Principally Quasi-Baer Modules

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Abstract In this paper, we give the equivalent characterizations of principally quasi-Baer modules, and show that any direct summand of a principally quasi-Baer module inherits the property and any finite direct sum of mutually subisomorphic principally quasi-Baer modules is also principally quasi-Baer. Moreover, we prove that left principally quasi-Baer rings have Morita invariant property. Connections between Richart modules and principally quasi-Baer modules are investigated.

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1. Introduction

The concept of principally quasi-Baer rings was first introduced in [1] by Birkenmeier, and further studied by many authors$^{[2−4]}$. Recall that a ring $R$ is called left (resp. right) principally quasi-Baer (or simply left (resp. right) p.q.-Baer) if the left (resp. right) annihilator of a principal left (resp. right) ideal is generated as a left (resp. right) ideal by an idempotent. This definition is not left-right symmetric. p.q.-Baer rings are the extensions of Baer and quasi-Baer rings$^{[5−11]}$. The class of p.q.-Baer rings include any domain, any semisimple ring, any Baer and quasi-Baer ring. Our work has been greatly motivated by these works, as mentioned above, and we try to extend these investigations to arbitrary modules.

We define principally quasi-Baer modules on the basis of p.q.-Baer rings. For a left $R$-module $M$, we call $M$ a principally quasi-Baer (or simply p.q.-Baer) module if the left annihilator in $M$ of any principal left ideal of $S$ is generated by an idempotent of $S$. It is easy to see that, when $M = R$, the notion coincides with the existing definition of left p.q.-Baer rings. Thus this definition is not left-right symmetric, either. Among examples of p.q.-Baer modules, we include any semisimple module, any Baer and quasi-Baer module, any finitely generated Abelian ring, any ideal direct summand of a left p.q.-Baer ring (Theorem 2.2), and any finitely generated...
projective left \( R \)-module, where \( R \) is a left \( p.q.-\)Baer ring (Corollary 2.1). Obviously, any left \( p.q.-\)Baer ring \( R \) is \( p.q.-\)Baer as an \( R \)-module.

In Section 2, we introduce the concept of a \( p.q.-\)Baer module, and show the equivalent characterizations of \( p.q.-\)Baer modules (Theorem 2.1). We prove that any finite direct sum of mutually subisomorphic \( p.q.-\)Baer modules is also \( p.q.-\)Baer. A natural question arises: for any algebraic property of modules, is the property inherited by direct summands of such a module? We give a positive answer to this question for the case of \( p.q.-\)Baer modules (Theorem 2.2). Among other results, we also include results on when direct sums of \( p.q.-\)Baer modules are \( p.q.-\)Baer (Theorem 2.3) and provide a characterization of \( p.q.-\)Baer modules in terms of the FI-strong summand intersection property.

In Section 3, our focus is on the endomorphism rings of \( p.q.-\)Baer modules and the connections between \( p.q.-\)Baer modules and Richart modules. We show that the endomorphism ring of a \( p.q.-\)Baer module is always left \( p.q.-\)Baer (Theorem 3.1) and that left \( p.q.-\)Baer rings have Morita invariant property. Various conditions on the equivalence of Richart modules and \( p.q.-\)Baer modules are discussed.

Throughout this paper, \( R \) denotes a ring with unity. For notation we use \( S_r(R) \) (resp. \( S_l(R) \)), \( \text{Cen}(R) \), \( M_n(R) \) for the right (resp. left) semicentral idempotents of \( R \), the center of \( R \), and the ring of \( n \times n \) matrices over \( R \), respectively. \( M \) is a left \( R \)-module and \( S = \text{End}_R(M) \) is the ring of \( R \)-endomorphisms of \( M \). Submodules of \( M \) will be left \( R \)-modules. Recall that a submodule \( X \) of \( M \) is called fully invariant if for every \( h \in S, h(X) \subseteq X \). So fully invariant submodules will be an \( R-S \)-bimodule. The notations \( l_R(\cdot) \) and \( r_M(\cdot) \) denote the left annihilator of a subset of \( M \) with elements from \( R \) and the right annihilator of a subset of \( R \) with elements from \( M \), respectively; while \( r_S(\cdot) \) and \( l_M(\cdot) \) stand for the right annihilator of a subset of \( M \) with elements from \( S \) and the left annihilator of a subset of \( S \) with elements from \( M \), respectively. Let \( N \subseteq M \). Then we use \( N \leq M, N \leq^e M, N \triangleleft M, N \triangleleft^e M, N \leq^* M \) to denote that \( N \) is a submodule, direct summand, fully invariant submodule, fully invariant direct summand, essential submodule of \( M \), respectively.

