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Approximation by Modified Summation Integral Type Operators in the L_p Spaces

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Abstract The generalized summation integral type operators with Beta basis functions are widely studied. At present, the investigations for the properties of these operators are only limited to the functions of bounded variation. Some authors studied the rate of point-wise rate of convergence, asymptotic formula of Voronovskaja type, and some direct results about these type of operators. The present paper considers the direct, inverse and equivalence theorems of modified summation integral type operators in the L_p spaces.

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1. Introduction

Beta operators are important and applied widely in probability theory and approximation theory[1−6]. The generalized summation integral type operators with Beta basis functions are widely studied[6−18]. At present, the investigations for the properties of these operators are only limited to the functions of bounded variation[6,9,17]. Some authors studied the rate of point-wise rate of convergence, asymptotic formula of Voronovskaja type, and some direct results about these type of operators[7,8,10−18].

In this paper, we will study the direct, inverse and equivalent theorems of modified summation integral type operators in the L_p spaces.

For $f \in L_p[0,\infty)$, a new kind of linear positive operators are defined as[6,13,15]

$$B_n(f,x) \equiv B_n(f(t),x) = \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)f(t)\mathrm{d}t, \quad x \in [0,\infty) \tag{1.1}$$

where

$$b_{n,k}(t) = \frac{(n+k)!}{(k-1)!\,n!}t^{k-1}(1+t)^{-n-k-1}, \quad k \geq 1.$$

Let

$$B(k,n+1) = \frac{(k-1)!\,n!}{(n+k)!}, \quad W_n(x,t) = \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k}(x)b_{n,k}(t).$$

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We can write

$$b_{n,k}(t)=\frac{t^{k-1}(1+t)^{-n-k-1}}{B(k,\ n+1)},\quad B_n(f,x)=\int_0^\infty W_n(x,t)f(t)\mathrm{d}t.$$

It is easy to verify that $B_n(1,x)=1$, and the operators $B_n(f,x)(1.1)$ are linear positive ones, the rate of convergence for these operators cannot be faster than $O(n^{-1})$[5,6,15]. Gupta[13] obtained the direct results for these operators.

Theorem A[13, Theorem 3.1] *Let $f\in C_r[0,\infty)$, $(C_r[0,\infty)=\{f: f\in C[0,\infty),\ |f(t)|\le Ct^r, r>0\})$, and suppose $f^{(r)}$ exists at a point $x\in(0,\infty)$. Then $\lim_{n\to\infty}B_n^{(r)}(f,x)=f^{(r)}(x)$.*

Theorem B[13, Theorem 3.3] *Let $f\in C_r[0,\infty)$, and $0<a<a_1<b_1<b<\infty$. Then for all n sufficiently large, we have,*

$$\|B_n^{(r)}(f,\cdot)-f^{(r)}\|_{C[a,b]}\le A_1\omega_2(f^{(r)},\frac{1}{\sqrt{n}};a,b)+A_2n^{-1}\|f\|_r,$$

where

$$A_1=A_1(r),\ A_2=A_2(r,f),\ \|f\|_r=\sup_{0<t<\infty}|f(t)|t^{-r},$$

$$\omega_k(f,\delta;a,b)=\sup\{\overrightarrow{\Delta}_h^kf(x):|h|\le\delta,\ x,x+kh\in[a,b]\},$$

$\overrightarrow{\Delta}_h^kf(x)$ is k-th forward difference with step length h.

The authors also gave the inverse result for these operators.

Theorem C[13, Theorem 4.3] *Let $0<\alpha<2$, $0<a_1<a_2<b_2<b_1<\infty$. If $f\in C_r[0,\infty)$, then in the following statements, the implication $1^\circ\Rightarrow 2^\circ$ holds:*

$$1^\circ\ \|B_n^{(r)}(f,\cdot)-f^{(r)}\|_{C[a_1,b_1]}=O(n^{-\frac{\alpha}{2}});\quad 2^\circ\ f^{(r)}\in Lip^*(\alpha,a_2,b_2),$$

where $\mathrm{Lip}^*(\alpha,a_2,b_2)=\{f:\forall\delta>0,\exists C>0\,,\ \text{such that}\ \omega_2(f,\delta;a,b)\le C\delta^\alpha\}$.

For convenience, we will introduce some of the notations. Let $f\in L_p[0,\infty),\ 1\le p\le\infty$. We write

$$\|f\|_p=\begin{cases}\{\int_0^\infty|f(x)|^p\mathrm{d}x\}^{\frac{1}{p}}, & 1\le p<\infty,\\ \sup_{x\in[0,\infty)}|f(x)|, & p=\infty,\end{cases}$$

$$\Delta_{h\varphi}^2f(x)=f(x+h\varphi(x))-2f(x)+f(x-h\varphi(x)),$$

$$\omega_\varphi^2(f,t)_p=\sup_{0<h\le t}\|\Delta_{h\varphi}^2f\|_p,$$

$$D=\{g\in L_p[0,\infty):g'\in A.C._{\mathrm{loc}},\varphi^2g''\in L_p\},$$

$$K_\varphi^2(f,t^2)_p=\inf_{g\in D}\{\|f-g\|_p+t^2\|\varphi^2g''\|_p\},\tag{1.2}$$

$$\overline{D}=\{g:g\in L_p[0,\infty),g'\in A.C._{\mathrm{loc}},g'',\varphi^2g''\in L_p[0,\infty)\},$$

$$\overline{K}_\varphi^2(f,t^2)_p=\inf_{g\in\overline{D}}\{\|f-g\|_p+t^2\|\varphi^2g''\|_p+t^4\|g''\|_p\}.$$

It is well known that[5]: there exist constants C_1, $C_2>0$, such that

$$C_1^{-1}\omega_\varphi^2(f,t)_p\le K_\varphi^2(f,t^2)_p\le C_1\omega_\varphi^2(f,t)_p,$$

$$C_2^{-1}\omega_\varphi^2(f,t)_p \le \overline{K}_\varphi^2(f,t^2)_p \le C_2\omega_\varphi^2(f,t)_p. \tag{1.3}$$

Since the first order moment of the operator $B_n(f,x)(1.1)$ is not equal to zero, and $\varphi^2(x) = x(1+x)$, we need to deal with the problem by a special method. Our main results can be stated as follows:

