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Hermitian Positive Definite Solutions of the Matrix Equation $X + A^*X^{-q}A = Q$ $(q \geq 1)$

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Abstract In this paper, Hermitian positive definite solutions of the nonlinear matrix equation $X + A^*X^{-q}A = Q$ $(q \geq 1)$ are studied. Some new necessary and sufficient conditions for the existence of solutions are obtained. Two iterative methods are presented to compute the smallest and the quasi largest positive definite solutions, and the convergence analysis is also given. The theoretical results are illustrated by numerical examples.

Keywords nonlinear matrix equations; positive definite solution; iterative method.

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1. Introduction

In this paper, we consider the nonlinear matrix equation

$$X + A^*X^{-q}A = Q, \tag{1}$$

where A is an $n \times n$ nonsingular complex matrix, Q is an $n \times n$ positive definite matrix, and $q \geq 1$. Eq.(1) has many applications in control theory, ladder networks, dynamic programming, queueing theory, stochastic filtering and statistics[1−3].

Recently, the matrix equation $X + A^*X^{-q}A = Q$ with the following cases have been investigated:

(a) $0 < q \leq 1$ and Q is a positive definite matrix[4−9];

(b) $q > 0$ and Q is the identity matrix[3−5],[9−11];

(c) q is a positive integer and Q is a positive definite matrix[3],[12−14].

Based on these, the matrix equation $X + A^*X^{-q}A = Q$ (i.e., Eq.(1)) is studied in this paper, where $q \geq 1$ and Q is a positive definite matrix.

This paper is organized as follows. In Section 2, we derive some new necessary and sufficient conditions for the existence of solutions. In Section 3, we construct two iterative methods to

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compute the smallest and the quasi largest positive definite solutions, and also give their convergence analysis. We use numerical examples to show that the iterative methods are feasible and effective in Section 4.

Throughout the paper, we write $B > 0$ $(B \geq 0)$ if the matrix B is a positive definite (semidefinite). If $B-C$ is a positive definite (semidefinite) matrix, then we write $B > C(B \geq C)$. We use $\| B \|$ and $\| B \|_F$ to denote the spectral norm and the Frobenius norm of a matrix B, respectively, and use $C^{n\times n}$, $C_n^{n\times n}$ and $U^{n\times n}$ to denote the set of all $n \times n$ complex matrices, nonsingular complex matrices and unitary matrices, respectively. The symbols $\lambda_1(B)$ and $\lambda_n(B)$ stand for the maximal and minimal eigenvalues of a positive definite matrix B. The symbols $\sigma_1(B)$ and $\sigma_n(B)$ stand for the maximal and minimal singular values of a matrix B.

Definition 1[6] *Let $X_S(X_L)$ be a positive definite solution of Eq.(1). If every positive definite solution X satisfies $X \geq X_S(X \leq X_L)$, then $X_S(X_L)$ is called the smallest (largest) positive definite solution of Eq.(1).*

Definition 2 *Let $\tilde{X}$ be a positive definite solution of Eq.(1). If every positive definite solution X satisfies $\lambda_n(Q^{-\frac{1}{2}}XQ^{-\frac{1}{2}}) < \lambda_n(Q^{-\frac{1}{2}}\tilde{X}Q^{-\frac{1}{2}})$, then $\tilde{X}$ is called the quasi largest positive definite solution of Eq.(1).*

Lemma 1[15] *If $A \geq B > 0$, then $A^\alpha \geq B^\alpha$ for all $\alpha \in (0,1]$ and $A^\alpha \leq B^\alpha$ for all $\alpha \in [-1,0)$.*

Lemma 2[15] *For any $MI \geq X, Y \geq mI > 0$, then $\| X^\alpha - Y^\alpha \| \leq |\alpha| m^{\alpha-1} \| X - Y \|$ for all $\alpha < 0$, $\| X^\alpha - Y^\alpha \| \leq \alpha m^{\alpha-1} \| X - Y \|$ for all $0 < \alpha < 1$ and $\| X^\alpha - Y^\alpha \| \leq \alpha M^{\alpha-1} \| X - Y \|$ for all $\alpha > 1$.*

Lemma 3[16] *Let $M_1 I \geq A \geq m_1 I > 0$, $M_2 I \geq B \geq m_2 I > 0$ and $B \geq A > 0$. Then*

$$A^\alpha \leq K_{1,\alpha} B^\alpha \leq (\frac{M_1}{m_1})^{\alpha-1} B^\alpha, \quad A^\alpha \leq K_{2,\alpha} B^\alpha \leq (\frac{M_2}{m_2})^{\alpha-1} B^\alpha$$

for all $\alpha \in [1,+\infty)$, where $K_{i,\ \alpha} = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha (M_i - m_i)} \frac{(M_i^\alpha - m_i^\alpha)^\alpha}{(m_i M_i^\alpha - M_i m_i^\alpha)^{\alpha-1}}$, $i = 1, 2$.