Before we discuss the properties of \( p.q.-\)Baer modules in Section 2, let us recall some related concepts.

**Definition 1.1** [12] A left \( R \)-module \( M \) is called a (quasi-) Baer module if for all \( I \leq S_S \ (I \leq S_S) \), \( l_M(I) = Me \) where \( e^2 = e \in S \).

**Definition 1.2** [14] A ring \( R \) is called a left Richart ring if for any element \( a \in R \), \( l_R(a) = Re \) where \( e^2 = e \in R \).

**Definition 1.3** [13] A left \( R \)-module \( M \) is called a Richart module if for any element \( \varphi \in S \), \( l_M(\varphi) = Me \) where \( e^2 = e \in S \).

**Definition 1.4** [2] An idempotent \( e \) of a ring \( R \) is called left (resp. right) semicentral if \( xe = exe \) (resp. \( ex = exe \)) for all \( x \in R \).

By [11, Proposition 9] and [1, Example 1.6], we can see that \( p.q.-\)Baer rings and Richart
Lemma 1.1 For an idempotent \( e \in R \), the following conditions are equivalent:

(i) \( e \in S_1(R) \);

(ii) \( 1 - e \in S_1(R) \);

(iii) \( Re \) is an ideal of \( R \);

(iv) \( (1 - e)R \) is an ideal of \( R \).

2. Principally quasi-Baer modules

In this section, we begin our investigations by first providing the equivalent characterizations of p.q.-Baer modules and give some properties of them.

Theorem 2.1 If \( M \) is a left \( R \)-module, then the following conditions are equivalent:

(i) \( M \) is p.q.-Baer;

(ii) The left annihilator in \( M \) of every finitely generated left ideal of \( S \) is generated by an idempotent of \( S \);

(iii) The left annihilator in \( M \) of every principal ideal of \( S \) is generated by an idempotent of \( S \);

(iv) The left annihilator in \( M \) of every finitely generated ideal of \( S \) is generated by an idempotent of \( S \).

Proof We only have to prove (i) \( \Rightarrow \) (ii) and the rest is clear.

Let \( I = \bigcap_{i=1}^{n} Sx_i \) (\( n \in N \)) be any finitely generated left ideal of \( S \). Then \( l_M(I) = \bigcap_{i=1}^{n} l_M(Sx_i) \). By hypothesis, we have \( l_M(Sx_i) = Me_i \) and \( e_i^2 = e_i \in S_r(S) (i = 1, 2, \ldots, n) \). Thus \( l_M(I) = \bigcap_{i=1}^{n} Me_i \). Then we assert that \( Me_1 \cap Me_2 = Me_1 e_2 \) and \( e_1 e_2 \in S_r(S) \).

First let \( x \in Me_1 \cap Me_2 \). It is easy to check that \( x = xe_1 = xe_2 = xe_1 e_2 \in Me_1 e_2 \). Since \( e_1 \in S_r(S) \), we have \( Me_1 e_2 = (Me_1 e_2)e_1 \) and \( Me_1 e_2 \subseteq Me_1 \cap Me_2 \). It follows that \( Me_1 e_2 = Me_1 \cap Me_2 \). Next, we have \( (e_1 e_2)^2 = (e_1 e_2)e_2 = e_1 e_2 \), and \( e_1 e_2 x = e_1 (e_2 x) e_2 = e_1 e_2 xe_1 e_2 (\forall x \in S) \) since \( e_i \in S_r(S) (i = 1, 2) \). Thus \( e_1 e_2 \in S_r(S) \).

Similarly, we have \( \bigcap_{i=1}^{n} Me_i = M(e_1 e_2 \cdots e_n) \) and \( (e_1 e_2 \cdots e_n) \in S_r(S) \). This completes the proof.

Theorem 2.2 Let \( M \) be a p.q.-Baer module. Then every direct summand \( N \) of \( M \) is also a p.q.-Baer module.

Proof Let \( N = Me \) where \( e^2 = e \in S \). Then \( \text{End}_R(N) = \text{End}_R(Me) \cong eSe \). For any element \( x \in \text{End}_R(N) \), we conclude that \( l_N(eSe \cdot x) \leq N \).