Theorem 1.1 (Direct result) *For $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi^2(x) = x(1+x)$, one has*

$$\|B_n(f,x) - f(x)\|_p \le C\Big(\omega_\varphi^2(f, \frac{1}{\sqrt{n}})_p + \omega_1(f, \frac{1}{n})_p + \frac{\|f\|_p}{n}\Big),$$

where

$$\omega_1(f, \frac{1}{n})_p = \sup_{0<t\le\frac{1}{n}} \|\Delta_t^1 f(x)\|_p, \quad \Delta_t^1 f(x) = f(x+\frac{t}{2}) - f(x-\frac{t}{2}).$$

Theorem 1.2 (The Bernstein-Type inequality) *For $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, one has*

$$\|\varphi^2 B_n'' f\|_p \le Cn\|f\|_p.$$

Theorem 1.3 (Inverse result) *For $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $0 \le \alpha < 2$, one has*

$$\|B_n(f,x) - f(x)\|_p = O(n^{-\frac{\alpha}{2}}) \Rightarrow 1^\circ.\ \omega_\varphi^2(f,t)_p = O(t^\alpha); \quad 2^\circ.\ \omega_1(f,t)_p = O(t^{\frac{\alpha}{2}}).$$

Combining the direct result Theorem 1.1 and the converse result Theorem 1.3, we can obtain the equivalence theorem.

Theorem 1.4 (Equivalence result) *For $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $0 \le \alpha < 2$, the following two statements are equivalent:*

(1) $\|B_n(f,x) - f(x)\|_p = O(n^{-\frac{\alpha}{2}})$;

(2) $1^\circ.\ \omega_\varphi^2(f,t)_p = O(t^\alpha)$; $2^\circ.\ \omega_1(f,t)_p = O(t^{\frac{\alpha}{2}})$.

Remark 1.1 From Theorem 1.1, one can get Theorems A and B in the case $r = 1$, $p = \infty$.

Remark 1.2 From Theorem 1.3, one can get Theorem C in the case $r = 1$, $p = \infty$.

Remark 1.3 Throughout this paper, C denotes a constant independent of n and x which is not necessarily the same in different cases.

2. The basic properties of the mixed type Beta operators

By simple calculations, one can get the boundedness and the estimates for the moments of the operators.

Lemma 2.1 *For $f \in L_p[0,\infty)$, $1 \le p \le \infty$, we have $\|B_n(f,\cdot)\|_p \le \|f\|_p$.*

Lemma 2.2 *Let $\delta_n^2(x) = \varphi^2(x) + \frac{1}{n}$, $\varphi^2(x) = x(1+x)$. We have*

$$B_n(1,x) = 1, \tag{2.1}$$

$$B_n(t-x,x) = \frac{2x+1}{n}, \tag{2.2}$$

$$B_n((t-x)^2,x) \le \frac{C}{n}\delta_n^2(x), \tag{2.3}$$

$$B_n((t-x)^4,x) \le \frac{C\delta_n^4(x)}{n^2}. \tag{2.4}$$

Lemma 2.3 *For the operator B_n, we have the estimation $B_n((1+t)^{-2},x) \le C(1+x)^{-2}$.*

Lemma 2.4[5] *For $\varphi^2(x) = x(1+x)$, and u between t and x, we have*

$$\frac{|t-u|}{\varphi^2(u)} \le \frac{|t-x|}{x}\left(\frac{1}{1+x}+\frac{1}{1+t}\right).$$

Lemma 2.5 *For u between t and x, $x \in E_n^c = [0, n^{-1})$, we have*

$$\frac{|t-u|}{\varphi^2(u)+\frac{1}{n}} \le \frac{|t-x|}{\varphi^2(x)+\frac{1}{n}}. \tag{2.5}$$

Proof All we have to do is to estimate $\frac{|t-u|}{\varphi^2(u)+\frac{1}{n}}$ in the following ways.

For $x < u < t$, we have (2.5) which in turn follows from $\frac{t-u}{u(1+u)} \le \frac{t-x}{x(1+x)}$ and $t-u \le t-x$.

For $t < u < x < \frac{1}{n}$, (2.5) follows from $\frac{u}{\varphi^2(u)+\frac{1}{n}} \le \frac{x}{\varphi^2(x)+\frac{1}{n}}$ and $\frac{u-t}{u} \le \frac{x-t}{x}$, and the fact that $h(x) = \frac{x}{\varphi^2(x)+\frac{1}{n}}$ increases in $[0, \frac{1}{n})$. □

Lemma 2.6 *For $f' \in A.C._{\mathrm{loc}}$, $1 \le p \le \infty$, we have $\|\varphi^2 B_n'' f\|_p \le 6\|\varphi^2 f''\|_p$.*

Proof Using the relation[13, Lemma 2.6]

$$B_n''(f,x) = \frac{n+2}{n(n-1)}\sum_{k=1}^{\infty} b_{n+2,k}(x)\int_0^{\infty} b_{n-2,k+2}(t)f''(t)\mathrm{d}t,$$

we write

$$|\varphi^2(x)B_n''(f,x)| \le \frac{1}{n}\sum_{k=1}^{\infty}\left[b_{n-1,k+1}(x)+b_{n,k+2}(x)\right]\int_0^{\infty}\left[b_{n-1,k+1}(t)+b_{n,k}(t)\right]|(t^2+t)f''(t)|\mathrm{d}t.$$

For $p=\infty$, $|\varphi^2(x)B_n''(f,x)| \le 6\|\varphi^2 f''\|_\infty$. For $p=1$,

$$\begin{aligned}
&\|\varphi^2(x)B_n''(f,x)\|_1\\
&\quad\le \frac{1}{n}\int_0^{\infty}\sum_{k=1}^{\infty}\left[b_{n-1,k+1}(x)+b_{n,k+2}(x)\right]\int_0^{\infty}\left[b_{n-1,k+1}(t)+b_{n,k}(t)\right]|\varphi^2(t)f''(t)|\mathrm{d}t\mathrm{d}x\\
&\quad\le 6\|\varphi^2 f''\|_1.
\end{aligned}$$

By the Riesz-Thorin interpolation theorem, we get Lemma 2.6. □

3. The direct theorem

We need the following lemmas to prove our direct theorem.