Lemma 4[17] *Let A and B be n-square positive semidefinite matrices. Then there exists an invertible matrix P such that P^*AP and P^*BP are both diagonal matrices. In addition, if A is nonsingular, then P can be chosen so that $P^*AP = I$ and P^*BP is diagonal.*

2. Conditions for the existence of solutions

In this section, we derive some new necessary and sufficient conditions for the existence of solutions of Eq.(1).

Theorem 1 *If Eq.(1) has a solution X, then*

$$X \in (\sqrt[q]{AQ^{-1}A^*}, Q - \frac{1}{K} A^* Q^{-q} A],$$

where $K = \frac{(q-1)^{q-1}}{q^q(\lambda_1(Q^{-1}) - \lambda_n(Q^{-1}))} \frac{(\lambda_1(Q^{-1})^q - \lambda_n(Q^{-1})^q)^q}{(\lambda_n(Q^{-1})\lambda_1(Q^{-1})^q - \lambda_1(Q^{-1})\lambda_n(Q^{-1})^q)^{q-1}}$.

Proof Let X be a solution of (1), then $X < Q$, $A^*X^{-q}A < Q$, which implies that $A^{-1}X^qA^{-*} >$

Q^{-1}, i.e., $X^q > AQ^{-1}A^*$. By Lemma 1, we have

$$X > \sqrt[q]{AQ^{-1}A^*}.$$

By Lemmas 1 and 3, we have

$$Q^{-q} \leq KX^{-q},\ X = Q - A^*X^{-q}A \leq Q - \frac{1}{K}A^*Q^{-q}A,$$

where $K = \frac{(q-1)^{q-1}}{q^q(\lambda_1(Q^{-1})-\lambda_n(Q^{-1}))} \frac{(\lambda_1(Q^{-1})^q - \lambda_n(Q^{-1})^q)^q}{(\lambda_n(Q^{-1})\lambda_1(Q^{-1})^q - \lambda_1(Q^{-1})\lambda_n(Q^{-1})^q)^{q-1}}$. Hence

$$X \in (\sqrt[q]{AQ^{-1}A^*}, Q - \frac{1}{K}A^*Q^{-q}A].$$

Theorem 2 *Eq.(1) has a solution X if and only if there exist $W \in C_n^{n\times n}$, $Z \in C^{n\times n}$ such that $A = (W^*W)^{\frac{q}{2}}Z$, where the columns of $\binom{WQ^{-\frac{1}{2}}}{ZQ^{-\frac{1}{2}}}$ are orthonormal. In this case, $X = W^*W$ is a solution of Eq.(1).*

Proof If X is a solution of Eq.(1), then there exists $W > 0$ such that $X = W^*W$. Eq.(1) can be rewritten as

$$W^*W + ((W^*)^{-q}A)^*(W^*)^{-q}A = Q,$$

or equivalently

$$\begin{pmatrix} WQ^{-\frac{1}{2}} \\ (W^*)^{-q}AQ^{-\frac{1}{2}} \end{pmatrix}^* \begin{pmatrix} WQ^{-\frac{1}{2}} \\ (W^*)^{-q}AQ^{-\frac{1}{2}} \end{pmatrix} = I. \tag{2}$$

Let $Z = (W^*)^{-q}A$. Then $A = (W^*W)^{\frac{q}{2}}Z$ and from (2) we have that the columns of $\binom{WQ^{-\frac{1}{2}}}{ZQ^{-\frac{1}{2}}}$ are orthonormal.