First we have \( x = exe \), and \( y = ye \) for any element \( y \in l_N(eSe \cdot x) \). Then \( l_N(eSe \cdot x) \subseteq l_M(Sx) \cap N \) since \( 0 = y \cdot Sx = ye \cdot S \cdot exe = y(eSe)x = 0 \). Secondly, let \( z \in l_M(Sx) \cap N \). We have \( z \in l_N(eSe \cdot x) \) since \( z = ze \in N \) and \( z \cdot eSe \cdot x = (ze)S(exe) = z \cdot Sx = 0 \). This implies \( l_N(eSe \cdot x) = l_M(Sx) \cap N \).

By assumption, we have \( l_M(Sx) = M_f \) where \( f^2 = f \in S_r(S) \). Then \( l_M(Sx) \cap N = \).
\( Mf \cap Me = Me(efe) \), and \( efe \) is an idempotent of \( eSe \) since \( f^2 = f \in S_r(S) \). Therefore, 
\[ l_N(eSe \cdot x) = Me(efe) \leq^* Me. \]

**Example 2.1** Let \( R \) be a left p.q.-Baer ring and let \( e^2 = e \in R \) be any idempotent of \( R \). Then \( M = Re \) is a left \( R \)-module which is p.q.-Baer.

**Theorem 2.3** If \( M_1 \) and \( M_2 \) are p.q.-Baer modules, and have the property that for any \( \psi \in \text{Hom}_R(M_i, M_j) \), \( \psi(x) = 0 \) implies \( x = 0 \) (i.e., \( j=1,2 \)). Then \( M_1 \oplus M_2 \) is a p.q.-Baer module.

**Proof** Let \( S = \text{End}_R(M_1 \oplus M_2) \) and \( I \) be any finitely generated ideal of \( S \). By [12, Lemma 1.10], we have \( l_{M_1 \oplus M_2}(I) \triangleleft M_1 \oplus M_2 \), and there exists \( N_i \triangleleft M_i \) (i = 1, 2) such that \( l_{M_1 \oplus M_2}(I) = N_1 \oplus N_2 \), where \( N_i = l_{M_i \oplus M_2}(I) \cap M_i \) (i = 1, 2).

As mentioned, \( S = S_1 \oplus \text{Hom}_R(M_1, M_2) \oplus \text{Hom}_R(M_2, M_1) \oplus S_2 \), where \( S_i = \text{End}_R(M_i) \) (i = 1, 2). Since \( I \) is a finitely generated ideal of \( S \), we have \( I = I_1 \oplus I_{12} \oplus I_{21} \oplus I_2 \), where \( I_1 \triangleleft S_1 \), \( I_2 \triangleleft S_2 \), \( I_{12} = \{ \varphi \in \text{Hom}_R(M_2, M_1) | \varphi = \xi_{12} \text{ with } (\xi_{ij})_{i,j=1,2} \in I \} \), \( I_{21} = \{ \varphi \in \text{Hom}_R(M_1, M_2) | \varphi = \xi_{21} \text{ with } (\xi_{ij})_{i,j=1,2} \in I \} \). It is easy to see that \( I_i \) is a finitely generated ideal of \( S_i \) (i = 1, 2).

Let us define \( l_{M_i}(I_i) = N_i' \) (i = 1, 2). It is easy to check that \( N_1 = N_1' \cap \bigcap_{\varphi \in I_{12}} \ker \varphi \).

Then we conclude that \( N_1 = N_1' \). For any element \( \psi_{12} \in \text{Hom}_R(M_2, M_1) \), \( \varphi \in I_{12} \), we have \( N_1' \varphi \psi_{12} = 0 \). Thus \( N_1' \varphi = 0 \Rightarrow N_1' \subseteq \bigcap_{\varphi \in I_{12}} \ker \varphi \). It follows that \( N_1 = N_1' \). Similarly, we have \( N_2 = N_2' \). Since \( M_1, M_2 \) are p.q.-Baer modules and \( I_i \) is a finitely generated ideal of \( S_i \), we have \( N_i' = l_{M_i}(I_i) \leq^* M_i \) (i = 1, 2). Therefore \( l_{M_1 \oplus M_2}(I) = N_1' \oplus N_2' \leq^* M_1 \oplus M_2 \). This completes the proof.

The proof of Theorem 2.3 is similar to [12, Theorem 3.18]. For the completion of this paper, we write down the whole process.

