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Some Large Deviation Results for Generalized Compound Binomial Risk Models

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Abstract This paper is a further investigation of large deviation for partial and random sums of random variables, where $\{X_n, n \geq 1\}$ is non-negative independent identically distributed random variables with a common heavy-tailed distribution function F on the real line R and finite mean $\mu \in R$. $\{N(n), n \geq 0\}$ is a binomial process with a parameter $p \in (0,1)$ and independent of $\{X_n, n \geq 1\}$; $\{M(n), n \geq 0\}$ is a Poisson process with intensity $\lambda > 0$, $S_n = \sum_{i=1}^{N(n)} X_i - cM(n)$. Suppose $F \in C$, we futher extend and improve some large deviation results. These results can apply to certain problems in insurance and finance.

Keywords generalized compound binomial risk model; large deviations; heavy-tailed distribution; ruin probability.

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1. Introduction and main results

Hu[1] introduced a generalized compound binomial risk model, which is based on the following independent objects:

(i) A binomial process $\{N(n); n = 0, 1, 2, \ldots\}$ with a parameter p, $0 < p < 1$, corresponding to the claim number process, $N(0) = 0$;

(ii) A sequence $\{X_n; n \geq 1\}$ of non-negative i.i.d random variables with common distribution function (df for short) F, corresponding to the claim size process, $0 < \mu = EX_1 < \infty$;

(iii) A Poisson process $\{M(n); n = 0, 1, 2, \ldots\}$ with intensity $\lambda > 0$, where $M(n)$ is corresponding to the number of customers who buy the insurance portfolios in the time interval $(0, n]$, $M(0)$=0;

(iv) $\{N(n); n = 0, 1, 2, \ldots\}$; $\{X_n; n \geq 1\}$ and $\{M(n); n = 0, 1, 2, \ldots\}$ are mutually independent.

For the GCBRM, the risk reserve process $\{R(n); n = 0, 1, 2, \ldots\}$ is then given by

$$R(n) = u + cM(n) - \sum_{i=1}^{N(n)} X_i, \quad n = 0, 1, 2, \ldots \tag{1}$$

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while the net total claim amount process $\{S(n); n = 0, 1, 2, \ldots\}$ is

$$S(n) = \sum_{i=1}^{N(n)} X_i - cM(n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $c > 0$ is the premium of a single insurance portfolio (i.e., the price of the insurance portfolio), and $u > 0$ is the initial capital of the company. For the GCBRM, the net profit condition becomes $c\lambda > pEX_1$.

The time of ruin for the GCBRM is described by

$$T(u) = \inf\{n; R(n) < 0\} = \inf\{n; S(n) > u\}. \tag{3}$$

In [1], Hu investigated the GCBRM with heavy-tailed claim sizes, namely in

(i) The probabilities of large deviations of $\{S(n)\}$;

(ii) The Lundberg type limiting results for the finite time ruin probabilities.

He obtained the following results.

Theorem A *For the GCBRM, let $\{S(n)\}$ be as in (2) and suppose that $\overline{F} \in ERV(-\alpha, -\beta)$ for some $1 < \alpha \le \beta < \infty$. Then $P(S(n) - ES(n) > x) \sim pn\overline{F}(x)$ holds uniformly for $x \ge \gamma pn$ for any fixed $\gamma > 0$ satisfying $\gamma p > c\lambda$, i.e.,*

$$\lim_{n\to\infty} \sup_{x\ge\gamma pn} |\frac{P(S(n) - ES(n) > x)}{pn\overline{F}(x)} - 1| = 0. \tag{4}$$

Theorem B *For the GCBRM, suppose that $\overline{F} \in ERV(-\alpha, -\beta)$ for some $1 < \alpha \le \beta < \infty$. Then*

(i) For every $x > 0$ and $y > 0$,

$$\liminf_{u\to\infty} \frac{1}{\log u} \log P(T(u) \le yu^x) \ge x - \beta \cdot \max\{1, x\}; \tag{5}$$

(ii) For either x=1 and $0 < y < (p\mu)^{-1}$or $0 < x < 1$ and $y > 0$,

$$\limsup_{u\to\infty} \frac{1}{\log u} \log P(T(u) \le yu^x) \le x - \alpha, \tag{6}$$

where $\mu = EX_1 < \infty$.

