$, $(2, 10, \infty, 11, 4, 1) \times 2$, $(0, 5, 3, 6, 11, 7) \times 2$.

$\underline{v = 21}$: $X = Z_{21}$, mod 21.

$C_2 :$ $(5, 2, 11, 0, 10, 9) \times 2$, $(2, 0, 8, 9, 14, 5) \times 2$, $(4, 0, 2, 9, 16, 8)$;
$C_3 :$ $(0, 10, 5, 14, 8, 11) \times 2$, $(4, 0, 7, 14, 6, 9) \times 2$, $(2, 0, 1, 9, 8, 4)$;
$C_4 : (0, 10, 1, 8, 3, 11) \times 2$, $(9, 0, 3, 7, 13, 8) \times 2$, $(2, 0, 3, 6, 10, 4)$;
$C_5 : (0, 10, 1, 7, 3, 11) \times 2$, $(0, 3, 12, 11, 5, 8) \times 2$, $(0, 5, 9, 16, 11, 7)$;
$C_6 : (2, 10, 0, 11, 4, 1) \times 2$, $(0, 5, 3, 6, 11, 7) \times 2$, $(0, 4, 12, 6, 7, 8)$.

$\underline{v = 52}$: $X = Z_{51} \bigcup \{\infty\}$, mod 51.

$C_2 : (7, 19, 27, \infty, 26, 8) \times 2$, $(25, 0, 6, 7, 20, 13) \times 2$, $(14, 0, 21, 6, 9, 4) \times 2$,
$(9, 0, 24, 14, 5, 23) \times 2$, $(22, 0, 17, 6, 23, 3) \times 2$, $(2, 0, 25, 9, 3, 24) \times 2$,
$(8, 0, 4, 20, 18, 16)$;
$C_3 : (23, 0, 16, 2, 7, 13) \times 2$, $(20, 0, 18, 1, 13, 22) \times 2$, $(\infty, 26, 17, 41, 16, 27) \times 2$,
$(27, 6, 26, 7, 28, 9) \times 2$, $(17, 14, 7, 30, 6, 0) \times 2$, $(12, 0, 15, 30, 5, 16) \times 2$,
$(2, 0, 1, 9, 8, 4)$;
$C_4 : (\infty, 26, 1, 13, 2, 27) \times 2$, $(10, 0, 15, 23, 17, 24) \times 2$, $(24, 0, 16, 21, 9, 22) \times 2$,
$(4, 0, 19, 20, 5, 21) \times 2$, $(23, 0, 9, 22, 4, 18) \times 2$, $(17, 0, 7, 19, 9, 20) \times 2$,
$(2, 0, 3, 6, 10, 4)$;
$C_5 : (3, 27, 4, 16, 1, 13) \times 2$, $(\infty, 26, 15, 24, 2, 27) \times 2$, $(0, 9, 5, 24, 1, 25) \times 2$,
$(0, 21, 1, 22, 9, 19) \times 2$, $(22, 0, 16, 19, 33, 15) \times 2$, $(11, 08, 7, 25, 5) \times 2$,
$(0, 5, 11, 18, 13, 7)$;

$C_6:(2,26,\infty,27,3,1)\times 2,\ (0,23,4,22,1,15)\times 2,\ (0,20,7,18,1,16)\times 2,$
$(0,17,4,6,26,7)\times 2,\ (0,5,17,7,30,14)\times 2,\ (0,12,33,11,6,3)\times 2,$
$(0,4,12,6,7,8).$

$\underline{v=53:}$ $X=Z_{53}$, $\bmod 53$.

$C_2:(7,19,27,0,26,8)\times 2,\ (25,0,6,7,20,13)\times 2,\ (14,0,21,6,9,4)\times 2,$
$(9,0,24,14,5,23)\times 2,\ (22,0,17,6,23,3)\times 2,\ (2,0,25,9,3,24)\times 2,$
$(8,0,4,20,18,16);$

$C_3:(0,26,17,41,16,27)\times 2,\ (23,0,16,2,7,13)\times 2,\ (20,0,18,1,13,22)\times 2,$
$(27,6,26,7,28,9)\times 2,\ (17,14,7,30,6,0)\times 2,\ (12,0,15,30,5,16)\times 2,$
$(2,0,1,9,8,4);$

$C_4:(0,26,1,13,2,27)\times 2,\ (10,0,15,23,17,24)\times 2,\ (24,0,16,21,9,22)\times 2,$
$(17,0,7,19,9,20)\times 2,\ (4,0,19,20,5,21)\times 2,\ (23,0,9,22,4,18)\times 2,$
$(2,0,3,6,10,4);$

