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Optimal Conditions on Generalized Solutions of Coupled Nonlinear Diffusion Systems

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Abstract This paper studies coupled nonlinear diffusion equations with more general nonlinearities, subject to homogeneous Neumann boundary conditions. The necessary and sufficient conditions are obtained for the existence of generalized solutions of the system, which extend the known results for nonlinear diffusion systems with more special nonlinearities.

Keywords nonlinear diffusion equations; generalized solution; classical solution.

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1. Introduction

In this paper, we consider the following coupled nonlinear diffusion system

$$\begin{cases} u_t = \Delta\varphi(u) + f(v)g_1(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta\psi(v) + f_1(v)g(u), & x \in \Omega,\ t > 0, \\ \dfrac{\partial\varphi(u)}{\partial\eta} = \dfrac{\partial\psi(v)}{\partial\eta} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$; η is the exterior normal vector on $\partial\Omega$; nonnegative and nondecreasing functions $f_1,\ g_1,\ f,\ g,\ \varphi,\ \psi \in C([0,\infty))$ satisfy $f_1g_1 > 0$ for $s \geq 0$ and $f(s)g(s)\varphi(s)\psi(s) > 0$ for $s > 0$ with $\varphi(0) = \psi(0) = 0$; initial data $u_0, v_0 \in C(\bar{\Omega})$ are positive and satisfy the compatibility conditions on the boundary. It is observed that the nonlinear diffusion system (1.1) possesses more general nonlinearities, and covers many models from population dynamics, chemical reactions, heat transfer, etc., where u and v represent the densities of two biological populations during a migration, the thickness of two kinds of chemical reactants, the temperatures of different materials during a propagation, etc.

Lair[6] considered (1.1) with $f_1 \equiv g_1 \equiv 1$, and obtained that classical solutions exist if and

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only if

$$\int^{\infty} \frac{1}{g(K^{-1}(J(s)))}\mathrm{d}s = \infty, \quad \text{or} \quad \int^{\infty} \frac{1}{f(J^{-1}(K(s)))}\mathrm{d}s = \infty, \tag{1.2}$$

where J^{-1}, K^{-1} are the inverse functions of $J(s) = \int_0^s f(\xi)\mathrm{d}\xi$, $K(s) = \int_0^s g(\xi)\mathrm{d}\xi$, respectively. Moreover, criteria (1.2) also can be used to the existence of generalized solutions. The scalar problem of (1.1) was discussed in [7].

There are some results on general scalar parabolic equations. Zhang[10] studied the parabolic equation $\beta_t(u) = Lu + f(u)$ with boundary condition $\frac{\partial u}{\partial \eta} + \sigma(x,t)b(u) = 0$ on $\partial\Omega$, where $Lu = a^{ij}(x)u_{ij} + b^i u_i$ is uniformly elliptic; positive functions $\beta \in C^3([0,+\infty))$, $f \in C^2([0,+\infty))$, and $b \in C^1([0,+\infty))$; nonnegative $\sigma \in C^1(\bar{\Omega}\times[0,T))$. The criteria on global and non-global solutions were obtained with some estimates in time.

For the system with localized terms, Chadam, Peirce, and Yin[3] considered the parabolic equation $u_t = \Delta u + f(u(x_0,t))$ with homogeneous Neumann boundary condition. They determined the conditions for non-global solutions with blow-up set and growth estimates in time. As for critical blow-up exponents and blow-up rates to semilinear parabolic equations with homogeneous Dirichlet boundary conditions, one can refer to, e.g.,[8,11]. Much more interesting results on critical blow-up exponents for various evolution equations can be found in [4] and the literature cited therein.

In this paper, we will establish the optimal existence criteria for the generalized solutions of the coupled nonlinear diffusion system (1.1) with more general nonlinearities to extend the main results in [6,7]. We will state the main results of the paper in Section 2, and then prove them in Section 3.

