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A Topological Minimax Theorem Involving Two Functions

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Abstract A topological minimax theorem involving two functions is derived. It generalizes Greco and Horvath'minimax theorem given by them in 2002.

Keywords minimax theorem; two functions; connectedness; a-b lower interconnected function about f finitely; a-b weakly lower interconnected function about f finitely; lower topologically concave function pair.

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1. Introduction

Greco and Horvath[1] introduced a minimax theorem in topological spaces. This paper gives a two-function minimax theorem which generalizes Greco and Horvath' minimax theorem.

We will introduce some notations and definitions, and recall Greco and Horvath' minimax theorem.

Throughout this paper, we always suppose that $f: X\times Y\to \bar{R}$ and $g: X\times Y\to \bar{R}$ are two functions defined on the product of two nonempty topological spaces and taking values in $\bar{R}:=R\bigcup\{\pm\infty\}$ with $f\le g$. For any real number μ and any $x\in X$, we denote $\{f\le\mu\}x:=\{y: f(x,y)\le\mu\}$.

Definition 1.1[1] *A function f is said to be lower interconnected on Y if, for any pair of real numbers μ and λ with $\mu<\lambda$ and any $\bar{x}\in X$ with $\{f<\mu\}\bar{x}\neq\emptyset$, there exists a neighborhood V of $\bar{x}$ such that, for every $x\in V$, there is a connected subset D of Y satisfying*

$$\{f<\mu\}x\subset D\subset\{f\le\lambda\}\bar{x}\bigcup\{f\le\lambda\}x \text{ and } D\bigcap\{f\le\lambda\}\bar{x}\neq\emptyset.$$

Remark 1.1 If f is lower semicontinuous on Y, and $\{f\le\lambda\}\bar{x}$ and $\{f\le\lambda\}x$ are close. If both of them are nonempty and $\{f<\mu\}x$ is also nonempety, then the intersection of them is nonempty by the connectedness of D.

Definition 1.2 *A function g is said to be lower interconnected on Y about f finitely if, for any set of real numbers $\{\mu,\lambda,\lambda_0\}$ with $\mu<\lambda\le\lambda_0$, any $\bar{x}\in X$, and every finite subset (possible*

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empty) $A = \{x_1, x_2, \ldots, x_n\} \subset X$ *with* $\{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i) \neq \emptyset$*, there exists a neighborhood* V *of* $\bar{x}$ *such that, for every* $x \in V$*, there is a connected subset* D *of* Y *satisfying*

$$\{g < \mu\}x \bigcap (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i) \subset D \subset \{g \leq \lambda\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i)$$

$$\bigcup \{g \leq \lambda\}x \bigcap (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i)$$

$$D \bigcap \{g \leq \lambda\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i) \neq \emptyset.$$

Definition 1.3 *A function* g *is said to be weakly lower interconnected on* Y *about* f *finitely if, for any set of real numbers* $\{\mu, \lambda, \lambda_0\}$ *with* $\mu < \lambda \leq \lambda_0$*, any* $\bar{x} \in X$*, and every finite subset (possible empty)* $A = \{x_1, x_2, \ldots, x_n\} \subset X$ *with* $\{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i) \neq \emptyset$*, there exists a neighborhood* V *of* $\bar{x}$ *such that, for every* $x \in V$*, there is a connected subset* D *of* Y *satisfying*

$$\{g < \mu\}x \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i) \subset D \subset \{g \leq \lambda\}\bar{x} (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i)$$

$$\bigcup \{g \leq \lambda\}x \bigcap (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i)$$

$$D \bigcap \{g \leq \lambda\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i) \neq \emptyset.$$

Remark 1.2 Since $\bigcap_{x_i \in A} \{f < \mu\} x_i = \bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i = Y$ when $A = \emptyset$, g is weakly lower interconnected on Y about f finitely implies that g is lower interconnected on Y.

Remark 1.3 Obviously, g is lower interconnected on Y about f finitely implies that it is weakly lower interconnected on Y about f finitely.

Definition 1.4 *A function* g *is said to be weakly lower connected on* Y *about* f *finitely if, for any set of real numbers* $\{\mu, \lambda, \lambda_0\}$ *with* $\mu < \lambda \leq \lambda_0$*, any* $\bar{x} \in X$*, and every finite subset (possible empty)* $A = \{x_1, x_2, \ldots, x_n\} \subset X$*, there is a connected subset* D *of* Y *satisfying*

$$\{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i) \subset D \subset \{g \leq \lambda\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f \leq \lambda_0\} x_i).$$

If $\{f < \mu\}$ is replaced by $\{f \leq \lambda_0\}$, then one obtains the definition of lower connected on Y about f finitely.