$C_5:(0,26,15,24,2,27)\times 2,\ (3,27,4,16,1,13)\times 2,\ (22,0,16,19,33,15)\times 2,$
$(0,21,1,22,9,19)\times 2,\ (0,9,5,24,1,25)\times 2,\ (11,08,7,25,5)\times 2,$
$(0,5,11,18,13,7);$

$C_6:(2,26,0,27,3,1)\times 2,\ (0,23,4,22,1,15)\times 2,\ (0,20,7,18,1,16)\times 2,$
$(0,17,4,6,26,7)\times 2,\ (0,5,17,7,30,14)\times 2,\ (0,12,33,11,6,3)\times 2,$
$(0,4,12,6,7,8).$

$\underline{v=68:}$ $X=Z_{67}\bigcup\{\infty\}$, $\bmod 67$.

$C_2:(9,2,35,\infty,34,19)\times 2,\ (7,1,31,1,29,32)\times 2,\ (11,6,38,7,6,34)\times 2,$
$(0,12,21,7,15,29)\times 2,\ (13,11,29,2,9,0)\times 2,\ (20,19,42,13,26,0)\times 2,$
$(24,22,12,17,20,0)\times 2,\ (21,25,6,17,18,0)\times 2,\ (16,0,8,40,46,32);$

$C_3:(\infty,35,16,6,2,34)\times 2,\ (31,0,18,7,30,8)\times 2,\ (35,6,15,33,0,11)\times 2,$
$(38,10,23,7,0,17)\times 2,\ (9,0,21,45,14,12)\times 2,\ (15,27,12,34,14,0)\times 2,$
$(26,0,27,2,25,20)\times 2,\ (16,0,32,4,30,19)\times 2,\ (2,0,1,9,8,4);$

$C_4:(\infty,34,1,32,2,35)\times 2,\ (32,0,28,18,38,29)\times 2,\ (19,0,26,20,2,27)\times 2,$
$(14,0,12,24,3,25)\times 2,\ (8,0,5,28,14,7,)\times 2,\ (5,0,11,26,2,32)\times 2,$
$(17,0,16,31,2,21)\times 2,\ (13,0,9,22,6,23)\times 2,\ (2,0,3,6,10,4);$

$C_5:(\infty,34,15,20,2,35)\times 2,\ (3,35,17,23,0,14)\times 2,\ (20,0,31,3,2,30)\times 2,$
$(27,0,22,18,31,15)\times 2,\ (5,3,30,13,38,29)\times 2,\ (24,0,32,16,36,3)\times 2,$
$(30,0,29,6,32,22)\times 2,\ (31,0,15,1,26,19)\times 2,\ (0,5,11,18,13,7);$

$C_6:(2,34,\infty,35,9,7)\times 2,\ (31,0,29,2,34,1)\times 2,\ (0,28,5,24,49,25)\times 2,$
$(0,22,1,19,6,18)\times 2,\ (7,0,10,21,6,23)\times 2,\ (0,12,38,10,1,21)\times 2,$
$(0,15,28,6,37,7)\times 2,\ (20,0,16,5,3,17)\times 2,\ (0,4,12,6,7,8).$

$\underline{v=69:}$ $X=Z_{69}$, $\bmod 69$.

$C_2:(9,2,35,0,34,19)\times 2,\ (7,1,31,1,29,32)\times 2,\ (11,6,38,7,6,34)\times 2,$
$(0,12,21,7,15,29)\times 2,\ (13,11,29,2,9,0)\times 2,\ (20,19,42,13,26,0)\times 2,$
$(24,22,12,17,20,0)\times 2,\ (21,25,6,17,18,0)\times 2,\ (16,0,8,40,46,32);$

$C_3:(0,35,16,6,2,34)\times 2,\ (31,0,18,7,30,8)\times 2,\ (35,6,15,33,0,11)\times 2,$

$(38,10,23,7,0,17)\times 2,\ (9,0,21,45,14,12)\times 2,\ (15,27,12,34,14,0)\times 2,$
$(26,0,27,2,25,20)\times 2,\ (16,0,32,4,30,19)\times 2,\ (2,0,1,9,8,4);$

$C_4: (0,34,1,32,2,35)\times 2,\ (32,0,28,18,38,29)\times 2,\ (19,0,26,20,2,27)\times 2,$
$(14,0,12,24,3,25)\times 2,\ (8,0,5,28,14,7,)\times 2,\ (5,0,11,26,2,32)\times 2,$
$(17,0,16,31,2,21)\times 2,\ (13,0,9,22,6,23)\times 2,\ (2,0,3,6,10,4);$