Conversely, assume there exist $W \in C_n^{n\times n}$, $Z \in C^{n\times n}$ such that $A = (W^*W)^{\frac{q}{2}}Z$ and the columns of $\binom{WQ^{-\frac{1}{2}}}{ZQ^{-\frac{1}{2}}}$ are orthonormal. Let $X = W^*W$. Then

$$\begin{aligned} X + A^*X^{-q}A &= W^*W + \left((W^*W)^{\frac{q}{2}}Z\right)^* (W^*W)^{-q}(W^*W)^{\frac{q}{2}}Z \\ &= W^*W + Z^*Z = Q. \end{aligned}$$

i.e., X is a solution of Eq.(1).

Theorem 3 *Eq.(1) has a solution X if and only if there exist $P, M, U \in U^{n\times n}$ and diagonal matrices Γ, Φ, $\Sigma > 0$ such that*

$$A = P^*\Gamma^q M\Phi U^* Q^{\frac{1}{2}},$$

where $\Sigma^2 + \Phi^2 = I$, $\Sigma^2 = (PQ^{-\frac{1}{2}}U)^*\Gamma^2 PQ^{-\frac{1}{2}}U$. In this case, $X = P^*\Gamma^2 P$ is a solution of Eq.(1).

Proof Assume there exist $P, M, U \in U^{n\times n}$ and diagonal matrices Γ, Φ, $\Sigma > 0$, $\Sigma^2 + \Phi^2 = I$, $\Sigma^2 = (PQ^{-\frac{1}{2}}U)^*\Gamma^2 PQ^{-\frac{1}{2}}U$ such that $A = P^*\Gamma^q M\Phi U^*Q^{\frac{1}{2}}$. Then it is easy to verify that $X = P^*\Gamma^2 P$ is a solution of Eq.(1).

Conversely, if Eq.(1) has a solution X. By the spectral decomposition theorem, there exists $P \in U^{n\times n}$ such that $X = P^*\Gamma^2 P$, where $\Gamma > 0$ is diagonal. Eq.(1) can be rewritten as

$$PA^*P^*\Gamma^{-2q}PAP^* = PQP^* - \Gamma^2. \tag{3}$$

By Lemma 4, there exists $U \in U^{n\times n}$ and diagonal matrix $\Sigma > 0$ such that

$$U^*Q^{-\frac{1}{2}}P^*(PQP^*)PQ^{-\frac{1}{2}}U = I$$

and

$$U^*Q^{-\frac{1}{2}}P^*\Gamma^2 PQ^{-\frac{1}{2}}U = \Sigma^2.$$

Then we have

$$U^*Q^{-\frac{1}{2}}P^*PA^*P^*\Gamma^{-2q}PAP^*PQ^{-\frac{1}{2}}U = I - \Sigma^2. \tag{4}$$

Let $K = \Gamma^{-q}PAQ^{-\frac{1}{2}}U \in C_n^{n\times n}$, $\Phi = (I-\Sigma^2)^{\frac{1}{2}} > 0$. From (4) it follows that $K^*K = \Phi^2$. Let $M = K\Phi^{-1}$. It is easy to verify that $M^*M = I$, i.e., M is an $n \times n$ unitary matrix. Since $K = \Gamma^{-q}PAQ^{-\frac{1}{2}}U = M\Phi$, we get $A = P^*\Gamma^q M\Phi U^* Q^{\frac{1}{2}}$.

3. Iterative methods and convergence analysis

In this section, we give two iterative methods to compute the smallest and the quasi largest positive definite solutions.

Consider the first iterative method

$$X_0 = sQ,\ X_{k+1} = \sqrt[q]{A(Q-X_k)^{-1}A^*}. \tag{5}$$

Theorem 4 *Assume* $\sigma_1^2(Q^{-\frac{q}{2}}AQ^{-\frac{1}{2}}) \le \frac{q^q}{(q+1)^{q+1}}$. *Let* α_1, β_1 *be solutions of equations* $x^q(1-x) = \sigma_n^2(Q^{-\frac{q}{2}}AQ^{-\frac{1}{2}})$, $x^q(1-x) = \sigma_1^2(Q^{-\frac{q}{2}}AQ^{-\frac{1}{2}})$, *respectively, on* $(0, \frac{q}{q+1}]$. *Consider* $\{X_k\}$ *defined by (5). Then*

(i) If $X_0 = \gamma Q$, $\gamma \in [0, \alpha_1]$, *then* $\{X_k\}$ *is monotonically increasing and converges to the smallest positive definite solution* X_S *of Eq.(1), and* $X_S \in [\alpha_1 Q, \beta_1 Q]$.