If $A = \emptyset$, then we obtain the definition of lower connected on Y[1].

Remark 1.4 Obviously, g is weakly lower interconnected on Y about f finitely implies that it is weakly lower connected on Y about f finitely.

Remark 1.5 If for real numbers a, b with $a < b$ we deal with μ, λ, λ_0 in the above definitions satisfying $a < \mu < \lambda < b$ and $a < \lambda_0 \leq b$, we can get the definitions about $a - b$ lower interconnected on Y, and $a - b$ (weakly) lower connected (interconnected) on Y about f finitely.

Definition 1.5 *A pair of two functions* (f, g) *with* $f \leq g$ *is said to be lower topologically concave between two points* x_1 *and* x_2 *of* X *if, for any pair of real numbers* μ *and* λ *with* $\mu < \lambda$ *there is a connected subset* C *of* X *such that* $\{x_1, x_2\} \subset C$ *and* $\forall x \in C$, $\{g < \mu\}x \subset \{g \leq \lambda\}x_1 \bigcup \{f \leq \lambda\}x_2$.

Moreover, (f,g) is said to be lower topologically concave on X if it is lower topologically concave between any two points of X.

Definition 1.6[1] *A function f is said to be finitely lower interconnected on Y if, for any real numbers μ,λ with $\mu<\lambda$, any $\bar{x}\in X$, and every finite subset $A=\{x_1,x_2,\ldots,x_n\}\subset X$ with $\{f<\mu\}\bar{x}\bigcap(\bigcap_{x_i\in A}\{f<\mu\}x_i)\neq\emptyset$, there exists a neighborhood V of $\bar{x}$ such that, for every $x\in V$, there is a connected subset D of Y satisfying*

$$\{f<\mu\}x\bigcap(\bigcap_{x_i\in A}\{f<\mu\}x_i)\subset D\subset\{f\leq\lambda\}\bar{x}\bigcap(\bigcap_{x_i\in A}\{f\leq\lambda\}x_i)$$
$$\bigcup\{f\leq\lambda\}x\bigcap(\bigcap_{x_i\in A}\{f\leq\lambda\}x_i)$$
$$D\bigcap\{f\leq\lambda\}\bar{x}\bigcap(\bigcap_{x_i\in A}\{f\leq\lambda\}x_i)\neq\emptyset.$$

Definition 1.7[1] *A function f is said to be finitely lower connected on Y if, for every $x\in X$, any pair of real numbers μ,λ with $\mu<\lambda$, and every finite subset $A=\{x_1,x_2,\ldots,x_n\}\subset X$, there exists a connected subset D of Y satisfying*

$$\{f<\mu\}x\bigcap(\bigcap_{x_i\in A}\{f<\mu\}x_i)\subset D\subset\{f\leq\lambda\}x(\bigcap_{x_i\in A}\{f\leq\lambda\}x_i).$$

Definition 1.8 *A pair of two functions (f,g) with $f\leq g$ is said to be finitely lower connected on Y if, for every $x\in X$, any pair of real numbers μ,λ with $\mu<\lambda$, and every finite subset $A=\{x_1,x_2,\ldots,x_n\}\subset X$, there exists a connected subset D of Y satisfying*

$$\{g<\mu\}x\bigcap(\bigcap_{x_i\in A}\{g<\mu\}x_i)\subset D\subset\{f\leq\lambda\}x\bigcap(\bigcap_{x_i\in A}\{f\leq\lambda\}x_i).$$

Let A be every subset of X, one gets the definitions of arbitrarily lower connected on Y.

Symmetrically, we can give the definitions about upper topologically concave on Y, (weakly) upper interconnected (connected) on X, (weakly) upper interconnected (connected) on X about g finitely and so on.

For example:

A function f is said to be arbitrarily upper connected on X if, $-f$ is arbitrarily lower connected on X.

A pair of two functions (f,g) with $f\leq g$ is said to be upper topologically concave between two points y_1 and y_2 of Y if, $(-g,-f)$ is lower topologically concave between two points y_1 and y_2 of Y.

A function f is said to be weakly upper interconnected on X about g finitely if $-f$ is weakly lower interconnected on X about $-g$.