$C_5: (0,34,15,20,2,35)\times 2,\ (3,35,17,23,0,14)\times 2,\ (20,0,31,3,2,30)\times 2,$
$(27,0,22,18,31,15)\times 2,\ (5,3,30,13,38,29)\times 2,\ (24,0,32,16,36,3)\times 2,$
$(30,0,29,6,32,22)\times 2,\ (31,0,15,1,26,19)\times 2,\ (0,5,11,18,13,7);$

$C_6: (2,34,0,35,9,7)\times 2,\ (31,0,29,2,34,1)\times 2,\ (0,28,5,24,49,25)\times 2,$
$(0,22,1,19,6,18)\times 2,\ (7,0,10,21,6,23)\times 2,\ (0,12,38,10,1,21)\times 2,$
$(0,15,28,6,37,7)\times 2,\ \ (20,0,16,5,3,17)\times 2,\ (0,4,12,6,7,8).$ □

Theorem B *For graph $G \in \{C_k : 2 \le k \le 6\}$, there exists a G-$GD_4(v) \Longleftrightarrow v \equiv 0,1 \pmod 4$ and $v \ge 8$.*

Proof The conclusion holds by Lemmas 1.6, 2.3 and 4.1–4.3. □

5. $\lambda = 8$

5.1 A constructing method for $\lambda = |E(G)|$

Let G be a connected graph, $|V(G)| = m$ and $|E(G)| = e$. Consider the graph design G-$GD_e(v) = (X, \mathcal{B})$. Let $n = 2\lceil \frac{v}{2} \rceil - 1$, which is odd. The vertex set X is denoted by Z_n for odd v or $Z_n \cup \{\infty\}$ for even v. The block set consists of $n \cdot \frac{n-1}{2}$ or $n \cdot \frac{n+1}{2}$ blocks. Let us construct $\frac{n-1}{2}$ (for odd v) or $\frac{n+1}{2}$ (for even v) base blocks as follows.

Step 1. Define a mapping from Z_n to $\{1, 2, \ldots, \frac{n-1}{2}\}$: $a \mapsto \langle 2a \rangle$, where $\langle t \rangle = t$ (if $t \le \frac{n-1}{2}$) or $n - t$ (if $t > \frac{n-1}{2}$). Then, the integers $1, 2, \ldots, \frac{n-1}{2}$ are partitioned into equivalent classes, each of which forms a cycle. The cycle contains the integer a ($1 \le a \le \frac{n-1}{2}$) and its length is denoted by (a) and $l(a)$ respectively, where the length $s = l(a)$ is the minimal positive integer satisfying $a \cdot 2^s \equiv \pm a \pmod n$. Obviously, $l(a) \le l(1)$ for $1 \le a \le \frac{n-1}{2}$. All the cycles form a graph H_n, which is 2-regular.

Step 2. For any $a \in [1, \frac{n-1}{2}]$ and $l(a) \ge 3$, take an injection f from $V(G)$ to $M = \{ma : -\frac{n-1}{2} \le m \le \frac{n-1}{2}\}$ such that for any edge $\{x, y\} \in E(G)$, the integer $\langle f(x) - f(y) \rangle$ is in the cycle (a). Note that f is an injection if and only if $f(x) \ne f(y)$ for any $x \ne y \in V(G)$. When $|V(G)| \le 7$, the set M may be restricted to the 7-set: $\{-2a, -a, 0, a, 2a\} \bigcup T$, where $T = \{3a, 4a\}$ or $\{-3a, -4a\}$, or $\{3a, -3a\}$, or $\{4a, -4a\}$. Then, for $x \ne y \in V(G)$, the equation $f(x) = f(y)$ holds only for the following cases:

1° $\ 0 = \pm 3a,\ \pm a = \pm 4a,\ \pm a = \mp 2a,\ \pm 2a = \mp 4a,\ 3a = -3a,$

$\Longrightarrow \quad n = 3a$, $l(a) = 1$ and (a) is the unique 1-cycle;

2° $\ \pm a = \mp 4a,\ \pm 2a = \mp 3a,$

$\Longrightarrow \quad n = 5a$, $l(a) = 2$ and $(a, 2a)$ is the unique 2-cycle.

Furthermore, there is another related case

3° $n = 15a$, there is a unique 1-cycle $(5a)$ and a unique 2-cycle $(3a, 6a)$.

