Green’s Relations on a Kind of Semigroups of Linear Transformations

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Abstract  Let $V$ be a linear space over a field $F$ with finite dimension, $L(V)$ the semigroup, under composition, of all linear transformations from $V$ into itself. Suppose that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ is a direct sum decomposition of $V$, where $V_1, V_2, \ldots, V_m$ are subspaces of $V$ with the same dimension. A linear transformation $f \in L(V)$ is said to be sum-preserving, if for each $i$ ($1 \leq i \leq m$), there exists some $j$ ($1 \leq j \leq m$) such that $f(V_i) \subseteq V_j$. It is easy to verify that all sum-preserving linear transformations form a subsemigroup of $L(V)$ which is denoted by $L^\oplus(V)$. In this paper, we first describe Green’s relations on the semigroup $L^\oplus(V)$. Then we consider the regularity of elements and give a condition for an element in $L^\oplus(V)$ to be regular. Finally, Green’s equivalences for regular elements are also characterized.

Keywords  linear spaces; linear transformations; semigroups; Green’s equivalence; regular semigroups.

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1. Introduction and preliminaries

Let $X$ be an arbitrary set, $T_X$ the full transformation semigroup on the set $X$ and $E$ be an equivalence relation on $X$. The first author observed in [6] a class of transformation semigroups determined by the equivalence $E$, namely

$$T_E(X) = \{f \in T_X : \forall (a, b) \in E, (f(a), f(b)) \in E\}.$$

$T_E(X)$ is obviously a subsemigroup of $T_X$. The common nature of all elements in $T_E(X)$ is that they preserve the decomposition induced by the equivalence $E$. In other words, all $f \in T_E(X)$ satisfy the condition that for each $E$-class $A$ there exists some $E$-class $B$ such that $f(A) \subseteq B$. In recent years, some properties for $T_E(X)$ are investigated in many papers. For example, [7] considered the Green’s equivalences, [9] and [10] discussed some subsemigroups of $T_E(X)$ inducing certain lattices of equivalences on the set $X$, and [8] investigated the rank of $T_E(X)$ for a special case of $X$ and $E$.

In this paper we examine a related semigroup defined as follows. Let $V$ be a linear space over a field $F$ and $L(V)$ be the semigroup, under composition, of all linear transformations on the
linear space $V$. Suppose that $V = \oplus\{V_i : i \in I\}$, where each $V_i$ is a subspace of $V$ with $|I| \geq 2$ and $\dim V_i \geq 2$ for each $i$. A linear transformation $f \in L(V)$ is called sum-preserving if for each $i \in I$, there exists some $j \in I$ such that $f(V_i) \subseteq V_j$. It is not hard to verify that if $f$ and $g$ are sum-preserving, then so is $fg$. Consequently, all sum-preserving linear transformations form a subsemigroup of $L(V)$ which will be denoted by $L^\oplus(V)$.

We notice that many conclusions for $T_X$ have their parallelism for $L(V)$. For example, in 1966, Howie\cite{howie} characterized the transformations in $T_X$ that can be written as a product of finite number idempotents in $T_X$. Since then Erdos\cite{erdos} and Dawlings\cite{dawlings} gave different proofs of the result that when $V$ is finite-dimensional, $\alpha \in L(V)$ is a finite product of proper idempotents in $L(V)$ if and only if $\dim(\alpha(V)) < \dim V$. Later in 1985, Reynolds and Sullivan\cite{reynolds} investigated the case of infinite-dimensional spaces and obtained the results similar to Howie’s.

We may compare the elements in $L^\oplus(V)$ with that in $T_E(X)$ and find that all they are transformations of a set (or a linear space) preserving some decomposition. Therefore, $L^\oplus(V)$ can be regarded as the linear transformation version of the semigroup $T_E(X)$.

In this paper, we are going to consider a special case for the direct sum decomposition, namely, we assume $\dim V_i = n \geq 2$ for each $i \in I = \{1, 2, \ldots, m\}$ with $m \geq 2$ while

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m, \quad \dim V_i = n \quad (1 \leq i \leq m).$$

Here we focus our attention to Green’s equivalence relations and the regularity for the semigroup $L^\oplus(V)$. Accordingly, in Section 2, we describe five Green’s relations and conclude that $D = J$. In Section 3, we consider the condition for an element $f \in L^\oplus(V)$ to be regular. By the way, we describe the Green’s relations for regular elements in the semigroup $L^\oplus(V)$.

In order to avoid repeat, in the remainder of the paper, the symbols $V_i, V_j, V_l, V_{j_1}, \ldots$ will always denote certain subspaces in the direct sum decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ without further mention. In addition, if we have defined a number of linear mappings $f_i : V_i \to V_{i'}$ where $i, i' \in I$, then there exists a unique linear transformation $f \in L^\oplus(V)$ satisfying $f|V_i = f_i$. Finally, for convenience, we do not distinguish the zero vector 0 and the singleton set \{0\}. As we have seen previously, we write $f(V_i) = 0$ to mean $f(V_i) = \{0\}$.

For standard concepts and notations in semigroup theory one can consult \cite{weid}}.

\section{Green’s relations}

In this section, we focus our attention on Green’s relations for the semigroup $L^\oplus(V)$. We begin with the relation $L$. Before stating the result, we need some notations.

Let $f \in L^\oplus(V)$ with $V_j \cap f(V) \neq 0$. Denote $W_j = \oplus\{V_i : 0 \neq f(V_i) \subseteq V_j\}$. Then it is easy to see that $f(W_j) = V_j \cap f(V)$. Suppose that all the subspaces $V_j$ such that $V_j \cap f(V) \neq 0$ are $V_{j_1}, V_{j_2}, \ldots, V_{j_t}$. Denote $K(f) = \{W_{j_1}, \ldots, W_{j_t}\}$. Denote by $\ker(f)$ the kernel of $f$, that is, $\ker(f) = \{x \in V : f(x) = 0\}$.

\textbf{Theorem 2.1} Let $f, g \in L^\oplus(V)$. Then $fLg$ if and only if $\ker(f) = \ker(g)$ and $K(f) = K(g)$. 


\textbf{Proof} Suppose }f \not\subseteq g. \text{ Then there exist } u, v \in L^\oplus(V), \text{ such that } uf = g \text{ and } vg = f. \text{ Hence }

\[ g(\ker(f)) = uf(\ker(f)) = u(0) = 0. \]

Thus, }\ker(f) \subseteq \ker(g). \text{ Similarly, } \ker(g) \subseteq \ker(f) \text{ and } \ker(f) = \ker(g). \text{ Suppose that }

\[ K(f) = \{W_{j_1}, \ldots, W_{j_s}\} \text{ and } K(g) = \{U_{l_1}, \ldots, U_{l_t}\}. \]

Without loss of generality, we may assume that }u(V_{j_1}) \subseteq V_{l_1}. \text{ So }

\[ g(W_{j_1}) = uf(W_{j_1}) \subseteq u(V_{j_1}) \subseteq V_{l_1}. \]