Lemma 1.1[1] *A function g is lower connected on Y if and only if, for any real number μ and any $x\in X$ the subset $\{g<\mu\}x$ of Y is connected.*

Remark 1.6 Checking the proof of Lemma 1.1 in [1], we have that g is $a-b$ lower connected on Y if and only if, for any real number μ with $a<\mu<b$ and any $x\in X$ the subset $\{g<\mu\}x$

of Y is connected.

Theorem 1.2[1] *Let either X or Y be connected, and let Y be compact. A function f is a minimax function ($\inf_Y \sup_X f = \sup_X \inf_Y f$), if the following conditions are satisfied:*

1) f is lower semicontinuous on Y; 2) f is arbitrarily upper connected on X;

3) f is finitely lower interconnected on Y.

2. Main results

Lemma 2.1 *Let X be connected and f, g be two functions with $f \leq g$. If there is a real number a satisfying*

1) For any real number $\lambda > a$ and any $x \in X$, $\{g < \lambda\}x \neq \emptyset$;

2) There exists a real number $b > a$, such that g is a-b lower interconnected on Y;

3) f and g are lower semicontinuous on Y;

4) (f, g) is lower topologically concave on X.

Then for any real number λ with $\lambda > a$ and any $x_1, x_2 \in X$, $\{g \leq \lambda\}x_1 \bigcap \{f \leq \lambda\}x_2 \neq \emptyset$.

Proof We only need to show that for any λ with $a < \lambda < b$, $\{g \leq \lambda\}x_1 \bigcap \{f \leq \lambda\}x_2 \neq \emptyset$.

Suppose that there exist $x_1, x_2 \in X$ and $b > \lambda > a$ such that $\{g \leq \lambda\}x_1 \bigcap \{f \leq \lambda\}x_2 = \emptyset$. By condition 4) for μ with $a < \mu < \lambda$ there is a connected subset C of X such that $\{x_1, x_2\} \subset C$ and $\forall x \in C$, $\{g < \mu\}x \subset \{g \leq \lambda\}x_1 \bigcup \{f \leq \lambda\}x_2$, where both $\{g \leq \lambda\}x_1$ and $\{f \leq \lambda\}x_2$ are close sets according to the lower semicontinuity of f and g. Consider the following sets: $A_1 = \{x \in C : \{g < \mu\}x \subset \{g \leq \lambda\}x_1\}$. $A_2 = \{x \in C : \{g < \mu\}x \subset \{f \leq \lambda\}x_2\}$. Obviously $x_i \in A_i$, $A_i \neq \emptyset$, where $i = 1, 2$. Since $\forall x \in C$, $\{g < \mu\}x \subset \{g \leq \lambda\}x_1 \bigcup \{f \leq \lambda\}x_2$ and $\{g < \mu\}x$ is connected by 2) and Lemma 1.1, we have that $\{g < \mu\}x \subset \{g \leq \lambda\}x_1$, or $\{g < \mu\}x \subset \{f \leq \lambda\}x_2$. This shows that $C = A_1 \bigcup A_2$. $C \bigcap \overline{A_1} \bigcap \overline{A_2} \neq \emptyset$ since C is connected. Pick a point $\bar{x}$ in it, we can assume that $\bar{x} \in A_1 \bigcap \overline{A_2}$. By 2) for μ_1 and μ_2 with $a < \mu_1 < \mu_2 < \mu$, there exists a neighborhood V of $\bar{x}$ such that, for every $x \in V$, there is a connected subset D of Y satisfying $\{g < \mu_1\}x \subset D \subset \{g \leq \mu_2\}\bar{x} \bigcup \{g \leq \mu_2\}x$ and $D \bigcap \{g \leq \mu_2\}\bar{x} \neq \emptyset$. Pick a point $\tilde{x} \in V \bigcap A_2$, we have that $\{g \leq \mu_2\}\bar{x} \subset \{g < \mu\}\bar{x} \subset \{g \leq \lambda\}x_1$, $\{g \leq \mu_2\}\tilde{x} \subset \{g < \mu\}\tilde{x} \subset \{f \leq \lambda\}x_2$. By the connectedness of D, $\{g \leq \mu_2\}\tilde{x} \bigcap \{g \leq \mu_2\}\bar{x} \neq \emptyset$. This implies $\{g \leq \lambda\}x_1 \bigcap \{f \leq \lambda\}x_2 \neq \emptyset$, thus, we have reached a contradiction. $\square$

Remark 2.1 If 2) is replaced by the following condition:

5) For any pair of real numbers μ and λ with $a < \mu < \lambda < b$ and any $\bar{x} \in X$ with $\{g < \mu\}\bar{x} \bigcap E_\mu \neq \emptyset$, there exists a neighborhood V of $\bar{x}$ such that, for every $x \in V$, there is a connected subset D of Y satisfying

$$\{g < \mu\}x \bigcap E_\mu \subset D \subset \{g \leq \lambda\}\bar{x} \bigcup \{g \leq \lambda\}x \;\text{ and }\; D \bigcap \{g \leq \lambda\}\bar{x} \neq \emptyset,\; \{g < \mu\}x \bigcap E_\mu \neq \emptyset.$$

E_μ is a subset of X relatived with μ. Then the conclution of Lemma 2.1 is still true (Checking the proof above and let $A_1 = \{x \in C : \{g < \mu\}x \bigcap E_\mu \subset \{g \leq \lambda\}x_1\}$. $A_2 = \{x \in C : \{g < \mu\}x \bigcap E_\mu \subset \{f \leq \lambda\}x_2\}$, one can get the conclusion).

Lemma 2.2 *A function g is weakly lower connected on Y about f finitely if and only if, for any* *real number* μ, *any* $\bar{x} \in X$, *and every finite subset (possible empty)* $A = \{x_1, x_2, \ldots, x_n\} \subset X$, $\{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i)$ *is connected.*

Proof We only need to show that if g is weakly lower connected on Y about f finitely, then $\{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i)$ is connected. The other part of this claim is obvious.

Take an increasing real number sequence $\{\mu_k\}$ with $\mu_k \to \mu$, $k \to \infty$. By the definition of weakly lower connected, there exists a sequence $\{D_k\}$ of connected sets of Y such that $\{g < \mu_1\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu_1\} x_i) \subset \{g < \mu_k\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu_k\} x_i) \subset D_k \subset \{g \leq \mu_{k+1}\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f \leq \mu_{k+1}\} x_i) \subset \{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i)$.

Therefore, $\{D_{2k-1}\}$ is an increasing sequence and $\{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i) = \bigcup_{k=1}^{\infty} D_{2k-1}$ is connected.

Indeed, it is easy to see that $\{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i) \supset \bigcup_{k=1}^{\infty} D_{2k-1}$. We will show that $\{g < \mu\}\bar{x} \bigcap (\bigcap_{x_i \in A} \{f < \mu\} x_i) \subset \bigcup_{k=1}^{\infty} D_{2k-1}$. If not so, there exists a point $y \in Y$ such that for all x_i, $f(x_i, y) < \mu$ and $g(\bar{x}, y) < \mu$ but for all k, $g(\bar{x}, y) \geq \mu_{2k-1}$ or $f(x_{i_k}, y) \geq \mu_{2k-1}$, for some $x_{i_k} \in A$. Since A is a finite set and $\mu_k \to \mu$ $(k \to \infty)$, we can take k_0 such that $\mu_{2k_0-1} > \max\{g(\bar{x}, y), f(x_1, y), \ldots, f(x_n, y)\}$. This is a contradiction. □

Theorem 2.1 *Let X be connected, Y be compact and f, g be two functions with $f \leq g$. Suppose that*

1) g is weakly lower interconnected on Y about f finitely;

2) f and g are lower semicontinuous on Y;

3) (f, g) is lower topologically concave on X.

Then $\inf_Y \sup_X f \leq \sup_X \inf_Y g$.

Proof Let $a = \sup_X \inf_Y g$. We can assume $a \neq +\infty$.

Firstly, we will show that $\bigcap_{x \in X} \{f \leq r\} x \neq \emptyset$, for any $r > a$.

Since Y is compact and f is lower semicontinuous on Y, it suffices to prove that for any finite subset A of X, $\bigcap_{x \in A} \{f \leq r\} x \neq \emptyset$. For this, it suffices to show that for any $x \in X$ and any finite subset $A = \{x_1, x_2, \ldots, x_n\} \subset X$ we have

$$\{g \leq r\} x \bigcap (\bigcap_{x_i \in A} \{f \leq r\} x_i) \neq \emptyset, \quad \text{for any} \quad r > a. \tag{1}$$

When $n = 1$ it is the conclusion of Lemma 2.1. Suppose that for any $x \in X$ and any $A = \{x_2, \ldots, x_n\} \subset X$ with card $A = n - 1$, (1) is right. But for some $r_0 > a$ and some $B = \{x_1, x_2, \ldots, x_n\} \subset X$ with card $B = n$, some $x_0 \in X$, $\{g \leq r_0\} x_0 \bigcap (\bigcap_{x_i \in B} \{f \leq r_0\} x_i) = \emptyset$. Let $\bar{Y} = \bigcap_{i=2}^{n} \{f \leq r_0\} x_i$. Then it is compact and nonempty. The conditions 3) and 4) of Lemma 2.1 are satisfied by $f|_{X \times \bar{Y}}$ and $g|_{X \times \bar{Y}}$. Now we show that the function $g|_{X \times \bar{Y}}$ also satisfies the conditions 1) and 5) in Lemma 2.1 and Remark 2.2 for a and $b = r_0$.