Lemma 3.1 *Let*

$$H_n(f,x) = \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n-1,k+1}(x)\int_0^{\infty} b_{n+1,k}(t)f(t)\mathrm{d}t.$$

Then $H_n((x-t)^2,x) = \frac{2x^2}{n+1} + \frac{2x}{(2n+1)^2} - x^2\frac{n}{n+1}(1+x)^{-n-1}$.

Proof Noticing that

$$H_n(1,x)=\frac{n}{n+1}-\frac{n}{n+1}(1+x)^{-n-1},\quad H_n(t,x)=\frac{n}{n+1}x+\frac{n}{(n+1)^2},$$

$$H_n(t^2,x)=\frac{n+2}{n+1}x^2+\frac{2x}{n+1},$$

and $H_n((x-t)^2,x)=H_n(t^2,x)-2xH_n(t,x)+x^2H_n(1,x)$, which imply Lemma 3.1.

Lemma 3.2 *For $g\in\overline{D}_p$, $\delta_n^2(x)=\varphi^2(x)+\frac{1}{n}$, $1\le p\le\infty$, we have*

$$\big\|B_n(\int_x^t(t-u)g''(u)\mathrm{d}u,x)\big\|_p\le\frac{C}{n}\|\delta_n^2g''\|_p.$$

Proof We separate the proof for $p=\infty$ and $p=1$ and prove the result for $p=\infty$ first.

When $x\in E_n=[\frac{1}{n},\infty)$, we have $\delta_n^2(x)\sim\varphi^2(x)$. Using Lemmas 2.2, 2.4 and Hölder inequality, we can deduce

$$\big|B_n(\int_x^t(t-u)g''(u)\mathrm{d}u,x)\big|\le\frac{C}{n}\|\varphi^2g''\|_\infty.$$

When $x\in E_n^c=[0,\frac{1}{n})$, by Lemma 2.5(2.5) and Lemma 2.2, we have

$$\begin{aligned}&\big|B_n(\int_x^t(t-u)g''(u)\mathrm{d}u,x)\big|\le\|\delta_n^2g''\|_\infty\Big|B_n(\int_x^t\frac{|t-u|}{\varphi^2(u)+\frac{1}{n}}\mathrm{d}u,x)\Big|\\&\le C\|\delta_n^2g''\|_\infty\Big|B_n(\frac{(t-x)^2}{\varphi^2(x)+\frac{1}{n}},x)\Big|\le C\|\delta_n^2g''\|_\infty\frac{\varphi^2(x)}{n-1}\cdot\frac{1}{\varphi^2(x)+\frac{1}{n}}\le\frac{C}{n}\|\delta_n^2g''\|_\infty.\end{aligned}$$

Therefore, Lemma 3.2 is valid for the case $p=\infty$.

To prove the rest of the result for $p=1$, we observe that

$$\begin{aligned}&\big\|B_n(\int_x^t(t-u)g''(u)\mathrm{d}u,x)\big\|_1\\&=\int_0^\infty\frac{1}{n+1}\Big|\sum_{k=1}^\infty b_{n,k}(x)(\int_0^x+\int_x^\infty)b_{n,k}(t)\int_x^t(t-u)g''(u)\mathrm{d}u\mathrm{d}t\Big|\mathrm{d}x\\&\le\frac{1}{n+1}\int_0^\infty|g''(u)|\big\{\int_u^\infty\int_0^u-\int_0^u\int_u^\infty\big\}(u-t)\sum_{k=1}^\infty b_{n,k}(x)b_{n,k}(t)\mathrm{d}t\mathrm{d}x\mathrm{d}u.\end{aligned}$$

We need to prove that

$$\begin{aligned}J&:=\big\{\int_u^\infty\int_0^u-\int_0^u\int_u^\infty\big\}(u-t)\sum_{k=1}^\infty b_{n,k}(x)b_{n,k}(t)\mathrm{d}t\mathrm{d}x\\&\le\frac{1}{2}\Big|\sum_{k=1}^\infty(\int_u^\infty\int_0^u-\int_0^u\int_u^\infty)b_{n,k}(t)\mathrm{d}(u-t)^2b_{n,k}(x)\mathrm{d}x\Big|\\&:=\frac{1}{2}(J_1+J_2)\le C\delta_n^2(u)=C\big(\varphi^2(u)+\frac{1}{n}\big).\end{aligned}$$

We will estimate J_1. Using the integration by parts and recalling the fact

$$b'_{n,k}(t)=(n+1)(b_{n+1,k-1}(t)-b_{n+1,k}(t)),\quad k\ge1,$$

we have

$$J_1 \le (n+1)u^2(1+u)^{-n-1} + \frac{n+1}{n}\sum_{k=1}^{\infty}\int_0^u (t-u)^2 b_{n+1,k}(t)b_{n-1,k+1}(u)\mathrm{d}t.$$

Similarly, $J_2 \le 2\sum_{k=1}^{\infty}\int_u^{\infty}(t-u)^2 b_{n+1,k}(t)b_{n-1,k+1}(u)\mathrm{d}t$.

Using Lemma 3.1,

$$\begin{aligned}\frac{1}{2}(J_1+J_2) &\le (n+1)u^2(1+u)^{-n-1} + \sum_{k=1}^{\infty}\int_0^{\infty}(t-u)^2 b_{n+1,k}(t)b_{n-1,k+1}(u)\mathrm{d}t\\ &\le u^2 + \varphi^2(u) \le C\delta_n^2(u).\end{aligned}$$

Thus,

$$J \le \frac{1}{2}(J_1+J_2) \le C\delta_n^2(u).$$

Lemma 3.2 is true for $p=1$. Again using the Riesz-Thorin interpolation theorem completes the proof of Lemma 3.2 for $1 \le p \le \infty$.

Now we can prove our direct theorem.