2. Main Results

Let (u_n, v_n) be a classical solution of the following nonlinear diffusion system

$$\begin{cases} u_{n,t} = \Delta\varphi_n(u_n) + f_n(v_n)g_{1,n}(u_n), & x \in \Omega,\ t > 0, \\ v_{n,t} = \Delta\psi_n(v_n) + f_{1,n}(v_n)g_n(u_n), & x \in \Omega,\ t > 0, \\ \dfrac{\partial\varphi_n(u_n)}{\partial\eta} = \dfrac{\partial\psi_n(v_n)}{\partial\eta} = 0, & x \in \partial\Omega,\ t > 0, \\ u_n(x,0) = u_{0,n}(x),\ v_n(x,0) = v_{0,n}(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where $f_n, g_n, \varphi_n, \psi_n, f_{1,n}, g_{1,n} \in C^\infty([0,+\infty))$ are nonnegative and increasing functions with $f_{1,n}g_{1,n} > 0$ for $s \geq 0$ and $f_n(s)g_n(s)\varphi_n(s)\psi_n(s) > 0$ for $s > 0$, $\varphi_n(0) = \psi_n(0) = 0$; $u_{0,n}, v_{0,n} \in C^\infty(\bar{\Omega})$ are nonnegative. The solution sequence $\{(u_n, v_n)\}$ of (2.1) is a sequence of approximating solutions to (1.1) if, for any compact subset $S \subset [0,+\infty)$,

$$\lim_{n\to\infty}(\|f_n - f\|_{\infty,S} + \|g_n - g\|_{\infty,S} + \|f_{1,n} - f\|_{\infty,S} + \|g_{1,n} - g\|_{\infty,S}) = 0,$$
$$\lim_{n\to\infty}(\|\varphi_n - \varphi\|_{\infty,S} + \|\psi_n - \psi\|_{\infty,S}) = 0,$$
$$\lim_{n\to\infty}(\|u_{0,n} - u_0\|_{\infty,\bar{\Omega}} + \|v_{0,n} - v_0\|_{\infty,\bar{\Omega}}) = 0.$$

Definition 1 *We call (u,v) a generalized solution of (1.1) if there exists a sequence $\{(u_n, v_n)\}$*

of (2.1), satisfying $\sup_n\{\|u_n\|_{\infty,\Omega\times(0,T)}+\|v_n\|_{\infty,\Omega\times(0,T)}\}<+\infty$ *and converging to* (u,v) *weakly in* $\mathbb{L}^1(\Omega\times(0,T))$ *norm for every* $T>0$.

Let (y,z) be a classical solution of

$$\begin{cases} y'(t)=2f(z)g_1(y), \ z'(t)=2f_1(z)g(y), \quad t>0, \\ y(0)=z(0)=M=\|u_0\|_\infty+\|v_0\|_\infty+1. \end{cases} \tag{2.2}$$

For convenience, we introduce some functions as follows,

$$F(s)=\int_0^s\frac{f(\xi)}{f_1(\xi)}\mathrm{d}\xi, \quad G(s)=\int_0^s\frac{g(\xi)}{g_1(\xi)}\mathrm{d}\xi,$$
$$\tilde F(z)=F(z)+G(M)-F(M), \quad \tilde G(y)=G(y)-G(M)+F(M),$$
$$H(y(t))=\int_M^{y(t)}\frac{\mathrm{d}s}{2g_1(s)f(F^{-1}(\tilde G(s)))}=t, \ Q(z(t))=\int_M^{z(t)}\frac{\mathrm{d}s}{2f_1(s)g(G^{-1}(\tilde F(s)))}=t.$$

We state the main results of the paper, namely, the necessary-sufficient conditions for the existence of generalized solutions to (1.1), and the growth rate estimates for the solutions.

Theorem 1 *System (1.1) has a nonnegative generalized solution* (u,v) *if and only if*

$$\int^\infty\frac{1}{f_1(s)g(G^{-1}(F(s)))}\mathrm{d}s=\infty, \quad \text{or} \quad \int^\infty\frac{1}{g_1(s)f(F^{-1}(G(s)))}\mathrm{d}s=\infty, \tag{2.3}$$

where F^{-1}, G^{-1} *are the inverse functions of* F, G, *respectively. Moreover,*

$$0\le u(x,t)\le H^{-1}(t), \quad 0\le v(x,t)\le Q^{-1}(t), \tag{2.4}$$

where H^{-1}, Q^{-1} *are the inverse functions of* H, Q, *respectively.*

Remark 1 Obviously, (2.3) is an optimal condition for generalized solutions to the nonlinear diffusion equations (1.1), corresponding to the so called critical global existence exponents. For example, if $g_1(u)=u^m$, $f(v)=v^p$, $f_1(v)=v^n$, $g(u)=u^q$, $\varphi(u)=u$, $\psi(v)=v$, then (2.3) becomes $pq\le(m-1)(n-1)$, a well known criterion for global solutions[8,11].