Clearly, }g(V_i) \neq 0 \text{ for each } V_i \subseteq W_{j_1}, \text{ since } \ker(f) = \ker(g). \text{ Thus } W_{j_1} \subseteq U_{l_1}. \text{ Assume } f(U_{l_1}) = vg(U_{l_1}) \subseteq v(V_{j_1}) \subseteq V_p \text{ for some } p. \text{ Notice that } f = vg = vuf, f(W_{j_1}) \subseteq V_{j_1} \text{ and }

\[ f(W_{j_1}) = vuf(W_{j_1}) \subseteq vuf(V_{j_1}) \subseteq v(V_{j_1}) \subseteq V_p, \]

we have }V_p = V_{j_1} \text{ and } f(U_{l_1}) \subseteq V_{j_1}. \text{ By } \ker(f) = \ker(g) \text{ again, } f(V_i) \neq 0 \text{ for each } V_i \subseteq U_{l_1}. \text{ Consequently, } U_{l_1} \subseteq W_{j_1} \text{ and } W_{j_1} = U_{l_1}. \text{ Similarly, one can verify that each } W \in K(f) \text{ is equal to some } U \in K(g) \text{ and } s = t. \text{ Therefore, } K(f) = K(g) \text{ and the necessity follows.}

In order to show the sufficiency, suppose }\ker(f) = \ker(g) \text{ and } K(f) = K(g). \text{ We must find some } u, v \in L^\oplus(V) \text{ satisfying } uf = g \text{ and } vg = f. \text{ Denote } f_i = f|V_i \text{ and } g_i = g|V_i \text{ (1 \leq i \leq m)}. \text{ Then } ker f_i = ker g_i. \text{ While for each } W \in K(f) = K(g), f|W \text{ and } g|W \text{ are linear mappings and }

\[ \ker(f|W) = \ker(g|W). \quad (2.1.1) \]

If }V_j \cap f(V) \neq 0, \text{ then there exists some } W \in K(f) = K(g) \text{ such that } f(W) = V_j \cap f(V), \text{ hence }

\[ g(W) = V_i \cap g(V). \]

Let }f(W) = V'_j \subseteq V_j \text{ and } g(W) = V'_i \subseteq V_i. \text{ From (2.1.1), } V'_j \text{ and } V'_i \text{ have the same dimension. Without loss of generality, we may assume } W = V_1 \oplus V_2 \oplus \cdots \oplus V_t. \text{ Take a basis } e_1, \ldots, e_r, e_{r_1}, \ldots, e_n \text{ for } V_1, \text{ a basis } \alpha_1, \ldots, \alpha_{r_2}, \alpha_{r_2 + 1}, \ldots, \alpha_{r_2 + r_1} \text{ for } V_2, \ldots, \alpha_n \text{ for } V_t, \text{ where } e_{r_1}, \ldots, e_n \text{ is a basis for } \ker(f_1), \text{ and } \alpha_{r_2 + 1}, \ldots, \alpha_{r_2 + r_1} \text{ is a basis for } \ker(f_2), \ldots, \alpha_n \text{ is a basis for } \ker(f_t). \text{ Then } \{e_i\} \cup \{\alpha_i\} \cup \cdots \cup \{\beta_i\} \text{ is a basis for } W. \text{ While in the subspace } V'_j \text{, } f(e_1), \ldots, f(e_{r_1}) \text{ are linearly independent, and so also are } f(\alpha_1), \ldots, f(\beta_{r_2}), \ldots, f(\beta_{r_1}). \text{ It is not difficult to see that }

\[ V'_j = \langle f(e_1), \ldots, f(e_{r_1}), f(\alpha_1), \ldots, f(\alpha_{r_2}), \ldots, f(\beta_{r_1}) \rangle. \]

Now we extend } f(e_1), \ldots, f(e_{r_1}) \text{ to obtain a basis for } V' \text{ by adding some } f(\alpha_s) \text{ (1 \leq s \leq r_2), \ldots, and } f(\beta_k) \text{ (1 \leq k \leq r_1). Without loss of generality, we assume the basis is }

\[ f(e_1), \ldots, f(e_{r_1}), f(\alpha_1), \ldots, f(\alpha_p), \ldots, f(\beta_1), \ldots, f(\beta_q). \quad (2.1.2) \]

We claim that

\[ g(e_1), \ldots, g(e_{r_1}), g(\alpha_1), \ldots, g(\alpha_p), \ldots, g(\beta_1), \ldots, g(\beta_q) \quad (2.1.3) \]

are linearly independent. Otherwise, suppose

\[ \sum_{s=1}^{r_1} a_sg(e_s) + \sum_{j=1}^{p} b_jg(\alpha_j) + \cdots + \sum_{k=1}^{q} c_kg(\beta_k) = 0 \]
for some \( a_i, b_j, c_k \in F \). Let

\[
\xi = a_1 e_1 + \cdots + a_r e_r + b_1 \alpha_1 + \cdots + b_p \alpha_p + \cdots + c_1 \beta_1 + \cdots + c_q \beta_q \in W.
\]

Then \( g(\xi) = 0 \) and \( \xi \in W \cap \ker(g) = W \cap \ker(f) \). Hence

\[
0 = f(\xi) = \sum_{i=1}^{r_1} a_i f(e_i) + \sum_{j=1}^{p} b_j f(\alpha_j) + \sum_{k=1}^{q} c_k f(\beta_k).
\]

Notice that (2.1.2) is linearly independent, the above equation implies that

\[
a_1 = \cdots = a_r = b_1 = \cdots = b_p = \cdots = c_1 = \cdots = c_q = 0.
\]

Thus, (2.1.3) are linearly independent, while being a basis for \( V' \).

Extend (2.1.2) to a basis \( B \) for \( V_j \) and define a linear mapping \( u_j : V_j \to V_i \) such that

\[
u_j(f(e_1)) = g(e_1), \ldots, u_j(f(e_r)) = g(e_r),
\]

\[
u_j(f(\alpha_1)) = g(\alpha_1), \ldots, u_j(f(\alpha_p)) = g(\alpha_p),
\]

\[
\ldots
\]

\[
u_j(f(\beta_1)) = g(\beta_1), \ldots, u_j(f(\beta_q)) = g(\beta_q),
\]

and for each \( \eta \in B \) out of (2.1.2), let \( u_j(\eta) = 0 \). For each \( V_i \), if \( V_i \cap f(V) \neq 0 \), then define \( u_i \) on \( V_i \) as above. If \( V_i \cap f(V) = 0 \), then let \( u_i(x) = 0 \) for each \( x \in V_i \). Thus, these \( u_i \) uniquely determine a linear transformation \( u \) on the linear space \( V \). Obviously, \( u \in L^\oplus(V) \).