Proof of Theorem 1.1 We construct a function $A_n(f,x)$ such that

$$A_n(f,x) = B_n(f,x) + L_n(f,x), \quad L_n(f,x) = f(x) - f(x+\frac{1}{n}). \tag{3.1}$$

Then $A_n(1,x)=1$, $A_n(t-x,x)=\frac{2x}{n}$, $A_n((t-x)^2,x) \le \frac{C\delta_n^2(x)}{n}$, $\|A_n\| \le 3$,

$$\|L_n(f,x)\|_p = \Big\{\int_0^{\infty} |f(x) - f(x+\frac{1}{n})|^p \mathrm{d}x\Big\}^{\frac{1}{p}} \le \omega_1(f,\frac{1}{n})_p.$$

By the definition of $\overline{K}_{\varphi}^2(f,t^2)_p$, we now choose $g \in \overline{D}$ satisfying

$$\|f-g\|_p + \frac{1}{n}\|\varphi^2 g''\|_p + \frac{1}{n^2}\|g''\|_p \le 2\overline{K}_{\varphi}^2(f,\frac{1}{n})_p. \tag{3.2}$$

Using the Taylor formula with integral remainder, we have

$$\begin{aligned}\|A_n(g,x) - g(x)\|_p &\le \|g'(x)A_n(t-x,x)\|_p + \big\|A_n(\int_x^t (t-u)g''(u)\mathrm{d}u, x)\big\|_p\\ &\le \frac{2}{n}\|xg'(x)\|_p + \big\|B_n(\int_x^t (t-u)g''(u)\mathrm{d}u, x)\big\|_p + \big\|\int_x^{x+\frac{1}{n}} (x+\frac{1}{n}-u)g''(u)\mathrm{d}u\big\|_p.\end{aligned}$$

Recalling

$$\big\|\int_x^{x+\frac{1}{n}} (x+\frac{1}{n}-u)g''(u)\mathrm{d}u\big\|_p \le \frac{1}{n^2}\|g''\|_p,$$

by the relation[20], $\|xf'\|_p \le C(\|x^2 f''\|_p + \|f\|_p)$, and Lemma 3.2 and the choice of g (3.2), we write

$$\|A_n(g,x) - g(x)\|_p \le C\overline{K}_{\varphi}^2(f,\frac{1}{n})_p + \frac{2}{n}\|f\|_p. \tag{3.3}$$

Combining (3.1)–(3.3) and the equivalence relation (1.3), we get

$$\begin{aligned}\|B_n(f,x) - f(x)\|_p &\le \|A_n(f,x) - f(x)\|_p + \|L_n(f,x)\|_p\\ &\le 4\|f-g\|_p + \|A_n g - g\|_p + \omega_1(f,\frac{1}{n})_p\end{aligned}$$

$$\leq C\big(\omega_{\varphi}^2(f,\frac{1}{\sqrt{n}})_p+\omega_1(f,\frac{1}{n})_p+\frac{\|f\|_p}{n}\big).$$

□

4. The inverse theorem

We also need the following lemmas to prove Theorems 1.2 and 1.3.

Lemma 4.1 *For $x\in E_n=[\frac{1}{n},\infty)$, we have*

$$|\varphi^2(x)b''_{n,k}(x)|\leq C\sum_{i=0}^{2}n^{1+\frac{i}{2}}b_{n,k}(x)|\frac{k-1-2x}{n}-x|^i\varphi^{-i}(x).$$

Proof Since

$$\begin{aligned}&b''_{n,k}(x)\\&=\frac{b_{n,k}(x)}{x^2(1+x)^2}[-(n+2)x(1+x)+(k-1-(n+2)x)^2-(1+2x)(k-1-(n+2)x)]\\&:=\frac{1}{x^2(1+x)^2}\sum_{i=0}^{2}[\frac{k-1}{n}-\frac{(n+2)x}{n}]^iQ_i(x)n^ib_{n,k}(x),\end{aligned}$$

where

$$Q_0(x)=-x(1+x)(n+2),\quad Q_1(x)=\frac{-nx(1+x)}{n+2},\quad Q_2(x)=\frac{n^2}{(n+2)^2},$$

on the other hand, for $i=0,1,2$, one has

$$|x^{-2}(1+x)^{-2}Q_i(x)n^i|\leq C\big(\frac{n}{x(1+x)}\big)^{1+\frac{i}{2}},$$

which completes the proof of Lemma 4.1. □

Lemma 4.2 *For $x\in E_n=[\frac{1}{n},\infty)$, $\varphi^2(x)=x(1+x)$, one has*

$$\int_{E_n}b_{n,k}(x)\varphi^{-2m}(x)(\frac{k}{n}-x)^{2m}\mathrm{d}x\leq\frac{C}{n^m},\quad m=0,1,2,\dots\ .$$

Proof We recall that $\int_{E_n}b_{n,k}(x)\mathrm{d}x\leq 1$, which is Lemma 4.2 for the case $m=0$.

We assume, using induction, that

$$\int_{E_n}b_{n,k}(x)\varphi^{-2l}(x)(\frac{k}{n}-x)^{2l}\mathrm{d}x\leq Cn^{-l}.$$

Now we shall show for the case $m=l+1$ that

$$\int_{E_n}b_{n,k}(x)\varphi^{-2(l+1)}(x)(\frac{k}{n}-x)^{2(l+1)}\mathrm{d}x\leq Cn^{-(l+1)}.\tag{4.1}$$

Using the integration by parts, we write

$$\begin{aligned}&\int_{E_n}\frac{b_{n,k}(x)}{x^l(1+x)^l}(\frac{k}{n}-x)^{2l}\mathrm{d}x=\frac{b_{n,k}(\frac{1}{n})}{(2l+1)(\frac{1}{n})^l(1+\frac{1}{n})^l}(\frac{k}{n}-\frac{1}{n})^{2l+1}+\\&\frac{2l+n+2}{2l+1}\int_{E_n}\frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}}(\frac{k}{n}-x)^{2l+1}(\frac{k-l-1}{2l+n+2}-x)\mathrm{d}x.\end{aligned}$$

By the assumption, one has

$$\begin{aligned}&\Big|\int_{E_n}\frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}}(\frac{k}{n}-x)^{2l+1}(\frac{k-l-1}{2l+n+2}-x)\mathrm{d}x\Big|\\&\quad\le C\big(n^{-l-1}+\frac{b_{n,k}(\frac{1}{n})}{n(\frac{1}{n})^l(1+\frac{1}{n})^l}(\frac{k}{n}-\frac{1}{n})^{2l+1}\big).\end{aligned}$$