Therefore, we only need to discuss the four cases:

Case 1 $\gcd(n, 15) = 1$, and the length $l(a) \geq 3$ for any cycle (a) in H_n. The injection f here gives a base block B_a. But the base blocks B_a $(1 \leq a \leq \frac{n-1}{2})$ will cover all differences in Z_n e times. In fact, let the cycle (a) be $(a, 2a, 4a, \ldots, 2^{s-1}a)$ and each $2^j a$, as edge-value $\langle f(x) - f(y) \rangle$, appear i_j times in the base block B_a, where $0 \leq j \leq s-1$ and $\sum_{j=0}^{s-1} i_j = e$. Then, all the edges in $B_a, B_{2a}, \ldots, B_{2^{s-1}a}$ will take edge-values as follows.

	a	$2a$	2^2a	$\cdots$	$2^{s-2}a$	$2^{s-1}a$
B_a	i_0	i_1	i_2	$\cdots$	i_{s-2}	i_{s-1}
B_{2a}	i_{s-1}	i_0	i_1	$\cdots$	i_{s-3}	i_{s-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{2^{s-1}a}$	i_1	i_2	i_3	$\cdots$	i_{s-1}	i_0

Table 1 Difference distribution

Thus, the base blocks $B_a, B_{2a}, \ldots, B_{2^{s-1}a}$ corresponding $a, 2a, \ldots, 2^{s-1}a$ in the cycle (a) cover the differences $a, 2a, \ldots, 2^{s-1}a$ e times.

Case 2 $n = 3b$ and $b \neq 0 \pmod 5$, there is a unique 1-cycle (b).

Case 3 $n = 5b$ and $b \neq 0 \pmod 3$, there is a unique 2-cycle $(b, 2b)$.

Case 4 $n = 15b$, there are a unique 1-cycle $(5b)$ and a unique 2-cycle $(3b, 6b)$.

Step 3. For the Cases 2, 3 and 4, the method stated in step 2 cannot be used for 1-cycle or 2-cycle because, replacing a by b, $2b$, $3b$, $5b$ or $6b$, the number of the available integers in the set M is less than six. We may change a few base blocks in $\mathcal{A}$ corresponding to cycle (1) (or b when $n = 15b$) and add some base blocks relating to the elements b, $2b$, $3b$, $5b$, $6b$. Note that the edges in these changed and added base blocks belong not yet to one cycle (but two or three cycles).

Step 4. For odd order v, the graph design G-$GD_e(v)$ will be obtained after Steps 2 and 3. For even order $v = n + 1$, we need to add one vertex ∞ to the vertex set Z_n, to change some base blocks in $\mathcal{A}$ corresponding to cycle (1), and to add some base blocks containing ∞.

Lemma 5.1 *There exists a C_k-$GD_8(v)$ for $v \geq 6$, $k = 4, 5, 6$.*

Proof Using the method mentioned above, we list the following table. First, the base block B_a for odd v and $l(a) \geq 3$, i.e., case 1 (odd), is given in the first row. The vertex sets are obviously in $\{0, \pm a, \pm 2a\} \bigcup T$ pointed in Step 2. We denote $\mathcal{A} = \{B_a : 1 \leq a \leq \frac{n-1}{2}\}$ for the B_a listed in the first row, then the base blocks for other cases will be uniformly denoted as $(\mathcal{A} \backslash \mathcal{C}) \bigcup \mathcal{C}' \bigcup \mathcal{D}$, where $\mathcal{C}$ is a few base blocks in $\mathcal{A}$ like B_1, B_2, B_4 or $B_b, B_{2b}, \ldots$, which is changed to $\mathcal{C}'$ (denoted by $\rightarrow$), and $\mathcal{D}$ is a few added base blocks.