In this paper, we will extend $\overline{F} \in ERV(-\alpha, -\beta)$ to $\overline{F} \in C$, and give some counterparts of (4), (5) and (6). First we give some definitions.

Definition 1 *The random variable X (or its d.f.F) is called heavy-tailed, if $Ee^{tX} = \infty$ holds for any fixed $t > 0$. The two important subclasses of heavy-tailed df are C and ERV.*

(i) We call $\overline{F} \in C$, if $\lim_{l\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(lx)}{\overline{F}(x)} = 1$.

(ii) We call $\overline{F} \in ERV(-\alpha, -\beta)$, if there exist constants $1 < \alpha \le \beta < \infty$ such that

$$y^{-\beta} \le \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \le \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \le y^{-\alpha}, \quad \forall y > 1.$$

Remark 1 Let N be a random variable with the geometric distribution, i.e., $P(N = k) = p(1-p)^{k-1}$, $k > 1$, $0 < p < 1$; U is another $r.v.$ with uniform distribution $U(0, 1)$ and it is independent of N. Write $X := 2^{2^N(1+U)}$. F denotes distribution of $r.v.X$, then $\overline{F} \in C$, but

$\overline{F} \notin ERV$. From Lemma 7 in [2], we have $ERV \subset C$. Thereby, C is a larger subclass than ERV.

Corresponding to the Proposition 2.1 in [1], we have the following result.

Theorem 1 *For the GCBRM, if $\overline{F} \in C$, then for every fixed $\gamma > 0$, $P(Y(n) - EY(n) > x) \sim pn\overline{F}(x)$holds uniformly for $x \geq \gamma pn$, that is,*

$$\lim_{n\to\infty} \sup_{x\geq\gamma pn} |\frac{P(Y(n) - EY(n) > x)}{pn\overline{F}(x)} - 1| = 0,$$

where $Y(n) = \sum_{i=1}^{N(n)} X_i$, $n = 0, 1, 2, \ldots$.

About the large deviations of $\{S(n)\}$ in (2), we have the following theorem.

Theorem 2 *For the GCBRM, let $\{S(n)\}$ be as in (2) and suppose that $\overline{F} \in C$. Then for any fixed $\gamma > 0$, $P(S(n) - ES(n) > x) \sim pn\overline{F}(x)$ holds uniformly for $x \geq \gamma pn$, where $\gamma p > c\lambda$, i.e.,*

$$\lim_{n\to\infty} \sup_{x\geq\gamma pn} |\frac{P(S(n) - ES(n) > x)}{pn\overline{F}(x)} - 1| = 0. \tag{7}$$

Remark 2 Obviously, (7) is equivalent to the following properties: for any fixed $\gamma > 0$, where $\gamma p > c\lambda$,

$$\liminf_{n\to\infty} \inf_{x\geq\gamma pn} \frac{P(S(n) - ES(n) > x)}{pn\overline{F}(x)} \geq 1, \quad \limsup_{n\to\infty} \sup_{x\geq\gamma pn} \frac{P(S(n) - ES(n) > x)}{pn\overline{F}(x)} \leq 1.$$

Before giving Theorem 3, we first see Lemma 1.

Lemma 1 *Let X be a non-negative random variable with its tail $\overline{F} \in C$ and $0 < \mu = EX_1 < \infty$. There exists some $1 < \beta < \infty$, such that $K_1 x^{-\beta} \leq \overline{F}(x) \leq \mu x^{-1}$ for all $x \geq x_0(\beta)$, where the constant $K_1 = K_1(\beta)$ is independent of x.*

Proof By Lemma 3.1 in [3], if $\overline{F} \in C$, then there exists $\beta > 1$, $K > 0$ and $x_0 > 0$, such that $\overline{F}(x) \geq Kx^{-\beta}$. For all $x > x_0$, where $K > 0$ is a constant and independent of x. Since $\infty > \mu = EX \geq EXI_{(X>x)} \geq x\overline{F}(x)$, $\overline{F}(x) \leq \mu x^{-1}$.