Since $\frac{b_{n,k}(\frac{1}{n})}{(\frac{1}{n})^l(1+\frac{1}{n})^l}(\frac{k}{n}-\frac{1}{n})^{2l+1}\le Cn^{-l}$, we get

$$\Big|\int_{E_n}\frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}}(\frac{k}{n}-x)^{2l+1}(\frac{k-l-1}{2l+n+2}-x)\mathrm{d}x\Big|=O(n^{-l-1}).$$

Let

$$G_{n,k}=\big\{x:\frac{nk-2(l+1)(n+k)}{n(n+2l+2)}\le x\le\frac{k+1}{n}\big\}\subset E_n=[\frac{1}{n},\infty).$$

On $E_n\setminus G_{n,k}$, we have the following relations

$$(\frac{k}{n}-x)(\frac{k-l-1}{2l+n+2}-x)>0,\tag{4.2}$$

$$|\frac{k}{n}-x|\le C|\frac{k-l-1}{2l+n+2}-x|.\tag{4.3}$$

Next, we will prove (4.1) for the case $m=0,1,2,\dots$. Write

$$\int_{E_n}b_{n,k}(x)\varphi^{-2(l+1)}(x)|\frac{k}{n}-x|^{2(l+1)}\mathrm{d}x=\int_{E_n\setminus G_{n,k}}+\int_{G_{n,k}}:=G_1+G_2.$$

From the relations (4.2) and (4.3), we write

$$\begin{aligned}G_1&\le C\int_{E_n\setminus G_{n,k}}\frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}}(\frac{k}{n}-x)^{2l+1}(\frac{k-l-1}{2l+n+2}-x)\mathrm{d}x\\&\le C\big(\int_{E_n}+\int_{G_{n,k}}\big)=O(n^{-l-1})+G_3.\end{aligned}$$

On $G_{n,k}$, for $x\in G_{n,k}$, we have

$$|\frac{k}{n}-x|=O\big(\frac{1}{n}(1+\frac{k}{n})\big),\tag{4.4}$$

$$|\frac{k-l-1}{2l+n+2}-x|=O\big(\frac{1}{n}(1+\frac{k}{n})\big),\tag{4.5}$$

$$\varphi^2(x)\ge C\frac{k}{n}(1+\frac{k}{n}).\tag{4.6}$$

Since the methods to estimate G_2, G_3 are similar, we deal here only with G_2. By (4.4), (4.6),

$$\begin{aligned}G_2&=\int_{G_{n,k}}\frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}}|\frac{k}{n}-x|^{2l+2}\mathrm{d}x\\&\le C\int_0^\infty b_{n,k}(x)\big(\frac{k}{n}(1+\frac{k}{n})\big)^{-(l+1)}\cdot\big(\frac{1}{n}(1+\frac{k}{n})\big)^{2(l+1)}\mathrm{d}x\le Cn^{-l-1}.\end{aligned}$$

□

Remark In the estimate of G_3, we use (4.5). From the estimates of G_1, G_2, G_3, (4.1) is valid for the case $m=l+1$. By induction, Lemma 4.2 is true for the case $m=0,1,2,\dots$.

Lemma 4.3 *For $x \in E_n = [\frac{1}{n}, \infty)$, $m = 0, 1, 2$, we have*

$$\sum_{k=1}^{\infty} b_{n,k}(x)(\frac{k}{n} - x)^{2m} \leq C\frac{\varphi^{2m}(x)}{n^{m-1}}.$$

Proof From the following equalities

$$\begin{aligned}
\sum_{k=1}^{\infty} b_{n,k}(x)\frac{k}{n} =& \frac{(n+1)(n+2)}{n}x + \frac{n+1}{n},\\
\sum_{k=1}^{\infty} b_{n,k}(x)\frac{k^2}{n^2} =& \frac{(n+1)(n+2)(n+3)}{n^2}x^2 + 3\frac{(n+1)(n+2)}{n^2}x + \frac{n+1}{n^2},\\
\sum_{k=1}^{\infty} b_{n,k}(x)\frac{k^3}{n^3} =& \frac{(n+1)(n+2)(n+3)(n+4)}{n^3}x^3 + 6\frac{(n+1)(n+2)(n+3)}{n^3}x^2 +\\
& 7\frac{(n+1)(n+2)}{n^3}x + \frac{n+1}{n^3},\\
\sum_{k=1}^{\infty} b_{n,k}(x)\frac{k^4}{n^4} =& \frac{(n+1)\cdots(n+5)}{n^4}x^4 + 10\frac{(n+1)\cdots(n+4)}{n^4}x^3 +\\
& 25\frac{(n+1)(n+2)(n+3)}{n^4}x^2 + 15\frac{(n+1)(n+2)}{n^4}x + \frac{n+1}{n^4},
\end{aligned}$$

we can obtain Lemma 4.3. □

Lemma 4.4 *For $f \in L_p[0, \infty)$ $(1 \leq p \leq \infty)$, we have $\|B_n'(f, x)\|_p \leq Cn\|f\|_p$.*

Proof Noticing that $b_{n,k}'(t) = (n+1)(b_{n+1,k-1}(t)) - b_{n+1,k}(t))$,

$$\begin{aligned}
B_n'(f, x) =& \sum_{k=1}^{\infty} b_{n+1,k}(x)\int_0^{\infty}[b_{n,k+1}(t)) - b_{n,k}(t)]f(t)\mathrm{d}t\\
\leq& \sum_{k=1}^{\infty} b_{n+1,k}(x)\int_0^{\infty}[b_{n,k+1}(t)) + b_{n,k}(t)] \cdot |f(t)|\mathrm{d}t,
\end{aligned}$$

which implies Lemma 4.4. □

Lemma 4.5 *For f, $f' \in L_p[0, \infty)$ $(1 \leq p \leq \infty)$, we have $\|B_n'(f, x)\|_p \leq C\|f'\|_p$.*