Now we verify that \( uf = g \). For each \( V_i \) and \( x \in V_i \), if \( f(x) = 0 \), then \( g(x) = 0 \) since \( \ker(f) = \ker(g) \), and \( uf(x) = g(x) \) in this case. If \( f(x) \neq 0 \), then there exists some \( W \in K(f) \) such that \( V_i \subseteq W \). Without loss of generality, we assume

\[
W = V_1 \oplus V_2 \oplus \cdots \oplus V_i,
\]

then \( f(x) \in f(W) = V'_j \subseteq V_j \). As above, we assume (2.1.2) to be a basis for \( V'_j \). Then

\[
f(x) = \sum_{i=1}^{r_1} a_i f(e_i) + \sum_{j=1}^{p} b_j f(\alpha_j) + \sum_{k=1}^{q} c_k f(\beta_k) = f(\xi),
\]

where

\[
\xi = a_1 e_1 + \cdots + a_r e_r + b_1 \alpha_1 + \cdots + b_p \alpha_p + \cdots + c_1 \beta_1 + \cdots + c_q \beta_q.
\]

Since \( \ker(f) = \ker(g) \), we have \( g(x) = g(\xi) \). By the definition of \( u \),

\[
u f(x) = u(\sum_{i=1}^{r_1} a_i f(e_i) + \sum_{j=1}^{p} b_j f(\alpha_j) + \sum_{k=1}^{q} c_k f(\beta_k)) = g(\xi) = g(x).
\]

Thus, \( uf(x) = g(x) \) holds for every \( x \in V_i \). Consequently, \( uf(x) = g(x) \) holds for every \( x \in V \) and \( uf = g \). Similarly, one can find \( v \in L^\oplus(V) \) such that \( vg = f \). Therefore, \( f L g \) holds. \( \square \)

Before describing the relation \( R \) on \( L^\oplus(V) \) some notations should be introduced. Let \( f \in L^\oplus(V) \). If \( V_j \cap f(V) \neq 0 \), then there exists some \( V_i \) such that \( 0 \neq f(V_i) \subseteq V_j \). Denote

\[
P_j(f) = \{ f(V_i) : 0 \neq f(V_i) \subseteq V_j \}
\]
and define a partial order $\leq$ on $P_j(f)$ by letting $A \leq B$ if and only if $A \subseteq B$. Denote by $M_j(f)$ the collection of all maximal elements in $P_j(f)$. Then for each $i$ with $0 \neq f(V_i) \subseteq V_j$, there exists some $s$ such that $f(V_i) \subseteq f(V_j) \in M_j(f)$.

Now we can state and prove the conclusion for the relation $R$.

**Theorem 2.2** Let $f, g \in L^\oplus(V)$. Then the following statements are equivalent:

1. $fRg$.
2. For each $i$ (1 $\leq i \leq m$) there exist $j, k$ such that $f(V_i) \subseteq g(V_j)$ and $g(V_i) \subseteq f(V_k)$.
3. $f(V) = g(V)$ and $M_j(f) = M_j(g)$ holds for each $j$ with $V_j \cap f(V) \neq 0$.

**Proof**

(1)$\implies$(2) Suppose $fRg$. Then there exist $u, v \in L^\oplus(V)$ such that $fu = g$ and $gv = f$. For each $i$, there exists some $j$ such that $v(V_i) \subseteq V_j$. Consequently, $f(V_i) = ge(V_i) \subseteq g(V_j)$. Similarly, there exists some $k$ such that $g(V_k) \subseteq f(V_i)$ holds.

(2)$\implies$(3) It is not difficult to see from (2) that $f(V) \subseteq g(V)$ and $g(V) \subseteq f(V)$, so $f(V) = g(V)$. Suppose $V_j \cap f(V) \neq 0$ and $f(V_i) \in M_j(f)$. Then there exist $i_1, i_2$ such that $f(V_i) \subseteq g(V_{i_1}) \subseteq f(V_{i_2})$. From $f(V_i) \subseteq V_j \cap f(V_{i_2})$, we see that $f(V_{i_2}) \subseteq V_j$. Since $f(V_i) \in M_j(f)$ and $f(V_i) \subseteq f(V_{i_2})$, we have $f(V_{i_2}) = g(V_{i_1}) = f(V_i)$. Take $g(V_{i_1}) \in M_j(g)$ such that $g(V_{i_1}) \subseteq g(V_{i_3})$. By (2) again, there exists $i_4$ such that $g(V_{i_3}) \subseteq f(V_{i_4})$. Thus,

$$f(V_i) \subseteq g(V_{i_1}) \subseteq g(V_{i_3}) \subseteq f(V_{i_4}) \subseteq V_j$$

which implies that $f(V_{i_4}) = f(V_i) = g(V_{i_3}) \in M_j(g)$ and that $M_j(f) \subseteq M_j(g)$. By symmetry, we have $M_j(g) \subseteq M_j(f)$ and therefore $M_j(f) = M_j(g)$ holds.

(3)$\implies$(1) Suppose $f(V) = g(V)$ and $M_j(f) = M_j(g)$ holds for each $j$ with $V_j \cap f(V) \neq 0$. We first look for some $h \in L^\oplus(V)$ such that $fh = g$. For each $V_i$, if $g(V_i) = 0$, then define $h(x) = 0$ for each $x \in V_i$. If there is some $j$ such that $0 \neq g(V_i) \subseteq V_j$, then there is some $A \in M_j(g) = M_j(f)$ such that $g(V_i) \subseteq A$. Denote $g_i = g|V_i$ and assume $A = f(V_i) = g(V_i)$. Take a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_n$ for $V_i$ where $e_{r+1}, \ldots, e_n$ is a basis for $\ker(g_i)$. Then $g(e_1), g(e_2), \ldots, g(e_r)$ are linearly independent. Let $f_s = f|V_s : V_s \to V_j$. Choose $e'_1, e'_2, \ldots, e'_r \in V_s$ such that

$$f_s(e'_1) = g(e_1), f_s(e'_2) = g(e_2), \ldots, f_s(e'_r) = g(e_r).$$

Then $e'_1, e'_2, \ldots, e'_r$ are linearly independent. Define a linear mapping $h_i : V_i \to V_s$ such that

$$h_i(e_1) = e'_1, \ldots, h_i(e_r) = e'_r, \quad h_i(e_{r+1}) = 0, \ldots, h_i(e_n) = 0.$$  

Then for each vector $x = a_1 e_1 + \cdots + a_r e_r + a_{r+1} e_{r+1} + \cdots + a_n e_n \in V_i$, we have

$$fh_i(x) = f(a_1 h_i(e_1) + \cdots + a_r h_i(e_r)) = f(a_1 e'_1 + \cdots + a_r e'_r)$$

$$= a_1 f(e'_1) + \cdots + a_r f(e'_r) = a_1 g(e_1) + \cdots + a_r g(e_r) = g(x).$$

These $h_i$ defined on each $V_i$ determine a linear transformation $h$ on $V$. It is obvious that $h \in L^\oplus(V)$ and $fh = g$. By symmetry, there exists $k \in L^\oplus(V)$ such that $gk = f$ holds. Therefore, $fRg$. $\Box$
As an immediate consequence of Theorems 2.1 and 2.2, we have the following

**Theorem 2.3** Let \( f, g \in L^\oplus(V) \). Then the following statements are equivalent:

1. \((f, g) \in \mathcal{H}\).
2. \(\ker(f) = \ker(g)\), \(K(f) = K(g)\) and for each \(1 \leq i \leq m\), there exist \(j, k\) such that \(f(V_i) \subseteq g(V_j)\), \(g(V_i) \subseteq f(V_k)\).

Let \( f \in L^\oplus(V) \) and assume that all the subspaces \(V_i\) with \(f(V) \cap V_i \neq 0\) are \(V_1, V_2, \ldots, V_s\).

Then, one easily verifies that \(f(V) = V_{i_1} \oplus V_{i_2} \oplus \cdots \oplus V_{i_s}\).

The following concept will be useful in describing the relations \(D\) and \(J\) on \(L^\oplus(V)\).

**Definition 2.4** Let \(U\) and \(W\) be two subspaces of \(V\) where

\[U = V_{i_1}' \oplus V_{i_2}' \oplus \cdots \oplus V_{i_h}'\]
and each \(V_{i_i}'\) is a non-zero subspace of \(V_{i_i}\) while each \(V_{j_j}'\) is a non-zero subspace of \(V_{j_j}\). If \(\phi : U \to W\) is an isomorphism such that for each \(1 \leq s \leq k\) there exists a unique \(r (1 \leq r \leq k)\) such that \(\phi(V_{i_i}') = V_{j_j}'\), then \(\phi\) is called a sum-preserving isomorphism.