Remark 2 From Theorem 1, one can check that the criteria (2.3) is independent of the diffusion terms in nonlinear diffusion equations, subject to homogeneous Neumann boundary conditions, which extends the results of [6,7].

3. Proof of Theorem 1

Using approximating methods[2], we prove Theorem 1 by three steps. The first step deals with the following ordinary differential system

$$\begin{cases} y'(t)=f(z(t))g_1(y(t)), & z'(t)=f_1(z(t))g(y(t)), & t>0, \\ y(0)=a\ge0, & z(0)=b\ge0, & a+b>0. \end{cases} \tag{3.1}$$

Proposition 1 *System (3.1) has a nonnegative classical solution* (u,v) *if and only if (2.3) is true.*

Proof Assume that system (3.1) has a classical solution $y=y(t)$. Let $a>0$. It follows from system (3.1) that $\frac{\mathrm{d}}{\mathrm{d}t}[G(y)-F(z)]=0$. Hence, $G(y)=F(z)+G(a)-F(b)$. It is easy to check that the inverse functions F^{-1}, G^{-1} exist. So, $y=G^{-1}(F(z)+G(a)-F(b))$. If $b>0$, we have

$$\int_b^{z(t)} \frac{\mathrm{d}s}{f_1(s)g(G^{-1}(F(s)+G(a)-F(b)))}=t. \tag{3.2}$$

The conclusion (2.3) can be obtained by letting $t\to\infty$ in (3.2). In fact, since $z'(t)\geq f_1(b)g(a)>0$, we have $z(t)\to\infty$ as $t\to\infty$. If $b=0$, we find some $\delta>0$ satisfying $z(\delta)>0$ and obtain (3.2) with b replaced by $z(\delta)$. Then (2.3) holds also by letting $\delta\to 0$.

Now assume (2.3) is true. Without loss of generality, suppose $a>0$. Define

$$\tilde F(s)=\int_b^s \frac{f(\xi)}{f_1(\xi)}\mathrm{d}\xi,\quad \tilde G(s)=\int_a^s \frac{g(\xi)}{g_1(\xi)}\mathrm{d}\xi,\quad H(s)=\int_a^s \frac{\mathrm{d}\sigma}{g_1(\sigma)f(\tilde F^{-1}(\tilde G(\sigma)))}.$$

It can be checked from (2.3) that $H:[a,\infty)\to[0,\infty)$ has an inverse function. Define $y(t)=H^{-1}(t):[0,\infty)\to[a,\infty)$ and $z(t)=\tilde F^{-1}(\tilde G(y(t)))$. A simple computation shows that (y,z) is a classical solution to (3.1).

The second step deals with the classical solutions of approximating system (2.1) by the similar method used in [6]. For simplicity, we still use f, g, φ, ψ, f_1, g_1, u_0, v_0 to denote f_n, g_n, φ_n, ψ_n, $f_{1,n}$, $g_{1,n}$, $u_{0,n}$, $v_{0,n}$ in (2.1), respectively.

Proposition 2 *System (2.1) has a nonnegative classical solution (u,v) if and only if (2.3) holds. Moreover, (u,v) satisfies (2.4).*

Proof At first we prove that (2.3) is necessary. Let (u,v) be a nonnegative classical solution of (2.1). Take $T_0>0$ such that $\inf_{\Omega\times[T_0,\infty)}\min\{u,v\}\geq a>0$. Let (y,z) be a nonnegative solution of the following ordinary differential system

$$\begin{cases} y'(t)=\dfrac12 f(z)g_1(y),\ z'(t)=\dfrac12 f_1(z)g(y), \quad T_0<t<t_0,\\ y(T_0)=z(T_0)=\dfrac a2, \end{cases} \tag{3.3}$$

where $t_0=\sup\big\{\tau|$ the solution of (3.3) exists for $t\in[T_0,\tau)\big\}$.