Proof Recalling that $b_{n,k}'(t) = (n+1)(b_{n+1,k-1}(t)) - b_{n+1,k}(t))$, and $b_{n,0}(x) = 0$, one has the estimates for $\|B_n'(f, x)\|_p$,

$$\begin{aligned}
B_n'(f, x) =& \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k}'(x)\int_0^{\infty} b_{n,k}(t)f(t)\mathrm{d}t\\
\leq& \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n+1,k}(x)\int_0^{\infty} b_{n-1,k+1}(t))|f'(t)|\mathrm{d}t.
\end{aligned}$$

□

Proof of Theorem 1.2 Using the technique of Lemmas 4.1 and 4.2, for $x \in E_n$, one has

$$\sum_{k=1}^{\infty} b_{n,k}(x)(\frac{k-1-2x}{n} - x)^{2m} \leq C\frac{\varphi^{2m}(x)}{n^{m-1}}, \tag{4.7}$$

$$|xb_{n,k}(x)''| \le C\sum_{i=0}^{2} n^{1+\frac{i}{2}} b_{n,k}(x)x^{-\frac{i}{2}}\Big|\frac{k-1-2x}{n}-x\Big|^i(1+x)^{-1-\frac{i}{2}}, \tag{4.8}$$

$$\int_{E_n}\varphi^{-2m}(x)b_{n,k}(x)\Big(\frac{k-1-2x}{n}-x\Big)^{2m}\mathrm{d}x \le Cn^{-m}, \quad m=0,1,2,\dots\ . \tag{4.9}$$

To estimate $\|\varphi^2 B_n''f\|_p$, we write

$$\|\varphi^2 B_n''(f,x)\|_p \le \|xB_n''(f,x)\|_p + \|x^2B_n''(f,x)\|_p := F_1+F_2.$$

First we will estimate F_1.

$$F_1 \le \|xB_n''(f,x)\|_{p(E_n)} + \|x^2B_n''(f,x)\|_{p(E_n^c)} := F_{11}+F_{12}.$$

For the case $p=1$, from relations (4.8), (4.9) and Hölder inequality, and recalling that

$$\int_0^\infty \frac{b_{n,k}(x)}{(1+x)^2}\mathrm{d}x \le C,$$

we have

$$\begin{aligned} F_{11} &\le \frac{1}{n+1}\int_{E_n}\Big|x\sum_{k=1}^{\infty}b_{n,k}''(x)\int_0^\infty b_{n,k}(t)f(t)\mathrm{d}t\Big|\mathrm{d}x \\ &\le C\sum_{i=0}^{2} n^{\frac{i}{2}}(n^{-i})^{\frac{1}{2}}(n+1)\int_0^\infty |f(t)|\mathrm{d}t \le Cn\|f\|_1. \end{aligned}$$

Using the expression of $B_n''(f,x)$[13, Lemma 2.6], for $n\ge 4$, $x<\frac{1}{n}$, we have

$$\begin{aligned} F_{12} \le& \frac{2}{n}\int_{E_n^c}\Big|x\sum_{k=1}^{\infty}b_{n+2,k}(x)\int_0^\infty b_{n-2,k+2}''(t)f(t)\mathrm{d}t\Big|\mathrm{d}x \\ \le& \frac{2}{n^2}\int_{E_n^c}\sum_{k=1}^{\infty}b_{n+2,k}(x)\int_0^\infty [n(n-1)b_{n,k}(t)-2n(n-1)b_{n,k+1}(t)+ \\ & n(n-1)b_{n,k+2}(t)]|f(t)|\mathrm{d}t\mathrm{d}x \le Cn\|f\|_1. \end{aligned}$$

Thus, $F_1 \le Cn\|f\|_1$ for the case $p=1$.

For the case $p=\infty$, by (4.7), we write

$$\begin{aligned} F_{11} \le& C\|f\|_\infty\sum_{i=0}^{2} n^{\frac{i}{2}}\Big(\sum_{k=1}^{\infty}b_{n,k}(x)\Big(\frac{k-1-2x}{n}-x\Big)^{2i}\Big)^{\frac{1}{2}}\times \\ & \Big(\sum_{k=1}^{\infty}b_{n,k}(x)\Big)^{\frac{1}{2}}\varphi^{-i}(x)(1+x)^{-2} \le Cn\|f\|_\infty. \\ F_{12} \le& \frac{2}{n^2}\sum_{k=1}^{\infty}b_{n+2,k}(x)\int_0^\infty [n(n-1)b_{n,k}(t)-2n(n-1)b_{n,k+1}(t)+ \\ & n(n-1)b_{n,k+2}(t)]|f(t)|\mathrm{d}t \le Cn\|f\|_\infty. \end{aligned}$$

Thus, $F_1 \le Cn\|f\|_\infty$ for the case $p=\infty$.

Using the Riesz-Thorin Theorem, we obtain $F_1 \le Cn\|f\|_p$ for $1\le p\le\infty$.

Next we will estimate $F_2 = \|x^2B_n''(f,x)\|_p \le Cn\|f\|_p$:

For $x \in E_n^c$, noticing that $x^2 \leq x$, then F_2 can be written as follows

$$\begin{aligned}
F_2 \leq & \|x^2 B_n''(f,x)\|_{p(E_n)} + \|x^2 B_n''(f,x)\|_{p(E_n^c)} \\
\leq & \Big\| \frac{x^2}{n+1} \sum_{k=1}^{\infty} b_{n,k}''(x) \int_{E_n} b_{n,k}(t) f(t) \mathrm{d}t \Big\|_{p(E_n)} + \\
& \Big\| \frac{x^2}{n+1} \sum_{k=1}^{\infty} b_{n,k}''(x) \int_{E_n^c} b_{n,k}(t) f(t) \mathrm{d}t \Big\|_{p(E_n)} + F_{12} \\
:= & F_{21} + F_{22} + F_{12}.
\end{aligned}$$