Next we consider the condition for two elements in \(L^\oplus(V)\) to be \(D\) equivalent.

**Theorem 2.5** Let \(f, g \in L^\oplus(V)\). Then \(fDg\) if and only if there exists a sum-preserving isomorphism \(\phi : f(V) \to g(V)\) such that for each \(i\) with \(f(V) \cap V_i \neq 0\), there exists some \(j\) such that \(\phi(f(V) \cap V_i) = g(V) \cap V_j\) and \(\phi(M_i(f)) = M_j(g)\).

**Proof** Suppose \(fDg\). Then there exists \(h \in L^\oplus(V)\) such that \(fLh\) and \(hRg\). From Theorems 2.1 and 2.2, we have \(\ker(f) = \ker(h)\), \(K(f) = K(h)\), \(h(V) = g(V)\) and \(M_j(h) = M_j(g)\) holds for each \(j\) with \(h(V) \cap V_j \neq 0\).

We first establish the isomorphism \(\phi\) from \(f(V)\) onto \(h(V)\). Suppose \(f(V) \cap V_j \neq 0\). Take \(W \in K(f) = K(h)\) such that \(f(W) = f(V) \cap V_j\). Then there is some \(j\) such that \(h(W) = h(V) \cap V_j\). Since \(\ker(f) = \ker(h)\), we have \(\ker(f|W) = \ker(h|W)\) and \(\dim f(W) = \dim h(W)\) which implies that \(f(W)\) and \(h(W)\) are isomorphic. Take a basis \(e_1, e_2, \ldots, e_r\) for \(f(W) = f(V) \cap V_i\) and choose \(w_1, w_2, \ldots, w_r \in W\) such that

\[f(w_1) = e_1, f(w_2) = e_2, \ldots, f(w_r) = e_r.\]

Then \(w_1, w_2, \ldots, w_r\) are linearly independent.

Let

\[e_1' = h(w_1), e_2' = h(w_2), \ldots, e_r' = h(w_r).\]

Then \(e_1', e_2', \ldots, e_r'\) are linearly independent while being a basis for \(h(W)\). Define a linear mapping
\begin{equation}
\phi_1: f(V) \cap V_i \rightarrow h(V) \cap V_j \text{ such that } \phi_1(e_1) = e'_i, \quad t = 1, 2, \ldots, r. \text{ Then } \phi_1 \text{ is an isomorphism and } \\
\phi_1 f(x) = h(x) \text{ for each } x \in W. \text{ Suppose } \\
M_i(f) = \{f(V_{i_1}), f(V_{i_2}), \ldots, f(V_{i_s})\}.
\end{equation}

By virtue of ker\(f) = \ker(h), \text{ one routinely verifies that } \\
M_j(h) = \{h(V_{i_1}), h(V_{i_2}), \ldots, h(V_{i_s})\}.

Besides, since \(V_{i_1}, V_{i_2}, \ldots, V_{i_s} \text{ are contained in } W \text{ and } \phi_i f = h \text{ on } W, \text{ we have } \\
\phi_i(f(V_{i_1})) = h(V_{i_1}), \ldots, \phi_i(f(V_{i_s})) = h(V_{i_s})

which implies that \(\phi_i(M_i(f)) = M_j(h). \text{ Notice that } h(V) = g(V) \text{ and } M_j(h) = M_j(g), \text{ it is evident that } \\
\phi_i : f(V) \cap V_i \rightarrow g(V) \cap V_j \text{ is an isomorphism satisfying } \phi_i(M_i(f)) = M_j(g).

Furthermore, we obtain the isomorphism \(\phi\) from \(f(V)\) onto \(g(V)\) determined by these \(\phi_i\) on \\
f(V) \cap V_i. \text{ Clearly, } \phi \text{ is a sum-preserving isomorphism as required.}

Conversely, suppose that there exists a sum-preserving isomorphism \(\phi : f(V) \rightarrow g(V)\) satisfying the condition of the theorem. \text{ Let } h = \phi f. \text{ Then } h \in L^{\oplus}(V), \text{ \(h(V) = g(V)\) and ker\(f) = \ker(h). \text{ Assume } W \in K(f) \text{ with } f(W) = f(V) \cap V_i \neq 0. \text{ Then there exists } j \text{ such that } \\
h(W) = \phi f(W) = \phi(f(V) \cap V_i) = g(V) \cap V_j = h(V) \cap V_j \subseteq V_j.

Notice that \(f(V_i) \neq 0\) for every \(V_i \subseteq W\) and that ker\(f) = \ker(h), \text{ it readily follows that } h(V_i) \neq 0 \text{ for every } V_i \subseteq W. \text{ Denote } W' = \oplus\{V_i : 0 \neq h(V_i) \subseteq V_j\}. \text{ Then } W' \in K(h) \text{ and } W \subseteq W'. \text{ Hence } \\
K(f) \text{ refines } K(h). \text{ Take } W^* \in K(h). \text{ Then there exists some } s, \text{ such that } \\
\phi f(W^*) = h(W^*) = h(V) \cap V_s = g(V) \cap V_s.

Since \(\phi\) is a sum-preserving isomorphism, there exists some \(t\) such that \\
\phi(f(W^*)) = g(V) \cap V_s = \phi(f(V) \cap V_i).

It follows that \(f(W^*) = f(V) \cap V_i\) and that \(W^*\) is contained in some \(W \in K(f)\). \text{ So } K(h) \text{ refines } \\
K(f) \text{ as well and } K(f) = K(h). \text{ Consequently, } f L h \text{ holds.}

Finally we verify that \(h \mathcal{R} g. \text{ As we have seen above that } h(V) = g(V). \text{ Now for each } V_i \text{ with } \\
g(V) \cap V_i \neq 0, \text{ there exists some } j \text{ such that } \phi(f(V) \cap V_j) = g(V) \cap V_i \text{ and } \phi(M_j(f)) = M_i(g). \text{ Then } \\
h(V) \cap V_i = \phi f(V) \cap V_i = g(V) \cap V_i = \phi(f(V) \cap V_j),

which together with ker\(f) = \ker(h) \text{ and } K(f) = K(h) \text{ implies that } M_i(h) = \phi(M_j(f)) = M_i(g) \text{ and } \\
h \mathcal{R} g. \text{ Consequently, } f \mathcal{D} g \text{ follows and the proof is completed.} \quad \Box

Now we consider the final Green relation \(J\) on the semigroup \(L^{\oplus}(V)\).