(ii) If $X_0 = \xi Q$, $\xi \in [\alpha_1, \beta_1]$, *and*

$$c_1 = \frac{1}{q}\Big(\frac{\|A\|\|Q^{-1}\|}{1-\beta_1}\Big)^2\Big(\frac{1-\alpha_1}{\lambda_n(AQ^{-1}A^*)}\Big)^{\frac{q-1}{q}} < 1, \tag{6}$$

then $\{X_k\}$ *converges to the smallest positive definite solution* X_S.

Proof It is easy to verify that $0 < \alpha_1 \le \beta_1 \le \frac{q}{q+1}$ and the function $f(x) = x^q(1-x)$ is monotonically increasing where $x \in [0, \frac{q}{q+1}]$. Thus for any $0 \le \alpha \le \alpha_1 \le \beta_1 \le \beta \le \frac{q}{q+1}$, we have

$$\alpha^q(1-\alpha)I \le Q^{-\frac{q}{2}}AQ^{-1}A^*Q^{-\frac{q}{2}} \le \beta^q(1-\beta)I,$$
$$\alpha^q(1-\alpha)Q^q \le AQ^{-1}A^* \le \beta^q(1-\beta)Q^q.$$

(i) We have $X_0 = \gamma Q \le \beta_1 Q$, $\gamma \in [0,\ \alpha_1]$. Thus

$$X_1 = \sqrt[q]{A(Q-\gamma Q)^{-1}A^*} = \sqrt[q]{\frac{AQ^{-1}A^*}{1-\gamma}} \le \sqrt[q]{\frac{1}{1-\beta_1}\beta_1^q(1-\beta_1)Q^q} = \beta_1 Q$$

and

$$X_1 = \sqrt[q]{A(Q-\gamma Q)^{-1}A^*} = \sqrt[q]{\frac{AQ^{-1}A^*}{1-\gamma}} \ge \sqrt[q]{\frac{1}{1-\gamma}\gamma^q(1-\gamma)Q^q} = \gamma Q = X_0,$$

and we have $X_0 \le X_1 \le \beta_1 Q$. Assume $X_{k-1} \le X_k \le \beta_1 Q$. Then

$$\sqrt[q]{A(Q-X_{k-1})^{-1}A^*} \le \sqrt[q]{A(Q-X_k)^{-1}A^*} \le \sqrt[q]{A(Q-\beta_1 Q)^{-1}A^*},$$

$$X_k \leq X_{k+1} \leq \sqrt[q]{\frac{AQ^{-1}A^*}{1-\beta_1}} \leq \sqrt[q]{\frac{\beta_1^q(1-\beta_1)}{1-\beta_1}Q^q} = \beta_1 Q.$$

Thus the sequence $\{X_k\}$ is monotonically increasing and converges to a solution $\hat{X}$ with $\hat{X} \leq \beta_1 Q$.

Let X be a solution of Eq.(1). Construct the sequence

$$\alpha_0 = 0,\ \alpha_{k+1} = \sqrt[q]{\frac{\sigma_n^2(Q^{-\frac{q}{2}}AQ^{-\frac{1}{2}})}{1-\alpha_k}},\ k = 0,1,2,\dots\ .$$

Obviously, we have $X \geq \alpha_0 Q = 0$ and $\alpha_0 \leq \frac{q}{q+1}$. Assume $X \geq \alpha_k Q$, $\alpha_k \leq \frac{q}{q+1}$. We have

$$X = \sqrt[q]{A(Q-X)^{-1}A^*} \geq \sqrt[q]{\frac{AQ^{-1}A^*}{1-\alpha_k}} \geq \sqrt[q]{\frac{Q^{\frac{q}{2}}(Q^{-\frac{q}{2}}AQ^{-1}A^*Q^{-\frac{q}{2}})Q^{\frac{q}{2}}}{1-\alpha_k}}$$

$$\geq \sqrt[q]{\frac{\sigma_n^2(Q^{-\frac{q}{2}}AQ^{-\frac{1}{2}})}{1-\alpha_k}Q^q} = \alpha_{k+1}Q,$$

$$\alpha_{k+1} = \sqrt[q]{\frac{\sigma_n^2(Q^{-\frac{q}{2}}AQ^{-\frac{1}{2}})}{1-\alpha_k}} \leq \sqrt[q]{\frac{q^q}{(q+1)^{q+1}}(q+1)} = \frac{q}{q+1}.$$

Then $X \geq \alpha_k Q$, $\alpha_k \leq \frac{q}{q+1}$ for $k = 0,1,2,\dots$.