By Theorems 2.2 and 2.3, we have the following result, which provides another source of examples for p.q.-Baer modules.

**Proposition 2.1** Let \( M = \bigoplus_{i=1}^n M_i \). If \( M_i \) is subisomorphic to (i.e., isomorphic to a submodule of) \( M_j \), \( \forall i \neq j; \, i, j = 1, 2, \ldots, n \). Then \( M \) is p.q.-Baer if and only if \( M_i \) is p.q.-Baer (i = 1, 2, ..., n).

It is easy to see that Proposition 2.1 also holds true when \( M = \prod_{i=1}^n M_i \). From Proposition 2.1 and Theorem 2.2, we have

**Corollary 2.1** A finitely generated projective module over a left p.q.-Baer ring is a p.q.-Baer module.

We know that Baer and quasi-Baer modules are p.q.-Baer modules. A natural question arises, is the p.q.-Baer module also a Baer or a quasi-Baer module? The \( n \times n \) (\( n > 1 \)) upper triangular matrix ring over a domain, which is not a division ring, is a p.q.-Baer ring but not Baer^[3,p16]. Let \( R = \{a_n \in \prod_{n=1}^{\infty} M_n | (a_n)_{n=1}^{\infty} \text{ is eventually constant} \} \), where \( W \) is the \( K \text{th Weyl algebra over a field of characteristic Zero}^{[11, Example 3.13]} \). Then \( R \) is p.q.-Baer but not quasi-Baer. So p.q.-Baer modules might be neither Baer nor quasi-Baer. We will ask: under what conditions might p.q.-Baer modules and quasi-Baer modules be equivalent? The following
Proposition answers this question. We define the FI-(strong) summand intersection property on the basis of (strong) summand intersection property\(^\text{[12]}\).

**Definition 2.2** A module \(M\) is said to have the FI-summand intersection property (FI-SIP) if the intersection of two fully invariant direct summands is again a direct summand. \(M\) has the FI-strong summand intersection property (FI-SSIP) if the intersection of any number of fully invariant direct summands is again a direct summand.

**Proposition 2.2** A module \(M\) is quasi-Baer if and only if \(M\) is p.q.-Baer and has the FI-strong summand intersection property (FI-SSIP).

**Proof** The first assertion of the necessary condition is clear.

For the second, let \(Me_i \triangleleft M, e_i^2 = e_i \in S, i \in \Lambda\) (\(\Lambda\) is an index set). Then \(e_i \in S_r(S), (1-e_i)S \triangleleft S, i \in \Lambda\). Let us define \(I = \sum_{i \in \Lambda} (1-e_i)S\). Then \(I \triangleleft S\) and \(l_M(I) = \bigcap_{i \in \Lambda} l_M((1-e_i)S) = \bigcap_{i \in \Lambda} Me_i \leq^\oplus M\). Thus, \(M\) satisfies the FI-SSIP.

Conversely, let \(I\) be any ideal of \(S\). Then we can write \(I = \sum_{i \in \Lambda} Sx_i S (x_i \in I, i \in \Lambda)\). So \(l_M(I) = l_M(\sum_{i \in \Lambda} Sx_i S) = \bigcap_{i \in \Lambda} l_M(Sx_i S)\). Since \(M\) is p.q.-Baer, we have \(l_M(Sx_i S) = Me_i \leq^\oplus M\) where \(e_i^2 = e_i \in S_r(S) (\forall i \in \Lambda)\). By assumption, \(l_M(I) = \bigcap_{i \in \Lambda} Me_i = Me \leq^\oplus M\). Hence \(M\) is quasi-Baer. \(\square\)

Recall from [12] that a module \(M\) is called \(K\)-nonsingular if, for all \(\varphi \in S, l_M(\varphi) = \ker \varphi \leq^e M\) implies \(\varphi = 0\).

By [12, Lemma 2.15] and [13, Theorem 2.4], we know that both Baer and Richart modules are \(K\)-nonsingular. The following theorem shows that under a certain condition, a p.q.-Baer module is also \(K\)-nonsingular.

**Proposition 2.3** Let \(M\) be a p.q.-Baer module. If every essential submodule of \(M\) is an essential extension of a fully invariant submodule of \(M\), then \(M\) is \(K\)-nonsingular.