**Theorem 2.6** \text{ Let } f, g \in L^{\oplus}(V). \text{ Then } f J g \text{ if and only if there exist sum-preserving isomorphisms } \\
\phi : f(V) \rightarrow g(V) \text{ and } \psi : g(V) \rightarrow f(V),

\text{such that for each } i, \text{ there exist } p, q \text{ such that } f(V_i) \subseteq \psi(g(V_p)), g(V_i) \subseteq \phi(f(V_q)).
**Proof** Suppose \( f \mathcal{J} g \). Then there exist \( h, k, u, v \in L^\oplus(V) \) such that \( hfk = g \) and \( ugv = f \). Thus, \( uhfkv(V) = f(V) \). Since \( fkv(V) \) is a subspace of \( f(V) \) and

\[
\dim f(V) = \dim uhfkv(V) \leq \dim fkv(V) \leq \dim f(V),
\]

we have \( \dim fkv(V) = \dim f(V) \) and \( fkv(V) = f(k(V) = f(V)) \). Similarly, \( g(V) = gv(V) \). Consequently, from \( h(f(V)) = h(k(V)) = g(V) \) we see that \( \dim g(V) \leq \dim f(V) \). By symmetry, \( \dim f(V) \leq \dim g(V) \). Thus, \( \dim f(V) = \dim g(V) \) and \( f(V) \) is isomorphic to \( g(V) \). Let \( \phi = h|f(V) \) and \( \psi = u|g(V) \). Then \( \phi : f(V) \to g(V) \) and \( \psi : g(V) \to f(V) \) are isomorphisms. Next we verify that both \( \phi \) and \( \psi \) are sum-preserving. Suppose

\[
f(V) = V'_1 \oplus V'_2 \oplus \cdots \oplus V'_{t}, \quad \text{and} \quad g(V) = V'_1 \oplus V'_2 \oplus \cdots \oplus V'_{s},
\]

where \( V'_p = f(V) \cap V_{ip}, \ 1 \leq p \leq t \) and \( V'_q = g(V) \cap V_jq, 1 \leq q \leq s \). Since \( h \) is sum-preserving, for each \( p \) there exists a unique \( q \) such that \( \phi(V'_{ip}) \subseteq V'_{jq} \). Notice that \( \phi \) is surjective, it must be the case that \( t \geq s \). By symmetry, \( s \geq t \) and \( t = s \). Thus, \( \phi(V'_{ip}) = V'_{jq} \) and \( \phi \) maps different \( V'_{ip} \) into different \( V'_{jq} \) isomorphically. Hence \( \phi \) is a sum-preserving isomorphism. Similarly, \( \psi \) is sum-preserving isomorphism as well.

Now for each \( i \), there exists some \( p \) such that \( v(V_i) \subseteq V_p \). Then \( f(V_i) = ugv(V_i) \subseteq ugv(V_p) = \psi(g(V_p)) \). By symmetry, there exists \( q \) such that \( g(V_i) \subseteq \phi(f(V_q)) \), and the necessity follows.

Conversely, suppose the condition holds and we need to show that \( f \mathcal{J} g \). We first look for some \( h, k \in L^\oplus(V) \) such that \( hfk = g \). For each \( i \), if \( g(V_i) = 0 \), then define \( k(x) = 0 \) for every \( x \in V_i \). If \( g(V_i) \neq 0 \), choose a basis \( e_1, \ldots, e_r, e_{r+1}, \ldots, e_n \) for \( V_i \) such that \( g(e_{r+1}) = 0, \ldots, g(e_n) = 0 \) and \( g(e_1), \ldots, g(e_r) \) are linearly independent. By hypothesis, there exists \( V_q \) such that \( g(V_i) \subseteq \phi(f(V_q)) \). Take linearly independent vectors \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r \) in \( V_q \) such that

\[
g(e_1) = \phi f(e_1), g(e_2) = \phi f(e_2), \ldots, g(e_r) = \phi f(e_r).
\]

Define a linear mapping \( k \) from \( V_i \) into \( V_q \) such that

\[
k(e_1) = \varepsilon_1, k(e_2) = \varepsilon_2, \ldots, k(e_r) = \varepsilon_r, k(e_{r+1}) = 0, \ldots, k(e_n) = 0.
\]

One easily verifies that \( g(x) = \phi f k(x) \) holds for each \( x \in V_i \). Thus, these \( k \) defined on each \( V_i \) determine uniquely a linear transformation \( k \) of \( V \). Clearly, \( k \in L^\oplus(V) \) and \( g(x) = \phi f k(x) \) for each \( x \in V \).

Now we define the linear transformation \( h \). For each \( V_j \) with \( V_j \cap f(V) = 0 \), define \( h(x) = 0 \) for every \( x \in V_j \). For those \( V_j \) with \( f(V) \cap V_j \neq 0 \), since \( \phi \) is sum-preserving, there exists some \( l \) such that \( \phi(f(V) \cap V_j) = g(V) \cap V_l \). Take a basis \( e_1, \ldots, e_r \) for \( f(V) \cap V_j \) and extend this to a basis

\[
e_1, \ldots, e_r, e_{r+1}, \ldots, e_n
\]

for \( V_j \). Define a linear mapping \( h \) from \( V_j \) into \( V_l \) such that

\[
h(e_1) = \phi(e_1), \ldots, h(e_r) = \phi(e_r), h(e_{r+1}) = 0, \ldots, h(e_n) = 0.
\]

Then one routinely verifies that \( h|(f(V) \cap V_j) = \phi|(f(V) \cap V_j) \). Consequently, there exists a unique linear transformation \( h \) on \( V \) determined by these linear mappings \( h \) defined on each
V. Clearly, $h \in L^{\oplus}(V)$, \(h f(V) = \phi\) and $g(x) = \phi f k(x) = h f k(x)$ holds for arbitrary $x \in V$. Consequently, $g = h f k$. By symmetry, there exist $u, v \in L^{\oplus}(V)$ such that $u g v = f$ and it follows that $f J g$.

It is well-known that $D \subseteq J$ for every semigroup. In what follows, we will soon see that $D = J$ for the semigroups $L^{\oplus}(V)$.

Suppose $f, g \in L^{\oplus}(V)$ and $f J g$. Assume that

$$f(V) = V_{i_1}' \oplus V_{i_2}' \oplus \cdots \oplus V_{i_s}' , \quad g(V) = V_{j_1}' \oplus V_{j_2}' \oplus \cdots \oplus V_{j_r}'$$

and $\phi : f(V) \rightarrow g(V), \psi : g(V) \rightarrow f(V)$ are both sum-preserving isomorphisms satisfying the condition in Theorem 2.6. Then we have the following two lemmas.

**Lemma 2.7** There exists a positive integer $r$ such that $(\psi \phi)^r : f(V) \rightarrow f(V)$ is a sum-preserving isomorphism such that

$$(\psi \phi)^r(V_{i_k}') = V_{i_k}' \quad \text{and} \quad (\psi \phi)^r(M_{i_k}(f)) = M_{i_k}(f)$$

holds for each $k$ ($1 \leq k \leq s$).

**Proof** It is clear that $\psi \phi : f(V) \rightarrow f(V)$ is a sum-preserving isomorphism and for each $i_k$, there exists a unique $i_k'$ such that

$$\psi \phi(V_{i_k}') = V_{i_k}' , \quad k = 1, 2, \ldots, s.$$ 

Thus, $\psi \phi$ induces a permutation $\rho$ of the set \(\{i_1, i_2, \ldots, i_s\}\) where

$$\rho = \begin{pmatrix} i_1 & i_2 & \cdots & i_s \\ i_1' & i_2' & \cdots & i_s' \end{pmatrix}.$$

By the property of permutations, there exists a positive integer $r$ such that $\rho^r$ is the identity permutation of the set \(\{i_1, i_2, \ldots, i_s\}\). Let $\xi = (\psi \phi)^r$. Then $\xi : f(V) \rightarrow f(V)$ is a sum-preserving isomorphism satisfying $\xi(V_{i_k}') = V_{i_k}' , \quad k = 1, 2, \ldots, s$.