The sequence $\{\alpha_k\}$ is monotonically increasing and bounded, hence it is convergent. Let $\hat{\alpha} = \lim_{k\to\infty}\alpha_k$. Then $\hat{\alpha} \leq \frac{q}{q+1}$ and $\hat{\alpha} = \sqrt[q]{\frac{\sigma_n^2(Q^{-\frac{q}{2}}AQ^{-\frac{1}{2}})}{1-\hat{\alpha}}}$, which means $\hat{\alpha}$ is one and only solution of the equation $x^q(1-x) = \sigma_n^2(Q^{-\frac{q}{2}}AQ^{-\frac{1}{2}})$ on $(0, \frac{q}{q+1}]$. Hence $\alpha_1 = \hat{\alpha}$ and $X \geq \alpha_1 Q$.

For $X_0 = \gamma Q$, $\gamma \in [0, \alpha_1]$, we have $X_0 \leq \alpha_1 Q \leq X$. Assume $X_k \leq X$. Then

$$X_{k+1} = \sqrt[q]{A(Q-X_k)^{-1}A^*} \leq \sqrt[q]{A(Q-X)^{-1}A^*} = X.$$

So the solution $\hat{X}$ is the smallest positive definite solution X_S of Eq.(1), i.e., $\hat{X} = X_S$, and $X_S \in [\alpha_1 Q,\ \beta_1 Q]$.

(ii) We have $X_0 = \xi Q$, $\xi \in [\alpha_1, \beta_1]$ and $\alpha_1 Q \leq X_0 = \xi Q \leq \beta_1 Q$. Assume $\alpha_1 Q \leq X_k \leq \beta_1 Q$. For X_{k+1} we compute

$$X_{k+1} = \sqrt[q]{A(Q-X_k)^{-1}A^*} \leq \sqrt[q]{A(Q-\beta_1 Q)^{-1}A^*} \leq \sqrt[q]{\frac{AQ^{-1}A^*}{1-\beta_1}} \leq \beta_1 Q,$$

$$X_{k+1} = \sqrt[q]{A(Q-X_k)^{-1}A^*} \geq \sqrt[q]{A(Q-\alpha_1 Q)^{-1}A^*} \geq \alpha_1 Q.$$

So $\alpha_1 Q \leq X_k \leq \beta_1 Q$ for $k = 0,1,2,\dots$. Furthermore, we have

$$A(Q-X_k)^{-1}A^* \geq \frac{AQ^{-1}A^*}{1-\alpha_1} \geq \frac{\lambda_n(AQ^{-1}A^*)}{1-\alpha_1}I,$$

$$(Q-X_k)^{-1} \leq \frac{1}{1-\beta_1}Q^{-1} \leq \frac{1}{1-\beta_1}\|Q^{-1}\|I,$$

$$X_{k+p} - X_k = \sqrt[q]{A(Q-X_{k+p-1})^{-1}A^*} - \sqrt[q]{A(Q-X_{k-1})^{-1}A^*}.$$

By Lemma 2, we obtain

$$\|X_{k+p} - X_k\| \leq \frac{1}{q}\Big(\frac{\lambda_n(AQ^{-1}A^*)}{1-\alpha_1}\Big)^{\frac{1}{q}-1}\|A\left((Q-X_{k+p-1})^{-1} - (Q-X_{k-1})^{-1}\right)A^*\|$$

$$
\begin{aligned}
&\le \frac{1}{q}\Big(\frac{1-\alpha_1}{\lambda_n(AQ^{-1}A^*)}\Big)^{\frac{q-1}{q}}\|A\|^2\left(\frac{\|Q^{-1}\|}{1-\beta_1}\right)^2\|X_{k+p-1}-X_{k-1}\|\\
&\cdots\\
&\le \Big(\frac{1}{q}\Big(\frac{\|A\|\|Q^{-1}\|}{1-\beta_1}\Big)^2\Big(\frac{1-\alpha_1}{\lambda_n(AQ^{-1}A^*)}\Big)^{\frac{q-1}{q}}\Big)^k\|X_p-X_0\|\\
&\le c_1^k\frac{1}{1-c_1}\|X_1-X_0\|.
\end{aligned}
$$

According to (6), the sequence $\{X_k\}$ is a Cauchy sequence on $[\alpha_1 Q, \beta_1 Q]$. Let $X_\xi=\lim_{k\to\infty}X_k$. Then $X_\xi\in[\alpha_1 Q,\beta_1 Q]$.