		C_4	C_5	C_6
odd v Case 1	$\mathcal{A}$	$(a, 0, 2a, 4a, 3a, -a)$	$(-a, a, 3a, 4a, 2a, 0)$	$(-2a, 0, a, -a, 3a, 2a)$
odd v	$\mathcal{C}$	B_1	B_1	B_1
Case 2 $n = 3b$	$\mathcal{D}'$	$(-b, 0, 2, 1, b+1, b)$ $(-b, 0, 2, 4, b+4, b)$	$(2b, b, b+2, 2b+2, 2, 0)$ $(2, 1, b+1, b+2, b, 0)$	$(-2, 2, b+2, b-2, b, 0)$ $(0, 2b, 2b+1, 1, b+1, b)$
odd v	$\mathcal{C}$	B_1, B_2	B_1, B_2	B_1, B_2
Case 3 $n = 5b$	$\mathcal{D}$	$(3ib, 0, 2i, 4i, ib+4i, ib)$ $(3ib, 0, 2i, i, ib+i, ib)$ $i = 1, 2$	$(-2, 0, b, b+1, b+2, 2) \times 2$ $(2b, 0, i, 2b+i, 4b+i, 4b)$ $i = 2, 4$	$(ib, 3ib, 3ib+i, ib+i, i, 0)$ $(2i, -2i, ib-2i, ib+2i, ib, 0)$ $i = 1, 2$
odd v	$\mathcal{C}$	B_b	B_b	B_b
Case 4 $n = 15b$	$\mathcal{D}$	$(2b, -b, -3b, -5b, 0, 5b)$ $(6b, 0, 9b, 3b, b, -3b)$ $(9b, 0, -5b, 5b, 2b, 3b)$ $(6b, 0, -5b, 5b, 4b, 3b)$	$(3b, 6b, 11b, 8b, 5b, 0)$ $(2b, 3b, 8b, 7b, 5b, 0)$ $(2b, 5b, 11b, 8b, 6b, 0)$ $(-b, 5b, 11b, 12b, 6b, 0,)$	$(3b, b, 13b, 7b, 12b, 0) \times 2$ $(-5b, -6b, 4b, b, 6b, 0)$ $(-5b, -3b, 7b, b, 6b, 0)$

Table 2 Some blocks of C_k-$GD_8(v), k = 4, 5, 6$, for odd v

		C_4	C_5	C_6
even v Case 1	$\mathcal{C}'$	$B_1 : 0 \rightarrow \infty$	$B_i : (i = 1,\ 2)$ $0 \rightarrow \infty$	$B_1 : -1 \rightarrow \infty$ $B_2 : 2 \rightarrow \infty$
	$\mathcal{D}$	$(0, \infty, 2, 3, -1, 1)$	$(\infty, -1, 0, 4, 2, 1)$	$(0, 2, -2, \infty, 3, 4)$
even v	$\mathcal{C}$	B_1	B_1	B_1
Case 2 $n = 3b$	$\mathcal{D}$	$(-b, \infty, 2, 4, 0, b) \times 2$ $(-b, 0, 2, 1, b+1, b)$	$(2b, b, \infty, 3, 1, 0) \times 2$ $(1, 2, b+2, \infty, b, 0)$	$(-b, b, b+2, \infty, 4, 0) \times 2$ $(1, 0, b, b+1, \infty, 2)$
even v Case 3 $n = 5b$	$\mathcal{D}$	$(b, 0, 3b, 4b, \infty, 2b) \times 2$ $(0, \infty, 3b, b, 4b, 2b)$	$(3b, 4b, 2b, b, 0, \infty) \times 2$ $(\infty, b, 3b, 2b, 0, 4b)$	$(\infty, 4b, 2b, 3b, b, 0) \times 2$ $(0, 3b, b, 4b, \infty, 2b)$
even v Case 4 $n = 15b$	$\mathcal{D}$	$(\infty, 0, 9b, 3b, 8b, 5b) \times 4$	$(\infty, 8b, 5b, -5b, 0, 3b) \times 2$ $(\infty, 12b, 9b, 3b, 0, 6b)$	$(-5b, 5b, \infty, -2b, 3b, 0) \times 2$ $(0, 9b, 12b, 6b, \infty, 3b) \times 2$

Table 3 Some blocks of C_k-$GD_8(v), k = 4, 5, 6$, for even v

In what follows, we point out some facts:

1) Obviously, the necessary condition for the existence of a C_k-$GD_8(v)$, $k = 4, 5, 6$, is $v \geq 6$. In addition, let $n = 2\lceil \frac{v}{2} \rceil - 1$. Then we have $n \geq 7,\ n \geq 9,\ n \geq 25$ or $n \geq 15$ for odd v or even v in Case 1, 2, 3, 4, in which $n = 2\lceil \frac{v}{2} \rceil - 1 \geq 5$ for even v in Case 3.

2) For Case 2 ($n = 3b$, odd b , $b \geq 3$, $b \not\equiv 0 \bmod 5$). Consider the blocks containing b. We know that the vertex-values are obviously distinct each other for odd v or even v with the

exception $(b, C_k) = (3, C_5)$ and even v. Here is C_5-$GD_8(10) = (X, \mathcal{B})$, where $X = Z_9 \bigcup \{\infty\}$, $\mathcal{B}:\ (4,2,6,1,3,\infty)\times 2,\ (1,0,3,7,6,2)\times 2,\ (\infty,0,1,2,4,6)$ mod 9.