If $t_0=\infty$, then (2.3) holds by Theorem 1. If $t_0<\infty$, we claim that

$$y(t)<u(x,t),\ \ z(t)<v(x,t),\quad (x,t)\in\bar\Omega\times[T_0,t_0). \tag{3.4}$$

Otherwise, there would exist some $(\tilde x,T)\in\bar\Omega\times(T_0,t_0)$ such that at least one of the inequalities in (3.4) fails.

Define $W(x,t)=u(x,t)-\varepsilon\xi(x)$ and $Z(x,t)=v(x,t)-\varepsilon\xi(x)$, where $\xi\in C^2(\bar\Omega)$ satisfies $\frac{\partial\xi}{\partial\eta}|_{\partial\Omega}<0$ and $\xi\geq 1$ on $\bar\Omega$, and $\varepsilon>0$ so small that $\xi(x)<\frac{m_0}{2\varepsilon}$,

$$-\varphi'(s)\Delta\xi(x)-\varepsilon\varphi''(s)|\nabla\xi(x)|^2\leq\frac{m_0}{4\varepsilon_0},\ \text{and}\ -\psi'(s)\Delta\xi(x)-\varepsilon\psi''(s)|\nabla\xi(x)|^2\leq\frac{m_0}{4\varepsilon_0}$$

with $m_0=\min\big\{a,f_1(\frac a2)g(\frac a2),f(\frac a2)g_1(\frac a2)\big\}$ and $s\in[0,s_0]$, $s_0=\max_{\bar\Omega\times[T_0,T]}\{u+v\}$. It is easy to check that $W(x,T_0)>y(T_0)$, $Z(x,T_0)>z(T_0)$, $x\in\bar\Omega$. Define

$$t_1=\sup\Big\{\tau\in[T_0,T]\mid W(x,t)>y(t),\ Z(x,t)>z(t),\ (x,t)\in\bar\Omega\times[T_0,\tau]\Big\}.$$

Then, for example, $W(x,t_1)\ge y(t_1)$ for $x\in\bar\Omega$, $Z(x_0,t_1)=z(t_1)=\min_{\bar\Omega}Z(\cdot,t_1)$ for some $x_0\in\bar\Omega$. Due to $\frac{\partial(Z-z)}{\partial\eta}|_{\partial\Omega\times(T_0,t_1)}>0$, we know $x_0\notin\partial\Omega$, and thus $\nabla Z(x_0,t_1)=0$, $\Delta Z(x_0,t_1)\ge 0$. At the point (x_0,t_1), we obtain a contradiction that

$$\begin{aligned}0\ge Z_t-z'=v_t-z'&=\Delta\psi(v)+g(u)f_1(v)-\frac{g(y)f_1(z)}{2}\\&\ge\varepsilon\psi'(v)\Delta\xi+\varepsilon^2\psi''(v)|\nabla\xi|^2+g(u)f_1(v)-\frac{g(y)f_1(z)}{2}\\&\ge-\frac{m_0}{4}+\frac{g(a/2)f_1(a/2)}{2}\ >0.\end{aligned}$$

This concludes (3.4). Hence the solutions of (3.3) can be extended to $t>t_0$, which contradicts the definition of t_0. We have $t_0=\infty$, and hence (2.3) holds by Proposition 1.

Now, assume that (2.3) holds. Obviously, the system (2.1) admits a local classical solution (u,v) in $\bar\Omega\times[0,T_0)$. In fact, by introducing the regularized systems of (2.1) with $\varphi(u)$ and $\psi(v)$ replaced by $\varphi(u)+1/m$ and $\psi(v)+1/m$, one can obtain a local classical solution of (2.1) as the limit of the approximating solutions of the regularized systems as $m\to+\infty$[9].

Let (y,z) be the classical solution of (2.2). Due to (2.3), (y,z) remains bounded for any time $T'<+\infty$. So, it suffices to prove that

$$u(x,t)<y(t),\quad v(x,t)<z(t),\quad (x,t)\in\bar\Omega\times(0,T'). \tag{3.5}$$