We assume that X, Y are solutions of Eq.(1) on $[\alpha_1 Q, \beta_1 Q]$ with $X\ne Y$. Then

$$\|X-Y\|\le c_1\|X-Y\|<\|X-Y\|.$$

Thus X_ξ is a unique solution of Eq.(1) on $[\alpha_1 Q,\beta_1 Q]$. According to (i), Eq.(1) has a smallest positive definite solution $X_S\in[\alpha_1 Q,\beta_1 Q]$. Hence $X_\xi=X_S$.

Consider the second iterative method

$$X_0=tQ,\ X_{k+1}=Q-A^*X_k^{-q}A. \tag{7}$$

Theorem 5 *Assume* $\sigma_1^2(Q^{-\frac{q}{2}}AQ^{-\frac12})\le\frac{q^q}{(q+1)^{q+1}}(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})})^{q-1}$. *Let* α_2, β_2 *be solutions of equations* $x^q(1-x)=(\frac{\lambda_1(Q^{-1})}{\lambda_n(Q^{-1})})^{q-1}\sigma_1^2(Q^{-\frac{q}{2}}AQ^{-\frac12})$, $x^q(1-x)=(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})})^{q-1}\sigma_n^2(Q^{-\frac{q}{2}}AQ^{-\frac12})$, *respectively, on* $[\frac{q}{q+1},1)$. *If*

$$c_2=\frac{q\|A\|^2\|Q^{-1}\|^{q+1}}{\alpha_2^{q+1}}<1, \tag{8}$$

then $\{X_k\}$ *defined by (7) converges to a unique solution* $\tilde X\in[\alpha_2 Q,\beta_2 Q]$ *of Eq.(1) for all* $\eta\in[\alpha_2,\beta_2]$, *and* $\tilde X$ *is the quasi largest positive definite solution of Eq.(1).*

Proof We have $X_0=\eta Q$, $\eta\in[\alpha_2,\beta_2]$ and $\alpha_2 Q\le X_0=\eta Q\le\beta_2 Q$. Thus

$$X_1=Q-\frac{1}{\eta^q}A^*Q^{-q}A\le Q-\frac{1}{\eta^q}\Big(\frac{\lambda_1(Q^{-1})}{\lambda_n(Q^{-1})}\Big)^{q-1}\beta_2^q(1-\beta_2)Q\le\beta_2 Q,$$

$$X_1=Q-\frac{1}{\eta^q}A^*Q^{-q}A\ge Q-\frac{1}{\eta^q}\Big(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\Big)^{q-1}\alpha_2^q(1-\alpha_2)Q\ge\alpha_2 Q.$$

Assume $\alpha_2 Q\le X_k\le\beta_2 Q$. By Lemmas 1 and 3, we obtain

$$X_k^{-1}\ge(\beta_2 Q)^{-1}=\frac{1}{\beta_2}Q^{-1},\ X_k^{-q}\ge\frac{1}{\beta_2^q}\Big(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\Big)^{q-1}Q^{-q}.$$

Thus

$$
\begin{aligned}
X_{k+1}&=Q-A^*X_k^{-q}A\\
&\le Q-\frac{1}{\beta_2^q}\Big(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\Big)^{q-1}A^*Q^{-q}A\\
&\le\Big(1-\frac{1}{\beta_2^q}\Big(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\Big)^{q-1}\Big(\frac{\lambda_1(Q^{-1})}{\lambda_n(Q^{-1})}\Big)^{q-1}\beta_2^q(1-\beta_2)\Big)Q\\
&=\beta_2 Q.
\end{aligned}
$$