3) For Case 3 ($n = 5b$, odd b, $b \not\equiv 0 \bmod 3$) and Case 4 ($n = 5b$, odd b), the vertex-values are obviously distinct each other for odd v or even v. □

5.2 Graphs C_k, $2 \le k \le 3$

Lemma 5.2 *There exists a C_2-$ID_8(8+w, w)$ for w=2, 3, 6, 7.*

Proof Let X=$Z_8 \bigcup \{\infty_1, \dots, \infty_w\}$.

<u>w=2:</u> $(2,x_1,4,x_2,3,0), (1,x_1,2,4,3,0), (1,0,4,5,7,3), (3,x_2,4,x_1,2,0), (4,x_2,7,6,3,0) \bmod 8$;
$(7,3,5,4,0,1),\ (0,4,6,5,1,2),\ (1,5,7,6,2,3),\ (2,6,0,7,3,4)$.

<u>w=3:</u> $(2,x_1,5,x_2,4,0),\ (3,x_1,0,1,4,7),\ (1,x_3,3,x_1,2,0),\ (3,x_2,4,x_3,1,0),\ (4,x_2,6,5,3,0)$,
$(6,x_3,1,0,2,5) \bmod 8$; $(7,3,5,4,0,1),\ (0,4,6,5,1,2),\ (1,5,7,6,2,3),\ (2,6,0,7,3,4)$.

<u>w=6:</u> $(1,0,x_1,2,4,x_2),\ (3,0,x_2,1,4,x_3),\ (1,0,x_3,2,4,x_4),\ (1,x_4,0,2,x_1,4),\ (3,x_5,0,1,x_2,6)$,
$(1,x_6,0,4,x_3,3),\ (3,x_1,0,1,x_4,7),\ (3,x_6,0,2,x_5,7),\ (1,x_5,0,5,x_6,3) \bmod 8$;
$(7,3,5,4,0,1),\ (0,4,6,5,1,2),\ (1,5,7,6,2,3),\ (2,6,0,7,3,4)$.

<u>w=7:</u> $(5,2,x_2,0,3,x_3),\ (4,2,x_3,0,1,x_4),\ (4,x_4,3,0,x_7,6),\ (4,x_5,3,0,x_6,6),\ (4,x_6,3,0,x_5,6)$,
$(2,x_7,1,0,x_4,4),\ (3,x_1,1,0,x_3,6),\ (7,x_7,4,0,x_2,5),\ (3,x_6,1,0,x_1,6) \bmod 8$;
$(4,0,x_1,1,5,x_5)+i,\ (4,0,x_2,5,1,x_5)+i,\ (4,0,x_1,5,1,x_2)+i \mod 8,\ i=0,1,2,3.$ □

Lemma 5.3 *There exists a C_3-$ID_8(8+w, w)$ for w=2, 3, 6, 7.*

Proof It suffices to give the following constructions. X=$Z_8 \bigcup \{x_1, \dots, x_w\}$.

<u>C_3-$ID_2(8+2,2)$:</u> $(\infty_1,0,4,\infty_2,3,1) \bmod 8$; $(1,0,7,5,6,3),\ (7,1,2,5,4,6),\ (7,2,0,5,3,4)$.

<u>C_3-$ID_2(8+3,3)$:</u> $(x_1,0,4,x_2,3,1) \mod 8$; $(7,0,5,4,1,x_3),\ (5,2,1,6,3,x_3),\ (5,6,0,2,4,x_3)$,
$(6,7,5,3,1,x_3),\ (0,x_3,2,7,4,3)$.

<u>C_3-$ID_4(8+6,6)$:</u> $(0,x_5,1,x_3,6,3),\ (0,x_2,2,4,5,1),\ (4,x_1,6,5,2,0),\ (0,x_6,2,x_4,3,1) \bmod 8$;
$(0,x_3,4,x_4,6,3),\ (0,x_4,3,x_3,1,6),\ (6,x_3,5,x_4,2,4),\ (x_3,0,x_4,7,5,2)$,
$(1,x_4,7,x_3,5,3),\ (x_3,7,2,x_4,4,1)$.