Otherwise, there would exist some $(x_0,T)\in\bar\Omega\times(0,T')$ such that at least one of the two inequalities in (3.5) fails. Set $p(x,t)=u(x,t)+\varepsilon\xi(x)$, $q(x,t)=v(x,t)+\varepsilon\xi(x)$, $(x,t)\in\Omega\times[0,T]$, where ξ is defined as above, and ε is so small that

$$\xi(x)<\frac{1}{\varepsilon}, \tag{3.6}$$

$$-\varphi'(s)\Delta\xi(x)+\varepsilon\varphi''(s)|\nabla\xi(x)|^2\le\frac{M_0}{\varepsilon_0}, \tag{3.7}$$

$$-\psi'(s)\Delta\xi(x)+\varepsilon\psi''(s)|\nabla\xi(x)|^2\le\frac{M_0}{\varepsilon_0} \tag{3.8}$$

with $M_0=\min\{f_1(M)g(M),f(M)g_1(M)\}$, $s\in[0,s_0]$ and $s_0=\max_{\bar\Omega\times[0,T]}\{u+v\}$. By (3.6), there exists $\tau_0\in(0,T_0]$ such that, e.g., $\max_{x\in\bar\Omega}p(x,\tau_0)=y(\tau_0)$ and $\max_{x\in\bar\Omega}q(x,\tau_0)\le z(\tau_0)$. Define $t_0=\sup\big\{\tau\in[0,T]\mid p(x,t)\le y(t),\ q(x,t)\le z(t),\ (x,t)\in\bar\Omega\times[0,\tau]\big\}$. Then $\max_{x\in\bar\Omega}p(x,t_0)=y(t_0)$ and $\max_{x\in\bar\Omega}q(x,t_0)\le z(t_0)$. Followed by $\frac{\partial(p-y)}{\partial\eta}=\varepsilon\frac{\partial\xi}{\partial\eta}<0$, there exists an $x_0\in\Omega$ such that $\nabla p(x_0,t_0)=0$ and $\Delta p(x_0,t_0)\le 0$. By (3.7) and (3.8), we obtain a contradiction at $(x,t)=(x_0,t_0)$ that

$$0\le p_t-y'=-\varepsilon\varphi'(s)\Delta\xi(x)+\varepsilon^2\varphi''(s)|\nabla\xi(x)|^2+f(v)g_1(u)-2f(z)g_1(y)<0.$$

This yields $T_0=\infty$, namely, the system (2.1) admits global solutions.

At last, we prove the growth rates for the global solution (u,v) of (2.1). We have shown $u(x,t)\le y(t)$, $v(x,t)\le z(t)$ for $(x,t)\in\bar\Omega\times[0,+\infty)$. It only needs to give the growth rates of (y,z). By using the similar arguments for the proof of Proposition 1, we obtain that

$$Q(z(t))=\int_{z(0)}^{z(t)}\frac{1}{2f_1(s)g(G^{-1}(\tilde F(s)))}\mathrm{d}s=t.$$

One can check that Q^{-1} exists since $Q'(s) > 0$. We conclude the growth rate estimate $0 \le v(x,t) < z(t) = Q^{-1}(t)$ (and also $0 \le u(x,t) < y(t) = P^{-1}(t)$ in the same way) for $(x,t) \in \bar{\Omega} \times [0,+\infty)$.

Now we deal with the generalized solutions of (1.1) to prove Theorem 1. The proof is motivated by [2, 6].

Proof of Theorem 1 At first we prove that (2.3) is necessary for generalized solutions of (1.1). Let (u,v) be a generalized solution of (1.1) with approximating solution sequence $\{(u_n,v_n)\}$, satisfying system (2.1) and $\sup_n \left\{\|u_n\|_{\mathbb{L}^\infty(\Omega\times(0,T))} + \|v_n\|_{\mathbb{L}^\infty(\Omega\times(0,T))}\right\} < +\infty$ for all $T > 0$.

Since (u_0,v_0) is positive and $\{(u_{0,n},v_{0,n})\}$ converges uniformly on $\bar{\Omega}$, there exists some positive constant a such that $\min\{u_n,v_n\} > a$ on $\bar{\Omega}$. Similarly to the proof of Proposition 1 with $T_0 = 0$, we obtain that $y_n(t) \le u_n(x,t)$, $z_n(t) \le v_n(x,t)$, $(x,t) \in \bar{\Omega} \times [0,+\infty)$, where (y_n,z_n) solves (3.3) with f, g, f_1, g_1 replaced by $f_n, g_n, f_{1,n}, g_{1,n}$, respectively. Hence,

$$y_n(t) + z_n(t) \le \sup_k\{\|u_k\|_{\infty,\Omega\times(0,T)}, \|v_k\|_{\infty,\Omega\times(0,T)}\} < +\infty, \quad n \in \mathbb{N},\ 0 \le t \le T. \tag{3.9}$$