Recall that $T(u)$ as in (3) is time of ruin for the GCBRM. By the corresponding Lundberg type limiting result, we have

Theorem 3 *For the GCBRM, suppose that $\overline{F} \in C$. Then*

(i) For every $0 < x \leq 1$ and $y > 0$, $\beta_0 = \inf\{\beta\}$, β satisfying Lemma 1, we have

$$\liminf_{u\to\infty} \frac{1}{\log u} \log P(T(u) \leq yu^x) \geq x - \beta_0. \tag{8}$$

(ii) For $0 < x < 1$ and $y > 0$, we get

$$\limsup_{u\to\infty} \frac{1}{\log u} \log P(T(u) \leq yu^x) \leq x - 1. \tag{9}$$

2. Proofs of main results

We first prove Theorem 1.

Proof of Theorem 1 For the GCBRM and $\overline{F} \in C$, from Lemma 9 in [2] we have

$$\lim_{n\to\infty} \sup_{x\geq \gamma n} \left|\frac{P(S_n - ES_n > x)}{n\overline{F}(x)} - 1\right| = 0,$$

where $S_n = \sum_{i=0}^{n} X_i, n = 1, 2, \ldots$.

Since $\{N(n); n \geq 0\}$ is a binomial process with a parameter $p \in (0,1)$, there exists a sequence $\{Y_i; i \geq 1\}$ which are i.i.d.r.v, independent of $\{X_n; n \geq 1\}$, and $P(Y_1 = 1) = p = 1 - P(Y_1 = 0)$, such that $N(n) = \sum_{i=1}^{n} Y_i$. Then

$$Y(n) = \sum_{i=1}^{N(n)} X_i = \sum_{i=1}^{Y_1} X_i + \sum_{i=Y_1+1}^{Y_1+Y_2} X_i + \cdots + \sum_{i=\sum_{k=1}^{n-1} Y_k+1}^{\sum_{k=1}^{n} Y_k} X_i := \sum_{j=1}^{n} Z_j,$$

where $\sum_{i=1}^{0} X_i := 0$ and $N(0) := 0$. Obviously, $\{Z_j, j = 1, \ldots, n\}$ are independent.

$$\begin{aligned} E\exp(r \sum_{i=\sum_{k=1}^{j} Y_k+1}^{\sum_{k=1}^{j+1} Y_k} X_i) &= \sum_{n_1=0}^{1} \cdots \sum_{n_j=0}^{1} E\exp(r \sum_{i=n_1+\cdots+n_j+1}^{n_1+\cdots+n_j+Y_{j+1}} X_i) I_{(Y_1=n_1,\ldots,Y_j=n_j)} \\ &= E\exp(r \sum_{i=1}^{Y_{j+1}} X_i) = E\exp(r \sum_{i=1}^{Y_1} X_i). \end{aligned}$$

Thus $\{Z_j, j \geq 1\}$ are i.i.d and $EZ_1 = EY_1 \cdot EX_1 = p\mu > 0$.

Since $P(Z_1 > x) = P(X_1 > x, Y_1 = 1) = p\overline{F}(x)$, $x > 0$, for any $y > 1$, we have

$$\frac{P(Z_1 > xy)}{P(Z_1 > x)} = \frac{p\overline{F}(xy)}{p\overline{F}(x)} = \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

By $\overline{F} \in C$, we obtain $\overline{Z}_1 \in C$. Then

$$\lim_{n\to\infty} \sup_{x\geq\gamma n} \left|\frac{P(\sum_{i=1}^{n} Z_i - E(\sum_{i=1}^{n} Z_i) > x)}{nP(Z_1 > x)} - 1\right| = 0,$$

thereby

$$\lim_{n\to\infty} \sup_{x\geq\gamma n} \left|\frac{P(Y(n) - E(Y(n)) > x)}{np\overline{F}(x)} - 1\right| = 0.$$

Thus the proof of Theorem 1 is completed. □

Proof of Theorem 2 Observe that $\{M(n); n = 0, 1, 2, \ldots\}$ is a Poisson process with intensity $\lambda > 0$, by [1], there exist a positive sequence $\{\varepsilon(n) \downarrow 0\}$ as $n \to \infty$ and

$$P(|M(n) - \lambda n| > \varepsilon(n)\lambda n) = o(1). \tag{10}$$

Noting that $Y(n) = \sum_{i=1}^{N(n)} X_i$, $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} P(S(n) - ES(n) > x) &= P(Y(n) - EY(n) > x - c\lambda n + cM(n)) \\ &= \sum_{k=0}^{\infty} P(Y(n) - EY(n) > x - c\lambda n + ck) P(M(n) = k). \end{aligned}$$

Therefore, Theorem 2 will be proved from the following three Lemmas 2–4.