In order to show the remainder, we assume $M_{i_1}(f) = M_1 \cup M_2 \cup \cdots \cup M_u$, where $M_r (1 \leq r \leq u)$ is the collection of those $A$ in $M_{i_1}(f)$ with $\dim A = m_r$, and $m_1 > m_2 > \cdots > m_u \geq 1$. By Theorem 2.6, for each $A \in M_{i_1}(f)$ there is some $p$ such that $A \subseteq \psi(g(V_p))$. While there is some $q$ such that $g(V_p) \subseteq \phi(f(V_q))$. Hence $A \subseteq \psi(f(V_q))$. Repeating the discussion, there exists some $p(A)$ ($1 \leq p(A) \leq m$) such that

$$A \subseteq (\psi \phi)^r(f(V_{p(A)})) = \xi(f(V_{p(A)})).$$

Since $\xi$ is sum-preserving and $\xi(V_{i_1}') = V_{i_1}'$, one routinely verifies that $f(V_{p(A)}) \subseteq V_{i_1}'$.

We first verify

$$\{f(V_{p(A)}) : A \in M_1\} = M_1.$$  \hspace{1cm} (2.7.2)

Suppose $A \in M_1$. Then $\dim f(V_{p(A)}) \leq m_1$ since $m_1$ is the maximal dimension of the elements in $M_{i_1}(f)$. Now by (2.7.1), we have

$$\dim f(V_{p(A)}) \geq \dim A = m_1.$$
Therefore, \( \dim f(V_{p(A)}) = m_1 \) and \( f(V_{p(A)}) \in M_1 \). Thus, \( \{f(V_{p(A)}) : A \in M_1\} \subseteq M_1 \). From (2.7.1) it follows that \( A = \xi(f(V_{p(A)})) \) for each \( A \in M_1 \). Notice that \( \xi \) is a sum-preserving isomorphism and that \( M_1 \) is a finite set, it is clear that (2.7.2) holds. Consequently, we have \( \xi(M_1) = M_1 \).

Next we verify that
\[
\{f(V_{p(B)}) : B \in M_2\} = M_2. \tag{2.7.3}
\]
Suppose \( B \in M_2 \). By (2.7.1) again, we have \( \dim f(V_{p(B)}) \geq \dim B = m_2 \). If \( \dim f(V_{p(B)}) > m_2 \), then there exists \( A \in M_1 \) such that \( f(V_{p(B)}) \subseteq A \). Consequently,
\[
B \subseteq \xi(f(V_{p(B)})) \subseteq \xi(A) \in M_1,
\]
which contradicts the hypothesis that \( B \) is a maximal element in \( P_1(f) \). Hence \( \dim f(V_{p(B)}) = m_2 \) and \( B = \xi(f(V_{p(B)})) \). While \( f(V_{p(B)}) \) cannot be contained in any element of \( M_1 \). Consequently, \( f(V_{p(B)}) \subseteq M_2 \) for each \( B \in M_2 \) and (2.7.3) follows. While we also have \( \xi(M_2) = M_2 \). Go on in this way, we can finally get
\[
\{f(V_{p(A)}) : A \in M_1\} = M_1 \text{ and } \xi(M_i) = M_i, \quad i = 1, 2, \ldots, u.
\]
Furthermore, \( M_i(f) = \xi(M_i(f)) \) holds. One similarly verifies that \( M_i_k(f) = \xi(M_i_k(f)) \) holds for \( k = 2, \ldots, s \). The proof is completed. \( \square \)

**Lemma 2.8** Let \( \theta = \phi(\psi\phi)^{-1} \). Then \( \theta : f(V) \to g(V) \) is a sum-preserving isomorphism. Moreover, if \( \theta(V_{i_k}) = V'_{j_k} \), then \( M_{j_k}(g) = \theta(M_{i_k}(f)) \).

**Proof** \( \theta \) is clearly a sum-preserving isomorphism and \( \xi = \psi \theta \). Denote
\[
M_{i_k}(f) = M_1 \cup M_2 \cup \cdots \cup M_u \text{ and } M_{j_k}(g) = N_1 \cup N_2 \cup \cdots \cup N_v,
\]
where \( \dim B = m_r \) for each \( B \in M_r \) (\( 1 \leq r \leq u \)) and \( \dim A = n_t \) for each \( A \in N_t \) (\( 1 \leq t \leq v \)) with \( m_1 > m_2 > \cdots > m_u \geq 1 \) and \( n_1 > n_2 > \cdots > n_v \geq 1 \). Suppose \( \theta(V_{i_k}) = V'_{j_k} \), then
\[
\psi(V'_{j_k}) = \psi(\theta(V'_{i_k})) = \xi(V'_{i_k}) = V'_{i_k}.
\]
For each \( A \in M_{j_k}(g) \) there exists some \( p \) such that \( f(V_p) \subseteq V'_{i_k} \) and \( A \subseteq \theta(f(V_p)) \). Moreover, there exists some \( B \in M_{i_k}(f) \) with \( f(V_p) \subseteq B \). Consequently,
\[
A \subseteq \theta(f(V_p)) \subseteq \theta(B). \tag{2.8.1}
\]
By Theorem 2.6, for this \( B \) there exists some \( q \) such that \( B \subseteq \psi(g(V_q)) \) and it is clear that \( g(V_q) \subseteq V'_{j_k} \). Thus there is \( A' \in M_{j_k}(g) \) such that \( g(V_q) \subseteq A' \). Hence we have
\[
B \subseteq \psi(g(V_q)) \subseteq \psi(A'). \tag{2.8.2}
\]
Suppose \( A \in N_1 \). Then \( \dim A = n_1 \). By (2.8.1) and (2.8.2), we have
\[
n_1 = \dim A \leq \dim B \leq \dim A' \leq n_1
\]
and \( \dim B = n_1 = \dim A' \). Notice that \( B \in M_{i_k}(f) \), so \( \dim B \leq m_1 \) and \( n_1 \leq m_1 \). Conversely, suppose \( B \in M_{i_k}(f) \) and \( \dim B = m_1 \). From the discussion above, there exist \( q \) and some
A' ∈ M_{jk}(g) such that B ⊆ ψ(g(V'_i)) ⊆ ψ(A'). Hence
\[ m_1 = \dim B ≤ \dim A' ≤ n_1 \]
and \( m_1 = n_1 \). Thus, (2.8.1) implies that \( A = \theta(B) \) and that every element \( A ∈ N_1 \) is an image of some \( B ∈ M_1 \) under the isomorphism \( \theta \). Consequently, \( |M_1| ≥ |N_1| \). Similarly, from (2.8.2), for each \( B ∈ M_1 \) there exists \( A' ∈ N_1 \) such that \( B = \psi(A') \), so \( |M_1| ≤ |N_1| \). Therefore, \( |M_1| = |N_1| \) and \( \theta(M_1) = N_1 \).