<u>C_3-$ID_4(8+7,7)$:</u> $(0,x_1,3,x_3,2,1),\ (0,x_2,1,x_4,4,2),\ (0,x_6,1,x_5,6,3),\ (4,x_7,5,7,3,0) \bmod 8$;
$(0,x_3,4,x_4,6,3),\ (0,x_4,3,x_3,1,6),\ (6,x_3,5,x_4,2,4),\ (0,x_5,4,3,2,1)$,
$(x_3,0,x_4,7,5,2),\ (x_3,7,2,x_4,4,1),\ (1,x_4,7,x_3,5,3),\ (4,x_5,0,7,6,5)$,
$(7,x_5,3,4,5,6),\ (3,x_5,7,0,1,2)$. □

Lemma 5.4 *(1) There exists a C_2-$GD_8(v)$ for all $v > 6$ except for $(v,15) = 3$ or $(v,15) = 5$ when v is odd and $v \equiv 1 \pmod{15}$ when v is even;*

(2) There exists a C_3-$GD_8(v)$ for all $v > 6$ except for $(v,15) = 3$ when v is odd and $v \equiv 1 \pmod{15}$ when v is even.

Proof Similarly to the proof of Lemma 5.1, we can list the following table.

		C_2	C_3
odd v Case 1	$\mathcal{A}$	$(0, a, 2a, -2a, -3a, -a)$	$(a, 0, -a, -3a, -2a, 2a)$
odd v Case 3 $v = 5b$	$\mathcal{C}$		B_1, B_2
	$\mathcal{D}$		$(2b, 0, 1, b+1, b, 3b) \times 2$ $(-1, , 0, b, b+2, 2, -2) \times 2$
odd v Case 4 $v = 15b$	$\mathcal{C}$	B_b	B_b
	$\mathcal{D}$	$(3b, 0, 6b, b, 4b, 9b) \times 2$ $(0, 5b, -5b, -3b, -2b, 3b)$ $(3b, 0, b, 7b, -2b, -b)$	$(b, 0, 3b, 8b, 5b, -5b)$ $(-b, 0, -6b, -4b, 2b, -2b)$ $(3b, 0, -3b, 2b, 5b, 6b)$ $(3b, 0, -5b, b, 6b, 9b)$
even v Case 1	$\mathcal{C}'$	$B_1, B_2 : 0 \to \infty$	$B_1 : 0 \to \infty$
	$\mathcal{D}$	$(2, 0, 4, \infty, -1, 1)$	$(0, \infty, 2, 3, -1, 1)$
even v Case 2 $v = 3b+1$	$\mathcal{C}$	B_1	B_1
	$\mathcal{D}$	$(b, -b, \infty, b+1, 1, 0) \times 2$ $(2, 0, 1, -1, \infty, -2)$	$(\infty, 0, 1, b+1, b, -b) \times 2$ $(2, \infty, -4, -2, 0, 4)$
even v Case 3 $v = 5b+1$	$\mathcal{D}$	$(2b, b, \infty, 4b, 3b, 0) \times 2$ $(0, 2b, 4b, \infty, b, 3b)$	$(3b, 0, b, \infty, 2b, 4b) \times 2$ $(0, \infty, 2b, 4b, 3b, b)$

Table 4 Some blocks of C_k-$GD_8(v), k = 2, 3$

Lemma 5.5 *There exist a C_k-$GD_8(v)$ for $k = 2, 3,\ v = 6, 10, 51$, and a C_2-$GD_8(55)$.*

Proof For each case, we list vertex set and blocks below.

<u>v=6:</u> $X{=}Z_5 \bigcup \{\infty\}$, mod 5.

$C_2 : (\infty, 0, 4, 1, 3, 2) \times 2,\ (1, 0, 4, 3, \infty, 2)$; $C_3 : (4, 0, 3, 1, 2, \infty) \times 2,\ (\infty, 0, 4, 3, 1, 2)$.

<u>v=10:</u> $X{=}Z_9 \bigcup \{\infty\}$, mod 9.

$C_2 : (3, 0, 8, 1, 4, 5) \times 2,\ (\infty, 3, 2, 0, 4, 1) \times 2,\ (4, 0, 3, 6, \infty, 8)$;

$C_3 : (4, 0, 3, 1, 2, 8) \times 2,\ (4, 2, 7, 0, 3, \infty) \times 2,\ (4, 3, 6, \infty, 8, 0)$.

<u>v=51:</u> $X{=}Z_{51}$, mod 51.