Lemma 2 *Let $\{\varepsilon(n)\}$ be as in (10). Then for any fixed $\gamma > 0$*

$$\sum_{|k-\lambda n|\le\varepsilon(n)\lambda n} P(M(n)=k)P(Y(n)-EY(n)>x-c\lambda n+ck)\sim pn\overline{F}(x)$$

holds uniformly for $x \ge \gamma pn$.

Lemma 3 *Let $\{\varepsilon(n)\}$ be as in (10). Then for any fixed $\gamma > 0$ satisfying $\gamma p > c\lambda$*

$$\sum_{k-\lambda n<-\varepsilon(n)\lambda n} P(M(n)=k)P(Y(n)-EY(n)>x-c\lambda n+ck)=o(pn\overline{F}(x))$$

holds uniformly for $x \ge \gamma pn$.

Lemma 4 *Let $\{\varepsilon(n)\}$ be as in (10). Then for any fixed $\gamma > 0$*

$$\sum_{k-\lambda n>\varepsilon(n)\lambda n} P(M(n)=k)P(Y(n)-EY(n)>x-c\lambda n+ck)=o(pn\overline{F}(x))$$

holds uniformly for $x \ge \gamma pn$.

Proof of Lemma 2 In view of Theorem 1, for fixed $\gamma > 0$, we have

$$P(Y(n)-EY(n)>x)\sim pn\overline{F}(x),$$

as $n\to\infty$, holds uniformly for $x \ge \gamma pn$.

Moreover, for $|k-\lambda n|\le\varepsilon(n)\lambda n$ with $\varepsilon(n)$ as in (10), and $x\ge\gamma pn$,

$$x-c\lambda n+ck=x+c(k-\lambda n)=x+o(x),\quad n\to\infty.$$

Then

$$\frac{\overline{F}(x-c\lambda n+ck)}{\overline{F}(x)}=\frac{\overline{F}(x+o(x))}{\overline{F}(x)}.$$

By Lemma 8 in [2], for any $\gamma > 0$, we have

$$\lim_{n\to\infty}\sup_{x\ge\gamma pn}\left|\frac{\overline{F}(x+o(x))}{\overline{F}(x)}-1\right|=0.$$

Thereby

$$P(Y(n)-E(Y(n))>x-c\lambda n+ck)\sim pn\overline{F}(x-c\lambda n+ck)$$

as $n\to\infty$, holds uniformly for $|k-\lambda n|\le\varepsilon(n)\lambda n$ and $x\ge\gamma pn$. Hence

$$\begin{aligned}
&\sum_{|k-\lambda n|\le\varepsilon(n)\lambda n} P(M(n)=k)P(Y(n)-EY(n)>x-c\lambda n+ck)\\
&\sim pn\overline{F}(x)\sum_{|k-\lambda n|\le\varepsilon(n)\lambda n} P(M(n)=k)\frac{\overline{F}(x-c\lambda n+ck)}{\overline{F}(x)}\\
&\sim pn\overline{F}(x)P(|M(n)-\lambda n|\le\varepsilon(n)\lambda n)\sim pn\overline{F}(x)
\end{aligned}$$

holds uniformly for $x\ge\gamma pn$. Lemma 2 is proved.