By the hypothesis of induction, it follows that $\emptyset \neq \{g \leq \bar{r}\} x \bigcap (\bigcap_{i=2}^{n} \{f \leq \bar{r}\} x_i) \subset \{g < r\} x \bigcap \bar{Y} = \{g|_{X \times \bar{Y}} < r\} x$ for any $x \in X$ and any $r > a$. Here $a < \bar{r} < \min\{r, r_0\}$, so the condition 1) of Lemma 2.1 is satisfied by $g|_{X \times \bar{Y}}$.

For any $\mu > a$, let $E_\mu = \bigcap_{i=2}^{n} \{f < \mu\} x_i$ and $a < \mu_0 < \mu$. By the hypothesis of induction, for any $x \in X$, $\emptyset \neq \{g \leq \mu_0\} x \bigcap (\bigcap_{i=2}^{n} \{f \leq \mu_0\} x_i) \subset \{g < \mu\} x \bigcap E_\mu$. According to 1), we have that for any pair of real numbers μ and λ with $a < \mu < \lambda < b = r_0$ and any $\bar{x} \in X$, there exists a neighborhood V of $\bar{x}$ such that, for every $x \in V$, there is a connected subset D of Y satisfying

$$\{g < \mu\}x \bigcap E_\mu = \{g < \mu\}x \bigcap E_\mu \bigcap \bar{Y} \subset D \subset \{g \leq \lambda\}\bar{x} \bigcap \bar{Y} \bigcup \{g \leq \lambda\}x \bigcap \bar{Y}$$

$$D \bigcap \{g \leq \lambda\}\bar{x} \bigcap \bar{Y} \neq \emptyset.$$

By Lemma 2.2, $\{g < \mu\}x \bigcap E_\mu \neq \emptyset$ is connected. Hence the condition 5) in Remark 2.2 is satisfied by $g|_{X \times \bar{Y}}$ for a and $b = r_0$.

Therefore by Remark 2.2 we have that

$$\{g \leq r_0\}x_0 \bigcap (\{f \leq r_0\}x_1 \bigcap \bar{Y}) = \{g \leq r_0\}x_0 \bigcap (\bigcap_{i=1}^{n} \{f \leq r_0\}x_i) \neq \emptyset.$$

This contradicts our assumption.

So for any $x \in X$ and any finite subset $A = \{x_1, x_2, \ldots, x_n\} \subset X$ we have

$$\{g \leq r\}x \bigcap (\bigcap_{x_i \in A} \{f \leq r\}x_i) \neq \emptyset, \text{ for any } r > a.$$

Secondly, we will prove $\inf_Y \sup_X f \leq \sup_X \inf_Y g$.

If $\inf_Y \sup_X f > \sup_X \inf_Y g = a$, take a real number $\epsilon > 0$, such that $r = \inf_Y \sup_X f - \epsilon > a$. According to the first part, there exists a point y_0 such that for any $x \in X$, $f(x, y_0) \leq r$. It follows that $\inf_Y \sup_X f \leq r$. This contradicts the definition of r.

This completes the proof of the theorem. □

When X is connected, f is arbitrarily upper connected on X if and only if (f, f) is topologically concave on X[1] and f is finitely lower interconnected on Y if and only if f is weakly lower interconnected on Y about f finitely. Therefore, when $f = g$ in Theorem 2.3, we obtain the case of Theorem 1.2, in which X is connected. Then we can obtain the other case of Theorem 1.2 in which Y is connected by the Lemma 4.2 of [1]. So Theorem 2.1 generalizes Theorem 1.2.

From Remark 1.3 we have the following corollary.

Corollary 2.1 *Let X be connected, Y be compact and f, g be two functions with $f \leq g$. Suppose that the following conditions are satisfied:*

1) g is lower interconnected on Y about f finitely;

2) f and g are lower semicontinuous on Y;

3) (f, g) is lower topologically concave on X.