Again using the expression of $B_n''(f,x)$[13, Lemma 2.6] and Lemma 4.1, we have

$$\begin{aligned}
F_{21} \leq & C \Big\| \frac{x^2(n+2)}{n(n-1)} \sum_{k=1}^{\infty} b_{n+2,k}(x) \int_{E_n} b_{n-2,k+2}''(t) f(t) \mathrm{d}t \Big\|_{p(E_n)} \\
\leq & C \Big\{ \| \sum_{k=1}^{\infty} b_{n,k+2}(x) \int_{E_n} \sum_{i=0}^{2} n^{\frac{i}{2}} b_{n,k+1}(t) \varphi^{-i}(t) | \frac{k+1}{n} - t|^i \cdot |f(t)| \mathrm{d}t \|_{p(E_n)} + \\
& \| \sum_{k=1}^{\infty} b_{n,k+2}(x) \int_{E_n} \sum_{i=0}^{2} n^{\frac{i}{2}} b_{n,k+1}(t) \varphi^{-i}(t) (\frac{2(k+1)}{n(n-2)})^i \cdot |f(t)| \mathrm{d}t \|_{p(E_n)} + \\
& \| \sum_{k=1}^{\infty} b_{n,k+2}(x) \int_{E_n} \sum_{i=0}^{2} n^{\frac{i}{2}} b_{n,k+1}(t) \varphi^{-i}(t) (\frac{2t}{n-2})^i \cdot |f(t)| \mathrm{d}t \|_{p(E_n)} \Big\} \\
:= & C\{E_1 + E_2 + E_3\}.
\end{aligned}$$

By (4.8), one has

$$\begin{aligned}
F_{22} \leq & C \Big\| \sum_{i=0}^{2} n^{\frac{i}{2}} \sum_{k=1}^{\infty} b_{n-1,k+1}(x) \varphi^{-i}(x) | \frac{k+1}{n} - x|^i \int_{E_n^c} [b_{n+1,k-1}(t) + b_{n,k}(t)] f(t) \mathrm{d}t \Big\|_{p(E_n)} + \\
& C \Big\| \sum_{i=0}^{2} n^{\frac{i}{2}} \sum_{k=1}^{\infty} b_{n-1,k+1}(x) \varphi^{-i}(x) (\frac{2x}{n})^i \int_{E_n^c} [b_{n+1,k-1}(t) + b_{n,k}(t)] f(t) \mathrm{d}t \Big\|_{p(E_n)} + \\
& C \Big\| \sum_{i=0}^{2} n^{\frac{i}{2}} \sum_{k=1}^{\infty} b_{n-1,k+1}(x) \varphi^{-i}(x) (\frac{2}{n})^i \int_{E_n^c} [b_{n+1,k-1}(t) + b_{n,k}(t)] f(t) \mathrm{d}t \Big\|_{p(E_n)} \\
:= & E_4 + E_5 + E_6.
\end{aligned}$$

Since the treatments for $E_1 \sim E_6$ are similar, we deal here only with E_1.

$$E_1 = \| \sum_{k=1}^{\infty} b_{n,k+2}(x) \int_{E_n} \sum_{i=0}^{2} n^{\frac{i}{2}} b_{n,k+1}(t) \varphi^{-i}(t) | \frac{k+1}{n} - t|^i |f(t)| \mathrm{d}t \|_{p(E_n)}.$$

By Hölder inequality, we discuss two cases $p = 1$ and $p = \infty$.

First, we will consider the caes $p = \infty$.

$$\begin{aligned}
E_1 \leq & C \|f\|_{\infty} \sum_{i=0}^{2} n^{\frac{i}{2}} \sum_{k=1}^{\infty} b_{n,k+2}(x) \Big\{ \int_{E_n} b_{n,k+1}(t) \varphi^{-2i}(t) | \frac{k+1}{n} - t|^{2i} \mathrm{d}t \Big\}^{\frac{1}{2}} \times \\
& \Big\{ \int_{E_n} b_{n,k+1}(t) \mathrm{d}t \Big\}^{\frac{1}{2}} \leq Cn \|f\|_{\infty}.
\end{aligned}$$

Next, for $p=1$, by [13, Lemma 2.1], and Hölder inequality, one has

$$\begin{aligned}E_1 &\le \int_{E_n}\sum_{i=0}^{2} n^{\frac{i}{2}}\sum_{k=1}^{\infty} b_{n,k+2}(x)b_{n,k+1}(t)\varphi^{-i}(t)|\frac{k+1}{n}-t|^i\cdot|f(t)|\mathrm{d}t\mathrm{d}x\\ &\le \sum_{i=0}^{2} n^{\frac{i}{2}}\big(\sum_{k=1}^{\infty} b_{n,k+1}(t)\varphi^{-2i}(t)\big)^{\frac{1}{2}}\cdot\big(\sum_{k=1}^{\infty} b_{n,k+1}(t)|\frac{k+1}{n}-t|^{2i}\big)^{\frac{1}{2}}\cdot\int_{E_n}|f(t)|\mathrm{d}t\\ &\le Cn\|f\|_1.\end{aligned}$$

Therefore, we have $1\le p\le\infty$, $F_2\le Cn\|f\|_p$.

Proof of Theorem 1.3 We will prove 1° first.

By the definition of K-functional, we can choose $g=B_n(f,x)$ such that

$$K^2_\varphi(f,t^2)_p\le\|f(x)-B_n(f,x)\|_p+t^2\|\varphi^2B''_n(f,x)\|_p.$$

The condition $\|B_n(f,x)-f(x)\|_p=O(n^{-\frac{\alpha}{2}})$, Theorem 1.2 and Lemma 2.6 imply

$$\begin{aligned}K^2_\varphi(f,t^2)_p &\le Cn^{-\frac{\alpha}{2}}+t^2\{\|\varphi^2B''_n(f-g,x)\|_p+\|\varphi^2B''_n(g,x)\|_p\}\\ &\le Cn^{-\frac{\alpha}{2}}+t^2\{n\|f-g\|_p+\|\varphi^2g''\|_p\}\le Cn^{-\frac{\alpha}{2}}+t^2nK^2_\varphi(f,\frac{1}{n})_p.\end{aligned}$$

By the well-known Berens-Lorentz Lemma[5, p 122], we have

$$K^2_\varphi(f,t^2)_p\le Ct^\alpha.$$

Combining the equivalence relation (1.2), one has $\omega^2_\varphi(f,t)_p=O(t^\alpha)$.

Next we will prove 2°. We introduce a new K-functional:

$$K_1(f,t)_p=\inf_{g\in\overline{D}_1}\{\|f-g\|_p+t\|g'\|_p\},$$

where $\overline{D}_1=\{g|\ \ g,g'\in L_p[0,\infty)\}$.