Proof of Lemma 3 For $x \geq \gamma pn$, we have

$$x - c\lambda n = x(1 - \frac{c\lambda n}{x}) \geq x(1 - \frac{c\lambda}{\gamma p}) := \gamma' x.$$

Since $\overline{F} \in C \subset D = \{\limsup_{x\to\infty} \frac{\overline{F}(\gamma' x)}{\overline{F}(x)} < \infty\}$ for any fixed $0 < \gamma' < 1$, again using Theorem 1 and choosing $\varepsilon(n)$ as in(10), we obtain that

$$\begin{aligned}
&\sum_{k-\lambda n < -\varepsilon(n)\lambda n} P(M(n) = k)P(Y(n) - EY(n) > x - c\lambda n + ck) \\
&\leq \sum_{k-\lambda n < -\varepsilon(n)\lambda n} P(M(n) = k)P(Y(n) - EY(n) > x - c\lambda n) \\
&\sim pn\overline{F}(x) \sum_{k-\lambda n < -\varepsilon(n)\lambda n} P(M(n) = k)\frac{\overline{F}(x - c\lambda n)}{\overline{F}(x)} \\
&\leq pn\overline{F}(x) \sum_{k-\lambda n < -\varepsilon(n)\lambda n} P(M(n) = k)\frac{\overline{F}(\gamma' x)}{\overline{F}(x)} \\
&\leq c_1 pn\overline{F}(x)P(M(n) - \lambda n \leq -\varepsilon(n)\lambda n) = o(1)pn\overline{F}(x) = o(pn\overline{F}(x))
\end{aligned}$$

uniformly for $x \geq \gamma pn$, where $\gamma > 0$ is a fixed constant, satisfying $\gamma p > c\lambda$ and $c_1 > 0$ is also a constant. Lemma 3 is proved.

Proof of Lemma 4 Using Theorem 1 once more and choosing $\varepsilon(n)$ as in(10), we have

$$\begin{aligned}
&\sum_{k-\lambda n > \varepsilon(n)\lambda n} P(M(n) = k)P(Y(n) - EY(n) > x - c\lambda n + ck) \\
&\leq \sum_{k-\lambda n > \varepsilon(n)\lambda n} P(M(n) = k)P(Y(n) - EY(n) > x) \\
&\sim pn\overline{F}(x) \sum_{k-\lambda n > \varepsilon(n)\lambda n} P(M(n) = k) \\
&= pn\overline{F}(x)P(M(n) - \lambda n > \varepsilon(n)\lambda n) = o(1)pn\overline{F}(x) = o(pn\overline{F}(x))
\end{aligned}$$

uniformly for $x \geq \gamma\lambda n$ where $\gamma > 0$ is a fixed constant. Lemma 4 is proved.

By Lemmas 2, 3 and 4, the proof of Theorem 2 is completed. □

Proof of Theorem 3 (i) Proof of (8). Let $0 < x \leq 1$ and $y > 0$, in view of Remark 2, for any $0 < \theta < 1$ we have uniformly for u large enough that

$$\begin{aligned}
P(T(u) \leq yu^x) &\geq P(S([yu^x]) > u) = P(S([yu^x]) - ES([yu^x]) > u - (p\mu - c\lambda)[yu^x]) \\
&\geq P(S([yu^x]) - ES([yu^x]) > u + c\lambda[yu^x]) \geq (1 - \theta)p[yu^x]\overline{F}(u + c\lambda[yu^x])
\end{aligned}$$

where $[y]$ stands for the integer part of $y \in R$. Consequently, let $1 < \beta < \infty$, by Lemma 1

$$\liminf_{u\to\infty} \frac{1}{\log u} \log P(T(u) \leq [yu^x]) \geq x + \liminf_{u\to\infty} \frac{1}{\log u}(-\beta)\log(u + c\lambda[yu^x]) \geq x - \beta_0,$$

thus (8) is proved.

(ii) Proof of (9). Let $0 < x < 1$ and $y > 0$. By Theorem 1, for every $0 < \theta < 1$, we have uniformly for u large enough that

$$P(T(u) \le [yu^x]) \le P(Y([yu^x] > u) = P(Y([yu^x]) - EY([yu^x]) > u - p[yu^x]\mu)$$
$$\le (1+\theta)p[yu^x]\overline{F}(u - p\mu[yu^x]).$$

Consequently, by Lemma 4

$$\limsup_{u\to\infty} \frac{1}{\log u} \log P(T(u) \le yu^x) \le x + \limsup_{u\to\infty} \frac{1}{\log u}(-1)\log(u - p\mu[yu^x]) = x - 1.$$

The proof of Theorem 3 is completed. □

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