**Proof** Let \(0 \neq \varphi \in S\) and \(l_M(\varphi) = \ker \varphi \leq^e M\). By hypothesis, there exists a fully invariant submodule \(N \triangleleft M\) such that \(N \leq^e l_M(\varphi)\). Then \(N \subseteq l_M(S\varphi) = Me (e^2 = e \in S)\) since \(NS\varphi = N\varphi = 0\) and \(M\) is p.q.-Baer. It follows that \(Me \leq^e M\). This implies that \(e = 1, \varphi = 0\), contradicting our assumption that \(\varphi \neq 0\). Thus \(M\) is \(K\)-nonsingular. \(\square\)

3. Endomorphism rings, connections between p.q.-Baer and Richart modules

In [12, 13] we can see that the endomorphism rings of any Baer, quasi-Baer and Richart modules are Baer, quasi-Baer and left Richart rings, respectively. This suggests that these modules property may be carried over to their endomorphism rings. In this section, we study the endomorphism rings of p.q.-Baer modules and the connections between p.q.-Baer modules and Richart modules.

**Theorem 3.1** If \(M\) is a p.q.-Baer module with \(S = \text{End}_R(M)\). Then \(S\) is a left p.q.-Baer ring.
**Proof** Let $I$ be any principal left ideal of $S$. We have $l_M(I) = Me$ where $e^2 = e \in S$. Then we conclude that $l_S(I) = Se$.

First, $Se \subseteq l_S(I)$ since $MSeI = MeI = 0$. Next, for any $0 \neq \varphi \in l_S(I)$, we have $M\varphi \subseteq l_M(I)$. Thus $\varphi = \varphi e$. This implies that $l_S(I) \subseteq Se \Rightarrow l_S(I) = Se$. This completes the proof. \qed

**Corollary 3.1** Let $R$ be a left p.q.-Baer ring and $e$ is an idempotent of $R$. Then $eRe$ is also a left p.q.-Baer ring.

**Theorem 3.2** The left p.q.-Baer condition is a Morita invariant property.

**Proof** Let $R$ be a left p.q.-Baer ring. By Proposition 2.1, we have $R^{(n)}$ is left p.q.-Baer. Since $M_n(R) \cong \text{End}_R(R^{(n)})$, we know that $M_n(R)$ is also left p.q.-Baer. \qed

**Proposition 3.1** Let $R$ be a commutative ring. Then the following conditions are equivalent:

(i) $R$ is left p.q.-Baer;

(ii) $R$ is left Richart;

(iii) $R$ is VN-regular.

**Proof** It is easy to see that when $R$ is commutative, left p.q.-Baer rings and left Richart rings are equivalent, and the rest is immediate from [13, Theorem 3.2]. \qed

**Corollary 3.2** Let $M$ be a left p.q.-Baer module. Then $\text{Cen}(S)$ is VN-regular.

**Definition 3.1**[13] A module $M$ is called quasi-retractable if $\text{Hom}_R(M, N) \neq 0$, where $N = Rm$, $\forall 0 \neq m \in M$ (or, equivalently, $\exists 0 \neq \varphi \in S$ with $M\varphi \subseteq N = Rm$).

**Proposition 3.2** Let $M$ be quasi-retractable. Then $M$ is p.q.-Baer if and only if $S$ is a left p.q.-Baer ring.

**Proof** We only have to prove the sufficient condition. Let $I$ be any principal left ideal of $S$. we assert that $l_M(I) = Me$.

First, by assumption, we have $l_S(I) = Se$ where $e^2 = e \in S$. Thus $Me \subseteq l_M(I)$ since $MeI \subseteq MSeI = 0$. Next, if $\exists 0 \neq m \in l_M(I) \setminus Me$, by quasi-retractability, there exists $0 \neq \beta \in S$ such that $M\beta \subseteq Rm$. It follows that $\beta = \beta(1 - e) \in S(1 - e)$. Also, we have $\beta \in l_S(I) = Se$ since $M\beta I \subseteq RmI = 0$. This implies that $\beta = 0$, a contradiction. Therefore, $l_M(I) = Me$. \qed

In the rest, we will consider the connections between p.q.-Baer modules and Richart modules. Similarly to the definitions of the insertion of factors property (IFP)\cite{16} and strongly bounded property \cite{1} of rings, we give the following definitions.

**Definition 3.2** A left $R$-module $M$ is said to satisfy the IFP (insertion of factors property) if $l_M(\varphi)$ is a fully invariant submodule of $M$ for all $\varphi \in S$ (or, equivalently, $r_s(m) \triangleleft S$ for all $m \in M$).