$C_2 : (4, 0, 17, 36, 18, 15),\ (2, 0, 23, 48, 25, 24),\ (3, 0, 20, 41, 21, 19),\ (6, 0, 10, 17, 23, 8),$
$(7, 0, 12, 35, 21, 10),\ (4, 0, 24, 48, 44, 23),\ (5, 0, 14, 30, 15, 12),\ (23, 0, 1, 11, 5, 21),$
$(20, 0, 17, 5, 10, 19),\ (7, 0, 18, 39, 34, 15),\ (3, 0, 20, 42, 29, 19),\ (8, 0, 9, 26, 50, 25),$
$(7, 0, 13, 19, 20, 12),\ (7, 0, 18, 11, 36, 17),\ (2, 0, 20, 8, 10, 19),\ (4, 0, 15, 7, 21, 14),$
$(1, 0, 24, 11, 6, 22),\ (6, 0, 15, 29, 17, 14),\ (6, 0, 22, 4, 19, 21),\ (2, 0, 24, 1, 19, 23),$
$(8, 0, 14, 6, 23, 13),\ (4, 0, 14, 34, 25, 13),\ (1, 0, 25, 3, 2, 26),\ (5, 0, 18, 31, 27, 16),$
$(9, 0, 12, 23, 3, 25),$

$C_3 : (19, 0, 24, 50, 21, 22),\ (23, 0, 22, 1, 15, 20),\ (19, 0, 2, 1, 18, 16),\ (18, 0, 17, 2, 24, 25),$
$(12, 0, 23, 48, 24, 25),\ (23, 0, 25, 6, 22, 19),\ (14, 0, 23, 3, 8, 10),\ (16, 0, 13, 1, 12, 11),$
$(20, 0, 14, 10, 13, 15),\ (20, 0, 8, 21, 9, 15),\ (24, 0, 9, 21, 10, 22),\ (21, 0, 9, 3, 8, 14),$

$(19,0,18,36,20,11)$, $(11,0,14,3,13,23)$, $(17,0,24,2,25,9$, $(20,0,17,37,18,5)$,
$(16,0,32,34,17,7)$, $(10,0,3,8,4,18)$, $(24,0,8,17,7,18)$, $(21,0,19,1,16,14)$
$(15,0,16,2,17,21)$, $(23,0,21,1,22,13)$, $(23,0,7,3,9,16)$, $(1,0,25,1,4,2)$,
$(11,0,1,4,25,8)$.

$\underline{C_2\text{-}GD_8(55)\text{:}}$ $X{=}Z_{55}$, mod 55.

$(1,0,27,47,11,25)$, $(2,0,23,1,12,22)$, $(3,0,19,1,10,18)$, $(11,0,26,2,1,24)$,
$(4,0,17,20,27,14)$, $(5,0,12,23,32,11)$, $(8,0,9,26,3,27)$, $(13,0,27,2,29,25)$,
$(5,0,15,40,21,14)$, $(3,0,22,45,29,21)$, $(4,0,20,2,22,17)$, $(9,0,14,1,27,10)$,
$(12,0,23,1,17,22)$, $(14,0,20,16,8,19)$, $(5,0,21,1,11,20)$, $(7,0,20,47,23,17)$,
$(2,0,26,4,19,25)$, $(6,0,12,26,28,11)$, $(8,0,7,10,11,2)$, $(4,0,23,11,3,22)$,
$(3,0,27,1,13,26)$, $(4,0,25,51,24,22)$, $(3,0,17,1,7,18)$, $(9,0,21,3,24,7)$,
$(8,0,16,1,11,15)$, $(9,0,12,16,14,11)$, $(1,0,27,2,9,25)$. □

Theorem C *For graph $G \in \{C_k : 2 \le k \le 6\}$, there exists a G-$GD_8(v)$ for $v \ge 6$.*

Proof From the following table, the existence of G-$GD_8(v)$ for $v \equiv 2,3 \pmod 4$ can be gotten, where $w = 2,3,6,7$.

Graph G	C_2, C_3	C_4, C_5, C_6
G-$GD_8(v)$	$v = 6,7,10,11,14,15,18,19,$ $22,23,50,51,54,55,66,67,$ $70,71$ (Lemma 5.4, 5.5)	
G-$ID_8(8r+w,w)$	$r=1$ (Lemma 5.2, 5.3)	
G-$HD_2(-)$ $\Longrightarrow$ G-$HD_4(-)$	$(8^q) : q = 3,4,5$ (Lemma 2.3)	
Conclusion	by Lemma 1.6	by Lemma 5.1

Table 5 Proof of Theorem C

Furthermore, by Theorem B, the conclusion follows. □

6. Conclusion

Proof of Theorem 1.2 Summarizing Lemma 1.1, Theorems A, B and C, we obtain the conclusion. □

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