Now suppose \( A ∈ N_2 \). By (2.8.1) again, there exists \( B ∈ M_{ik}(f) \) such that \( A ⊆ θ(B) \). If \( B ∈ M_1 \), then there is some \( A' ∈ N_1 \) such that \( A ⊆ θ(B) = A' \) which contradicts the fact that \( A \) is maximal. Thus, it must be the case that \( B /∈ M_1 \) and \( \dim(B) < m_1 \). While from (2.8.2) we see that there exists some \( A' ∈ M_{jk}(g) \) such that \( B ⊆ ψ(A') \). If \( \dim A' = n_1 (= m_1) \), since \( θ(M_1) = N_1 \), then there exists some \( B' ∈ M_1 \) such that \( A' = θ(B') \). Therefore there exists some \( B'' ∈ M_1 \) such that \( B ⊆ ψ(A') ⊆ ψ(θ(B')) = B'' \) holds, contradicting the fact that \( B \) is maximal. So \( \dim A' < n_1 (= m_1) \) and
\[ n_2 = \dim A ≤ \dim B ≤ \dim A' ≤ n_2. \]

Consequently, \( \dim B = n_2 \), \( A = θ(B) \) and \( n_2 = m_2 \). Similarly, we have \( |N_2| = |M_2| \) and \( θ(M_2) = N_2 \). Repeating the discussion above, we finally obtain that
\[ u = v, \ |N_i| = |M_i|, \ θ(M_i) = N_i, \ n_i = m_i, \ i = 1, 2, \ldots, u. \]
Consequently, \( M_{jk}(g) = θ(M_{ik}(f)) \) holds. The proof is completed. \( \square \)

By Lemma 2.8 and Theorem 2.5, we can prove the following

**Theorem 2.9** In the semigroup \( L^0(V), D = J \).

**Proof** We only need to show that \( J ⊆ D \). Suppose \((f, g) ∈ J \). From Theorem 2.6, there exist sum-preserving isomorphisms \( φ : f(V) → g(V) \) and \( ψ : g(V) → f(V) \) satisfying the condition in Theorem 2.6. Let \( ξ = (ψφ)^{-1} \). By Lemma 2.7, \( ξ : f(V) → f(V) \) is a sum-preserving isomorphism satisfying that \( ξ(V'_{ik}) = V'_{ik}, ξ(M_{ik}(f)) = M_{ik}(f) \) (1 ≤ k ≤ s). Denote \( θ = φ(ψφ)^{-1} \). By Lemma 2.8, \( θ : f(V) → g(V) \) is a sum-preserving isomorphism and if \( θ(V'_{ik}) = V'_{ik}, \) then \( M_{jk}(g) = θ(M_{ik}(f)) \). Thus \( θ \) satisfies the condition of Theorem 2.5, hence \( (f, g) ∈ D \) and \( J = D \) holds. \( \square \)

3. Regular elements in \( L^0(V) \)

In this section we consider the condition under which an element in \( L^0(V) \) is regular and when the semigroup \( L^0(V) \) is a regular semigroup. And then we investigate the Green’s relations for regular elements in the semigroup \( L^0(V) \).

For \( f ∈ L^0(V) \), denote \( \text{Fix}(f) = \{ x ∈ V : f(x) = x \} \). The following result is routinely verified and the proof is omitted.

**Lemma 3.1** Let \( f ∈ L^0(V) \). Then \( f \) is idempotent if and only if \( f(V) = \text{Fix}(f) \).

**Lemma 3.2** Suppose \( f ∈ L^0(V) \) is an idempotent. Then for each \( W ∈ K(f) \) there exits some
Let \( V_i \subseteq W \) such that \( f(V_i) = f(W) = V_i \cap f(V) \).

**Proof** Suppose \( f(W) = V_i \cap f(V) \). Then for each \( x \in V_i \cap f(V) \), by Lemma 3.1, \( x = f(x) \in f(V_i) \) which implies that \( V_i \cap f(V) \subseteq f(V_i) \). Hence \( 0 \neq f(V_i) \subseteq V_j \) for some \( j \). Notice that \( V_i \cap f(V) \subseteq f(V_i) \) and \( V_i \cap f(V) = f(V_i \cap f(V)) \subseteq V_j \), so \( V_i = V_j \). Consequently, \( f(V_i) \subseteq V_i \) and \( f(V_j) = V_i \cap f(V) \). While \( V_i \subseteq W \) follows from the definition of \( K(f) \). \( \square \)

**Theorem 3.3** Let \( f \in L^\oplus(V) \). Then \( f \) is regular if and only if for each \( i \) with \( V_i \cap f(V) \neq 0 \) there exists some \( j \) such that \( f(V_j) = V_i \cap f(V) \).

**Proof** If \( f \) is regular, then there exists an idempotent \( g \) in \( L^\oplus(V) \) such that \( fLg \). By Theorem 2.1 we have \( \ker(f) = \ker(g) \) and \( K(f) = K(g) \). Take a subspace \( V_i \) such that \( V_i \cap f(V) \neq 0 \). Then there exists \( W \in K(f) = K(g) \) such that \( f(W) = V_i \cap f(V) \). By Lemma 3.2, we can choose \( V_j \subseteq W \) such that \( g(V_j) = g(W) = V_j \cap g(V) \). Now \( \ker(f) = \ker(g) \) and \( g(V_j) = g(W) \) implies that \( f(V_j) = f(W) = V_i \cap f(V) \) and the necessity holds.

Now suppose that \( f \) satisfies the condition and we shall find some idempotent \( g \) such that \( fLg \) which of course implies that \( f \) is regular. We first define \( g \) on each \( W \in K(f) \). By hypothesis, there exist \( i \) and \( j \) such that \( V_i \subseteq W \) and \( V_j \subseteq W \). Take a basis \( \{e_u\} \) for \( V_i \) and \( \{e'_u\} \) for \( V_j \) such that \( f(e'_u) = e_u \) for each \( u \). Then \( \{e'_u\} \) is linearly independent. Extend this to a basis \( \{e_u\} \cup \{d_v\} \) for \( W \). Then \( f(d_v) = 0 \) for each \( v \). Now define a linear mapping \( g : W \to V_j \) such that \( g(e'_u) = e'_u \) for each \( u \) and \( g(d_v) = 0 \) for each \( v \). For those \( V_i \) (if exists) with \( f(V_i) = 0 \), define \( g(x) = 0 \) for each \( x \in V_i \). Thus, we have defined the linear transformation \( g \). It is obvious that \( g \in L^\oplus(V) \) and \( g^2 = g \). By definition of \( g \) it readily follows that \( K(f) = K(g) \) and \( \ker(f) = \ker(g) \). Consequently, \( fLg \) and \( f \) is regular in \( L^\oplus(V) \). \( \square \)

The following example tells us that the semigroup \( L^\oplus(V) \) is not, in general, a regular semigroup.

**Example** Let \( V = V_1 \oplus V_2 \oplus V_3 \) where \( V_1 \) has a basis \( e_1, e_2, \ldots, e_n \) \((n \geq 3)\), \( V_2 \) has a basis \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( V_3 \) has a basis \( \beta_1, \beta_2, \ldots, \beta_n \). Define a linear transformation \( f : V \to V \) such that

\[
f(e_1) = f(\beta_1) = \alpha_1, f(\alpha_1) = f(e_1) = \alpha_2, f(\alpha_i) = f(\beta_i) = \alpha_3 \quad (\text{for } i \neq 1).
\]

Then \( f \in L^\oplus(V) \) and \( V_2 \cap f(V) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \). However, \( f(V_1) = \langle \alpha_1, \alpha_2 \rangle, f(V_2) = \langle \alpha_2, \alpha_3 \rangle \) and \( f(V_3) = \langle \alpha_1, \alpha_3 \rangle \). It is clear that there is no \( j \) \((1 \leq j \leq 3)\) satisfying \( V_2 \cap f(V) = f(V_j) \). By Theorem 3.3, \( f \) is not regular in the semigroup \( f \in L^\oplus(V) \).