**Definition 3.3** A left $R$-module $M$ is strongly bounded if every nonzero submodule of $M$ contains a nonzero fully invariant submodule.
Proposition 3.3 Let $M$ be p.q.-Baer and strongly bounded. Then $M$ is Richart and satisfies the IFP.

Proof Let $\varphi \in S$. We have $Me = l_M(S\varphi S) \subseteq l_M(\varphi)$ ($e^2 = e \in S$). Hence, $l_M(\varphi) = Me \oplus A$ for some $A \subseteq M$. If $A \neq 0$, by assumption, there exists a fully invariant submodule $0 \neq B \subseteq A$. Then, $B \subseteq l_M(\varphi) \Rightarrow BS \subseteq l_M(\varphi) \Rightarrow BS\varphi = 0 \Rightarrow BS\varphi S = 0$. Thus $B \subseteq Me$, this is impossible. Therefore, $l_M(\varphi) = l_M(S\varphi S) \triangleleft M$. $M$ is Richart and satisfies the IFP.

Proposition 3.4 Let $M$ be a left $R$-module that satisfies the IFP. Then
(i) $M$ is Richart if and only if $M$ is p.q.-Baer;
(ii) $S$ is Abelian.

Proof (i) First, for any $\varphi \in S$, we have $l_M(S\varphi) \subseteq l_M(\varphi)$. Next, for any element $m \in l_M(\varphi)$, we have $\varphi \in r_S(m)$. It follows that $m \in l_M(S\varphi)$ since $r_S(m) \triangleleft M$ and $S\varphi \subseteq r_S(m)$. Thus $l_M(S\varphi) = l_M(\varphi)$, Richart and p.q.-Baer modules are equivalent;
(ii) The proof is routine.

Theorem 3.3 Let $M$ be a left $R$-module, $S = \text{End}_R(M)$. Then the following conditions are equivalent:
(i) $M$ is a Richart module and $S$ is Abelian;
(ii) $M$ is a p.q.-Baer module which satisfies the IFP.

Proof (i)$\Rightarrow$(ii). First, for any $\varphi \in S$, we have $l_M(S\varphi) \subseteq l_M(\varphi)$ and $l_M(\varphi) = Me$ ($e^2 = e \in \text{Cen}(S)$). Then, $eS\varphi = 0$ since $eS\varphi = Se\varphi$ and $e\varphi \subseteq Me\varphi = 0$. It follows that $Me \subseteq l_M(S\varphi)$. Thus $l_M(S\varphi) = Me$. Since $S$ is Abelian, we have $l_M(S\varphi) = Me \triangleleft M$;
(ii)$\Rightarrow$(i). This is immediate from Proposition 3.4.

Proposition 3.5 Let $M$ be a left $R$-module, $S = \text{End}_R(M)$. Consider the following conditions:
(a) $M$ satisfies the IFP;
(b) $S$ is reduced;
(c) $S$ satisfies the IFP;
(d) $S$ is Abelian.

The following statements hold true:
(i) If $S$ is a left Richart ring, then (b) through (d) are equivalent;
(ii) If $M$ is a Richart module, then (a) through (d) are equivalent;
(iii) If $S$ is a VN-regular ring, then (a) through (d) are equivalent.

Proof (i) For any ring $S$, it is easy to get (b)$\Rightarrow$(c)$\Rightarrow$(d). Now, we only have to prove (d)$\Rightarrow$(b). Let $x^2 = 0$. Then $r_R(x) = eS$ where $e^2 = e \in \text{Cen}(S)$. Thus $x = ex = xe = 0$ since $x \in r_R(x) = eS$;
(ii) By [13, Theorem 3.1], we know that $S$ is left Richart. Thus, we only have to prove that (a)$\Rightarrow$(d). By Proposition 3.4 and Theorem 3.3, we know that (a)$\Leftrightarrow$(d);
(iii) We only have to prove that if $S$ is VN-regular, then $M$ is Richart.
For any \( \varphi \in S \), there exists \( \psi \in S \) such that \( \varphi = \varphi \psi \varphi \). Let us define \( \pi = \varphi \psi \varphi \). Then \( \pi^2 = \pi \) and \( \varphi = \pi \varphi \). This implies that \( \ker \varphi = \ker \pi = M(1-\pi) \leq M \). \( \square \)

References