Since f_n, g_n, $f_{1,n}$, $g_{1,n}$ converge uniformly on compact subsets of $[0,+\infty)$ to f, g, f_1, g_1, respectively, $(y_n(t), z_n(t))$ converges to $(y(t), z(t))$ uniformly on compact subsets of $[0,t_0)$, where t_0 represents the maximal existence time of (3.3). Clearly, $(y(t), z(t))$ satisfies (3.9) too, and hence $t_0 = +\infty$. By Proposition 1, condition (2.3) holds.

Now consider the sufficient condition for generalized solutions of (1.1). Assume that (2.3) holds. By using mollifiers and the properties of f, g, φ, ψ, f_1, g_1, u_0, v_0, one can construct sequences $\{f_n\}$, $\{g_n\}$, $\{\varphi_n\}$, $\{\psi_n\}$, $\{f_{1,n}\}$, $\{g_{1,n}\}$, $\{u_{0,n}\}$, $\{v_{0,n}\}$ in (2.1), respectively. Furthermore, choose $\{f_n\}$, $\{g_n\}$, $\{\varphi_n\}$, $\{\psi_n\}$, $\{f_{1,n}\}$, $\{g_{1,n}\}$, for each n, to satisfy (2.3) with f, g, f_1, g_1 replaced by f_n, g_n, $f_{1,n}$, $g_{1,n}$, respectively. Let (u_n,v_n) be the solution of system (2.1), and (y_n,z_n) be the solution of

$$\begin{cases} y_n'(t) = 2f_n(z_n)g_{1,n}(y_n), \quad z_n'(t) = 2f_{1,n}(z_n)g_n(y_n), \quad 0 < t < +\infty, \\ y_n(0) = z_n(0) = M = \sup\limits_n\{\|u_{0,n}\|_\infty + \|v_{0,n}\|_\infty\} + 1. \end{cases} \tag{3.10}$$

Similarly to the discussion for Proposition 2, we have $0 \le u_n(x,t) \le y_n(t)$, $0 \le v_n(x,t) \le z_n(t)$, $(x,t) \in \Omega \times [0,+\infty)$. Since (y_n,z_n) converges to (y,z) locally, we know that (y_n,z_n) is bounded locally so that $\sup_n \left\{\|u_n\|_{\mathbb{L}^\infty(\Omega\times(0,T))} + \|v_n\|_{\mathbb{L}^\infty(\Omega\times(0,T))}\right\} < +\infty$ for any $0 < T < +\infty$. By the diagonal process, one can choose the subsequence of $\{(u_n,v_n)\}$, which converges to (u,v) weakly in $\mathbb{L}^1(\Omega\times(0,T))$ norm for every $T > 0$. Therefore, (u,v) is the generalized solution of system (1.1) by Definition 1.

By Proposition 2, we obtain that

$$0 \le u_n(x,t) \le H_n^{-1}(t), \quad 0 \le v_n(x,t) \le Q_n^{-1}(t), \quad (x,t) \in \bar{\Omega} \times [0,+\infty), \tag{3.11}$$

where H_n^{-1}, Q_n^{-1} are the inverse functions of

$$H_n(y_n(t)) = \int_{y_n(0)}^{y_n(t)} \frac{\mathrm{d}s}{2g_{1,n}(s)f_n(F_n^{-1}(\tilde{G}_n(s)))} = t,$$

$$Q_n(z_n(t)) = \int_{z_n(0)}^{z_n(t)} \frac{\mathrm{d}s}{2f_{1,n}(s)g_n(G_n^{-1}(\tilde{F}_n(s)))} = t,$$

respectively; (y_n, z_n) satisfies (3.10). Similarly to the approximating process above, one can obtain that $H_n(y_n(t))$ and $Q_n(z_n(t))$ converge to $H(y(t))$ and $Q(z(t))$ in $\mathbb{L}^1(\Omega \times (0,t))$ norm, respectively. Then by (3.11), we have the growth rate estimates in (2.4).

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