Next we investigate when the semigroup \( L^\oplus(V) \) is a regular semigroup.

**Theorem 3.4** The semigroup \( L^\oplus(V) \) is regular if and only if \( m = 1 \) or \( \dim V_i = 1 \) for each \( i \).

**Proof** If \( m = 1 \), then \( V = V_1 \) is an \( n \) dimensional space. Thus, \( L^\oplus(V) = L(V) \) is a regular semigroup. If \( \dim V_i = 1 \) for each \( i \), then \( V \) is a direct sum of \( m \) one dimensional spaces. Let \( f \in L^\oplus(V) \). If \( V_i \cap f(V) \neq 0 \), then we have \( V_i \cap f(V) = V_i \) since the subspace \( V_i \cap f(V) \) must be one dimensional. Notice that there must be some \( j \) such that \( 0 \neq f(V_j) = V_i \), otherwise, we would
conclude that \( V_i \cap f(V) = 0 \), a contradiction. Consequently, we have \( f(V_j) = V_i \cap f(V) \neq 0 \). By Theorem 3.3, \( f \) is regular and \( L^{\oplus}(V) \) is a regular semigroup.

Conversely, suppose that \( m > 1 \) and \( n \geq 2 \). Take a basis \( e_1,e_2,\ldots,e_n \) for \( V_1 \), a basis \( g_1,g_2,\ldots,g_n \) for \( V_2 \). Define \( f : V \to V \) such that \( f(e_k) = e_1, f(g_k) = e_2 \) for each \( k \) and \( f(x) = 0 \) for any \( x \in V_s \) (\( s \neq 1,2 \)). Clearly, \( f \in L^{\oplus}(V) \) and \( f(V) = \langle e_1,e_2 \rangle \subseteq V_1 \). Thus, \( V_1 \cap f(V) = \langle e_1,e_2 \rangle \). However, there is no \( j \) satisfying \( V_1 \cap f(V) = f(V_j) \) which implies that \( f \) is not a regular element. Consequently, \( L^{\oplus}(V) \) is not a regular semigroup. \( \square \)

Finally, we describe Green’s equivalences for regular elements in the semigroups \( L^{\oplus}(V) \). We first make some observations.

**Theorem 3.5** Let \( f,g \in L^{\oplus}(V) \) be regular. If \( \ker(f) = \ker(g) \), then \( K(f) = K(g) \).

**Proof** Suppose
\[
W = \oplus \{ V_i : 0 \neq f(V_i) \subseteq V_j \} \in K(f).
\]
Then \( f(W) = V_j \cap f(V) \). Since \( f \) is regular, by Theorem 3.3, there exists some \( l \) such that \( f(W) = V_j \cap f(V) = f(V_l) \). Suppose \( 0 \neq g(V_i) \subseteq V_k \) for some \( k \). Denote
\[
U = \oplus \{ V_s : 0 \neq g(V_s) \subseteq V_k \}.
\]
By Theorem 3.3 again, there exists some \( u \) such that \( g(U) = V_k \cap g(V) = g(V_u) \). We claim that \( W = U \). Actually, from \( \ker(f) = \ker(g) \) one routinely verifies that, for each \( V_i \subseteq W \), \( f(V_i) \subseteq f(V) \) implies \( 0 \neq g(V_i) \subseteq V_k \). Thus, \( V_i \subseteq U \) and \( W \subseteq U \) holds.

On the other hand, since \( g(V_u) = V_k \cap g(V) \) and \( g(V_i) \subseteq V_k \), we have \( g(V_i) \subseteq g(V_u) \) which together with \( \ker(f) = \ker(g) \) implies that \( f(V_i) \subseteq f(V_u) \). Therefore,
\[
f(V_i) = V_j \cap f(V) = f(V_u).
\]
By \( \ker(f) = \ker(g) \) again, we have \( g(V_i) = g(V_u) \). Now for each \( V_s \subseteq U \), we have \( 0 \neq g(V_s) \subseteq g(V_i) \). Hence \( 0 \neq f(V_s) \subseteq f(V_i) \). Thus, \( V_s \subseteq W \) and \( U \subseteq W \) holds. Consequently, \( U = W \) and \( K(f) \subseteq K(g) \). By symmetry, \( K(g) \subseteq K(f) \), so \( K(f) = K(g) \). \( \square \)

**Theorem 3.6** Let \( f,g \in L^{\oplus}(V) \) be regular elements. If \( f(V) = g(V) \), then, for each \( i \), there exist \( j,k \) such that \( f(V_i) \subseteq g(V_j) \), \( g(V_i) \subseteq f(V_k) \).

**Proof** If \( f(V_i) = 0 \), then \( f(V_i) \subseteq g(V_j) \) holds for arbitrary \( j \). If \( 0 \neq f(V_i) \subseteq V_i \), then
\[
V_i \cap g(V) = V_i \cap f(V) \neq 0.
\]
Since \( g \) is regular, there exists \( j \) such that \( V_i \cap g(V) = g(V_j) \). Consequently,
\[
f(V_i) \subseteq V_i \cap f(V) = V_i \cap g(V) = g(V_j).
\]
By symmetry, for each \( i \), there exists \( k \) such that \( g(V_i) \subseteq f(V_k) \). \( \square \)

As an immediate consequence of Theorems 2.1, 2.2 and 3.3, we have the following result.

**Theorem 3.7** Let \( f,g \in L^{\oplus}(V) \) be regular elements. Then

1. \( f \not\leq g \) if and only if \( \ker(f) = \ker(g) \).
Finally, we observe the relation $D$ for regular elements.

**Theorem 3.8** Let $f, g \in L_{\oplus}(V)$ be regular elements. Then $fDg$ if and only if there exists a sum-preserving isomorphism from $f(V)$ onto $g(V)$.

**Proof** Suppose $fDg$. Then there exists some $h \in L_{\oplus}(V)$ such that $fLh$ and $hRG$. By Theorem 2.1, $\ker(f) = \ker(h)$ and $K(f) = K(h)$. While by Theorem 2.2, $h(V) = g(V)$. Denote $K(f) = \{W_1, \ldots, W_t\} = K(h)$. Denote $V_i' = f(W_i) = V_i \cap f(V)$ and $V_j' = h(W_j) = V_j \cap h(V)$, $1 \leq r \leq t$. Then

$$f(V) = V_1' \oplus V_2' \oplus \cdots \oplus V_t',$$

$$h(V) = V_1' \oplus V_2' \oplus \cdots \oplus V_t' = g(V).$$

By the proof of Theorem 2.5, there exists a sum-preserving isomorphism from $f(V)$ onto $h(V) = g(V)$.

Conversely, if there exists a sum-preserving isomorphism $\phi$ from $f(V)$ onto $g(V)$, define $h : V \to V$ by $h = \phi f$. Then it is clear that $h \in L_{\oplus}(V)$, $\ker(f) = \ker(h)$ and $K(f) = K(h)$. By Theorem 2.1, $fLh$. Hence $h$ is also regular. While from the definition of $h$ one easily verifies that $h(V) = g(V)$ and $hRG$ follows from Theorem 3.7. Consequently, $fDg$ holds. $\square$

**References**