Similarly, we have $X_{k+1} \geq \alpha_2 Q$. Thus $\alpha_2 Q \leq X_k \leq \beta_2 Q$ for $k = 0, 1, 2, \ldots$. Since $X_k \geq \alpha_2 Q$, we have $X_k \geq \frac{\alpha_2}{\|Q^{-1}\|} I$. By Lemma 2, we obtain

$$\begin{aligned}\|X_{k+p} - X_k\| &= \|A^*(X_{k+p-1}^{-q} - X_{k-1}^{-q})A\| \\ &\leq q\|A\|^2 (\frac{\|Q^{-1}\|}{\alpha_2})^{q+1} \|X_{k+p-1} - X_{k-1}\| \\ &\leq (\frac{q\|A\|^2 \|Q^{-1}\|^{q+1}}{\alpha_2^{q+1}})^k \|X_p - X_0\| \\ &\leq c_2^k \frac{1}{1 - c_2} \|X_1 - X_0\|.\end{aligned}$$

According to (8), the sequence $\{X_k\}$ is a Cauchy sequence on $[\alpha_2 Q, \beta_2 Q]$, and converges to a unique solution $\tilde{X} \in [\alpha_2 Q, \beta_2 Q]$ of Eq.(1).

Assume that X', X'' are solutions of (1) on $[\alpha_2 Q, Q]$ with $X' \neq X''$. Then

$$\|X' - X''\| \leq c_2 \|X' - X''\| < \|X' - X''\|.$$

Thus $\tilde{X}$ is unique on $[\alpha_2 Q, Q]$. In view of Definition 2, we know that $\tilde{X}$ is the quasi largest positive definite solution of Eq.(1).

4. Numerical examples

In this section, we have made two numerical experiments to compute the positive definite solutions of Eq.(1). All the computations are implemented on a PC with 1.4 GHz Pentium IV and 512 MB SDRAM using MATLAB 6.5, where the stopping criterion $\|X_k - X_{k+1}\| < 10^{-15}$ is used.

Example 1 Consider the matrix equation $X + A^*X^{-1.3}A = Q$ with

$$A = \begin{pmatrix} 0.37 & 0.13 & 0.32 \\ 0.30 & 0.34 & 0.12 \\ 0.11 & 0.17 & 0.29 \end{pmatrix}, \quad Q = \begin{pmatrix} 7.15 & 3.02 & 0.11 \\ 3.02 & 6.20 & 2.01 \\ 0.11 & 2.01 & 6.50 \end{pmatrix}.$$

It is easy to verify that A, Q and q satisfy the conditions of Theorem 4. We also can get $\alpha_1 = 0.0022, \beta_1 = 0.0151$. Consider the iterative method (5) with $X_0 = 0.01Q$. After 8 iterations, we get the smallest positive definite solution of Eq.(1)

$$X_S \approx X_8 = \begin{pmatrix} 0.0772 & 0.0274 & 0.0340 \\ 0.0274 & 0.0504 & 0.0203 \\ 0.0340 & 0.0203 & 0.0339 \end{pmatrix}$$

and $\| X_8 + A^*X_8^{-1.3}A - Q \|_F = 5.3688 \times 10^{-15}$.

Example 2 Consider the matrix equation $X + A^*X^{-2.1}A = Q$ with

$$A = \begin{pmatrix} 1.8 & 0.9 & 0 & 0.9 \\ 0.9 & 2.7 & 1.8 & 0.9 \\ 0 & 0.9 & 0 & 0.9 \\ 0.9 & 0 & 0.9 & 1.8 \end{pmatrix}, \quad Q = \begin{pmatrix} 7 & -1 & 2 & 1 \\ -1 & 9 & -3 & 1 \\ 2 & -3 & 9 & 2 \\ 1 & 1 & 2 & 8 \end{pmatrix}.$$

It is easy to verify that A, Q and q satisfy the conditions of Theorem 5. We also can get $\alpha_2 = 0.8082$, $\beta_2 = 0.9998$. Consider the iterative method (7) with $X_0 = 0.9Q$. After 14 iterations, we get the quasi largest positive definite solution of Eq.(1)

$$\tilde{X} \approx X_{14} = \begin{pmatrix} 6.9192 & -1.0567 & 1.9758 & 0.9574 \\ -1.0567 & 8.7656 & -3.0824 & 0.9476 \\ 1.9758 & -3.0824 & 8.9539 & 1.9632 \\ 0.9574 & 0.9476 & 1.9632 & 7.9484 \end{pmatrix}$$

and $\| X_{14} + A^* X_{14}^{-2.1} A - Q \|_F = 1.1213 \times 10^{-15}$.

The above examples show that the iterative methods (5) and (7) are feasible and effective for computing the smallest and the quasi largest positive definite solutions.

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