Then $\inf_Y \sup_X f \leq \sup_X \inf_Y g$.

3. Comments on conditions in Theorem 2.1

Proposition 3.1 *Suppose X, Y are two topological spaces and f, g are two functions with $f \leq g$.*

1) If for any $B \subset Y$ and any pair of real numbers μ and λ with $\mu < \lambda$, there is a connected set $C \subset X$ such that

$$\{x \in X : \{g \leq \lambda\}x \subset B\} \subset C \subset \{x \in X : \{g < \mu\}x \subset B\}, \tag{2}$$

then (f, g) is lower topologically concave on X.

2) If (f,g) is lower topologically concave on X, then for any $B \subset Y$ and any pair of real numbers μ and λ with $\mu < \lambda$, there is a connected set $C \subset X$ such that

$$\{x \in X : \{f \le \lambda\}x \subset B\} \subset C \subset \{x \in X : \{g < \mu\}x \subset B\}. \tag{3}$$

Proof 1) Suppose that for any $B \subset Y$ and any pair of real numbers μ and λ with $\mu < \lambda$, there is a connected subset $C \subset X$ such that (2) is right. Now for any $x_1, x_2 \in X$ and any pair of real numbers μ and λ, let $B = \{g \le \lambda\}x_1 \bigcup \{f \le \lambda\}x_2 \subset Y$. There is a connected subset $C \subset X$ such that (2) is right. Obviously, $\{g < \mu\}(C) \subset B = \{g \le \lambda\}x_1 \bigcup \{f \le \lambda\}x_2$. Since $\{g \le \lambda\}x_1 \subset B$ and $\{g \le \lambda\}x_2 \subset \{f \le \lambda\}x_2 \subset B$, $\{x_1, x_2\} \subset C$. Hence (f,g) is lower topologically concave on X.

2) Now suppose that (f,g) is lower topologically concave on X. For any $B \subset Y$ and any pair of real numbers μ and λ with $\mu < \lambda$, any $x_0, \bar{x} \in \Delta := \{x \in X : \{f \le \lambda\}x \subset B\}$, there exists a connected $C_{x_0,\bar{x}} \subset X$ such that

$$\{x_0, \bar{x}\} \subset C_{x_0,\bar{x}} \quad \text{and} \quad \{g < \mu\}C_{x_0,\bar{x}} \subset \{g \le \lambda\}x_0 \bigcup \{f \le \lambda\}\bar{x}. \tag{4}$$

Let $C = \bigcup_{\bar{x} \in \Delta} C_{x_0,\bar{x}}$. It is connected and $\Delta \subset C$. By (4) we have that $\{g < \mu\}C \subset \{g \le \lambda\}x_0 \bigcup (\bigcup_{\bar{x} \in \Delta} \{f \le \lambda\}\bar{x}) \subset B$ ($\{g \le \lambda\}x_0 \subset \{f \le \lambda\}x_0$). Therefore (3) is true. □

Remark 3.1 Let $B = Y$. Then $C = X$ in (3). So (f,g) being lower topologically concave on X implies that X is connected.

Considering the conditions in Theorem 2.1 symmetrically, we have the following theorem.

Theorem 3.1 *Let Y be connected, X be compact and f, g be two functions with $f \le g$. Suppose that*

1) f is weakly upper interconnected on X about g finitely;

2) f and g are upper semicontinuous on X;

3) (f,g) is upper topologically concave on Y.

Then $\inf_Y \sup_X f \le \sup_X \inf_Y g$.

Proposition 3.2 *If a pair of functions (f,g) with $f \le g$ is lower topologically concave on X, then it is arbitrarily upper connected on X.*

Proof According to Proposition 3.1, we only need to show that (3) implies

$$\bigcap_{y \in Y \setminus B} \{f > \lambda\}y \subset C \subset \bigcap_{y \in Y \setminus B} \{g \ge \mu\}y. \tag{5}$$

In fact, let $x \in \bigcap_{y \in Y \setminus B} \{f > \lambda\}y$. Then for any $y \in Y \setminus B$, $f(x,y) > \lambda$. Hence $\{f \le \lambda\}x \subset B$. By (3) $x \in C$.

Now let $x \in C$. Then $x \in \{x \in X : \{g < \mu\}x \subset B\}$. It follows that for any $y \in Y \setminus B$, $y \notin \{g < \mu\}x$. Therefore, $g(x,y) \ge \mu$, $x \in \bigcap_{y \in Y \setminus B} \{g \ge \mu\}y$. □

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