By the definition of the new K-functional, for $t=\frac{1}{n}$, we can choose $g\in\overline{D}_1$ such that

$$\|f-g\|_p+\frac{1}{n}\|g'\|_p\le CK_1(f,\frac{1}{n})_p.$$

On the other hand, by Lemmas 4.4 and 4.5, one has

$$\begin{aligned}K_1(f,t)_p &\le\|f-B_nf\|_p+t\|B'_nf\|_p\\ &\le Cn^{-\frac{\alpha}{2}}+t\{n\|f-g\|_p+\|g'\|_p\}\le Cn^{-\frac{\alpha}{2}}+tnK_1(f,\frac{1}{n})_p.\end{aligned}$$

Again using Berens-Lorentz Lemma[5, Lemma 9.3.4], we have $K_1(f,t)_p\le Ct^{\frac{\alpha}{2}}$.

Finally we will estimate $\omega_1(f,t)_p=O(t^{\frac{\alpha}{2}})$. Write

$$\|\Delta^1_tf(x)\|_p\le\|\Delta^1_t(f-g)\|_p+\|\Delta^1_tg\|_p:=T_1+T_2,$$

where $T_1=\|(f-g)(x+\frac{t}{2})-(f-g)(x-\frac{t}{2})\|_p\le2\|f-g\|_p$.

Using Hölder inequality, one has

$$T_2=\big\|\int_{-\frac{t}{2}}^{\frac{t}{2}}g'(x+u)\mathrm{d}u\big\|_p=\big\{\int_0^\infty|\int_{-\frac{t}{2}}^{\frac{t}{2}}g'(x+u)\mathrm{d}u\big|^p\mathrm{d}x\big\}^{\frac{1}{p}}$$

$$\leq \Big\{ \int_0^{\infty} \Big(\int_{-\frac{t}{2}}^{\frac{t}{2}} |g'(x+u)|^p \mathrm{d}u \Big) \cdot \Big(\int_{-\frac{t}{2}}^{\frac{t}{2}} \mathrm{d}u \Big)^{\frac{p}{q}} \mathrm{d}x \Big\}^{\frac{1}{p}} \leq Ct \|g'\|_p.$$

Therefore, $\|\Delta_t^1 f(x)\|_p \leq 2\|f-g\|_p + Ct\|g'\|_p \leq CK_1(f,t)_p \leq Ct^{\frac{\alpha}{2}}$, which implies $\omega_1(f,t)_p = O(t^{\frac{\alpha}{2}})$. □

References

[1] UPRETI R. *Approximation properties of beta operators* [J]. J. Approx. Theory, 1985, **45**(2): 85–89.

[2] UPRETI R. *On sequence-to-sequence transformations* [J]. India J. Pure Appl. Math., 1982, **13**(4): 454–457.

[3] ZHOU Dingxuan. *L_p-inverse theorem for Beta operators* [J]. J. Approx. Theory, 1991, **66**(3): 279–287.

[4] XIE Linsen. *L_p-approximation by beta operators* [J]. J. Math. Res. Exposition, Suppl., 1999, **19**: 258–260. (in Chinese)

[5] DITZIAN Z, TOTIK V. *Moduli of Smoothness* [M]. Springer-Verlag, New York, 1987.

[6] SRIVASTAVA H M, GUPTA V. *A certain family of summation-integral type operators* [J]. Math. Comput. Modelling, 2003, **37**(12-13): 1307–1315.

[7] GUPTA M K, VASISHTHA V. *The iterative combinations of a new sequence of linear positive operators* [J]. Math. Comput. Modelling, 2004, **39**(4-5): 521–527.

[8] GUPTA V, MOHAPATRA R. N, FINTA Z. *A certain family of mixed summation-integral type operators* [J]. Math. Comput. Modelling, 2005, **42**(1-2): 181–191.

[9] GUPTA V, LUPAS A. *Direct results for mixed Beta-Szász type operators* [J]. Gen. Math., 2005, **13**(2): 83–94.

[10] GUPTA V, NOOR M A. *Convergence of derivatives for certain mixed Szasz-beta operators* [J]. J. Math. Anal. Appl., 2006, **321**(1): 1–9.

[11] GOVIL N K, GUPTA V. *Simultaneous approximation for the Bézier variant of Baskakov-beta operators* [J]. Math. Comput. Modelling, 2006, **44**(11-12): 1153–1159.

[12] GUPTA V. *Error estimation for mixed summation-integral type operators* [J]. J. Math. Anal. Appl., 2006, **313**(2): 632–641.

[13] GUPTA M K, GUPTA V, KUMAR M. *Direct and inverse estimates for a new family of linear positive operators* [J]. J. Math. Anal. Appl., 2007, **330**(2): 799–816.

[14] FINTA Z, GOVIL N K, GUPTA V. *Some results on modified Szász-Mirakjan operators* [J]. J. Math. Anal. Appl., 2007, **327**(2): 1284–1296.

[15] GUPTA V, AGRAWAL P N. *Inverse theorem in simultaneous approximation by Szász-Durrmeyer operators* [J]. J. Indian Math. Soc. (N.S.), 1997, **63**(1-4): 99–113.

[16] GUPTA V, SRIVASTAVA G S. *Convergence of derivatives by summation-integral type operators* [J]. Rev. Colombiana Mat., 1995, **29**(1): 1–11.

[17] ZENG Xiaoming. *On the rate of convergence of the generalized Szász type operators for functions of bounded variation* [J]. J. Math. Anal. Appl., 1998, **226**(2): 309–325.

[18] GUPTA V, GUPTA M K, VASISHTHA V. *Simultaneous approximation by summation-integral type operators* [J]. Nonlinear Funct. Anal. Appl., 2003, **8**(3): 399–412.

[19] TOTIK V. *Some properties of a new kind of modulus of smoothness* [J]. Z. Anal. Anwendungen, 1984, **3**(2): 167–178.

[20] DITZIAN Z, TOTIK V. *K-functionals and weighted moduli of smoothness* [J]. J. Approx. Theory, 1990, **63**(1): 3–29.