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On a Class of Analytic Functions Defined by Ruscheweyh Derivatives

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Abstract In the present paper a class of extended close-to-convex functions $Q_{k,\lambda}(\alpha,\beta,\rho)$ defined by making use of Ruscheweyh derivatives is introduced and studied. We provide integral representations, distortion theorem, radius of close-to-convexity and Hadamard product properties for this class.

Keywords Ruscheweyh derivatives; close-to-convex function; Hadamard product.

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1. Introduction

Suppose that the parameters λ, α, β, ρ satisfy $\lambda > -1$, $\alpha \geq 0$, $0 \leq \beta \leq 1$, $0 \leq \rho < 1$. Let H_k $(k = 1, 2, \ldots)$ be the class of functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{k+n} z^{k+n}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Let $P_k(\beta)$ denote the class of functions of the form $p(z) = 1 + p_k z^k + \cdots$ which are analytic in U and satisfy $\mathrm{Re}\, p(z) > \beta$. Let $S_k^*(\beta)$ and $K_k(\beta)$ stand for β class starlike function and β class convex function in H_k, respectively.

A function $f(z) \in H_k$ is said to be in the class $C_k(\beta, \rho)$ if and only if there exists $g(z) \in S_k^*(\beta)$ such that

$$\mathrm{Re} \frac{z f'(z)}{g(z)} > \rho, \quad z \in U.$$

From [9], we know that

$$f(z) \in K_k(\beta) \Leftrightarrow z f'(z) \in S_k^*(\beta).$$

For fixed real number $\lambda > -1$, the operator D^λ is defined by

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \quad f(z) \in H_k, \tag{1.1}$$

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where the operation $*$ stands for Hadamard product. The operator D^λ is the Ruscheweyh derivative introduced in [1,10] and is of the following properties:

$$D^\lambda f(z) = z + \sum_{n=1}^{\infty} \frac{(\lambda+1)\cdots(\lambda+k+n-1)}{(k+n-1)!} a_{k+n} z^{k+n}; \tag{1.2}$$

$$z(D^\lambda f(z))' = (\lambda+1)D^{\lambda+1}f(z) - \lambda D^\lambda f(z). \tag{1.3}$$

Next we introduce new functions class.

Definition 1.1 *If a function $f(z) \in H_k$ satisfies condition*

$$\mathrm{Re}\Big\{(1-\alpha)\frac{D^\lambda f(z)}{z} + \alpha (D^\lambda f(z))'\Big\} > \beta, \quad z \in U, \tag{1.4}$$

then we denote $f(z) \in V_{k,\lambda}(\alpha,\beta)$.

Definition 1.2 *Suppose $f(z) \in H_k$. If there exists a function $g(z) \in V_{k,\lambda}(\alpha,\beta)$ such that*

$$\mathrm{Re}\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda g(z)} > \rho, \quad z \in U, \tag{1.5}$$

then we denote $f(z) \in Q_{k,\lambda}(\alpha,\beta,\rho)$.

In [2], the functions class $Q_{1,0}(0,\frac{1}{2},0)$ was studied and distortion theorem, univalent radius and rotation theorem were obtained, but Hadamard product has not been solved. We will study the close-to-convex function class $Q_{k,\lambda}(\alpha,\beta,\rho)$ introduced above which is a great extension of [2].

As in [3], we introduce linear operator $L(a,c)$ which is more general than D^λ. Let

$$\phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad z \in U, \ c \neq 0,-1,-2,\dots$$

$$L(a,c)f(z) = \phi(a,c;z) * f(z), \quad f(z) \in H_k \tag{1.6}$$

where $(\zeta)_n = \frac{\Gamma(\zeta+n)}{\Gamma(\zeta)}$. From [4], we know that $L(a,c)$ is continuous mapping from H_k to H_k. It is easy to see that

$$\phi(2(1-\alpha),1;z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{1.7}$$

and for $c > a > 0$, we have

$$L(a,c)f(z) = \int_0^1 u^{a-1} f(uz) \mathrm{d}\eta(a,c-a)(u), \tag{1.8}$$

where η is $\mathbf{B}$ distribution

$$\mathrm{d}\eta(a,c-a)(u) = \frac{u^{a-1}(1-u)^{c-a-1}}{\mathrm{B}(a,c-a)}\mathrm{d}u. \tag{1.9}$$

If $a \neq 0,-1,-2,\dots$, then $L(c,a)$ is the inverse mapping of $L(a,c)$, so $L(a,c)$ is one-to-one mapping from H_k to H_k. It is obvious that

$$L(a,c) = L(a,b)L(b,c) = L(b,c)L(a,b), \quad b,c \neq -1,-2,\dots .$$

If $g(z) = zf'(z)$, then $g(z) = L(2,1)f(z)$, $f(z) = L(1,2)g(z)$.

By (1.6) and (1.7), we have

$$L(\lambda+1,1)f(z)=D^{\lambda}f(z). \tag{1.10}$$

In view of the operator $L(a,c)$ and (1.10), we may write (1.5) as:

$$\mathrm{Re}\frac{L(2,1)L(\lambda+1,1)f(z)}{L(\lambda+1,1)g(z)}>\rho,\quad z\in U. \tag{1.11}$$

In the present paper, we deduce integral representations of function in $Q_{k,\lambda}(\alpha,\beta,\rho)$. Distortion theorems, radius of close-to-convexity and Hadamard product properties are obtained for functions belonging to this class. Then we solve the closeness of Hadamard product in [2].

2. Integral representations

If $g(z)\in V_{k,\lambda}(\alpha,\beta)$, then it is not difficult to verify that there exists $p(z)=1+p_kz^k+\cdots\in P_k(\beta)$ such that

$$\begin{aligned}g(z)\in V_{k,\lambda}(\alpha,\beta)\Leftrightarrow zp(z)=L(\frac{1}{a},\frac{1}{a}+1)L(\lambda+1,1)g(z)\Leftrightarrow L(1,\lambda+1)g(z)\\ =L(\frac{1}{a},\frac{1}{a}+1)(zp(z)).\end{aligned}$$

In view of the Herglotz formula[5] of positive real part and the property of $L(\lambda+1,1)$, we prove the following result:

Theorem 2.1 *If $g(z)\in V_{k,\lambda}(\alpha,\beta)(\alpha>0)$, then there exists left continuous probability measure $\eta(x)$ on $X=\{x:|x|=1\}$ such that*

$$g(z)=L(1,\lambda+1)\Big\{\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^z t^{\frac{1}{\alpha}-1}\Big[\int_{|x|=1}\frac{1+(1-2\beta)tx}{1-tx}\mathrm{d}\eta(x)\Big]\mathrm{d}t\Big\}, \tag{2.1}$$

or there exists $p(z)\in P_k(\beta)$ such that

$$g(z)=L(1,\lambda+1)\Big\{\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^z t^{\frac{1}{\alpha}-1}p(t)\mathrm{d}t\Big\}.$$

For fixed parameters λ,α,β, $V_{k,\lambda}(\alpha,\beta)$ and left continuous probability measure points $\{\eta(x)\}$ on X are one-to-one correspondence through the relation expressed by (2.1).

Theorem 2.2 *A function $f(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)(\alpha>0)$ if and only if there exists left continuous probability measures $\eta(x),\mu(x)$ on $X=\{x:|x|=1\}$ such that*

$$\begin{aligned}f(z)=&L(1,\lambda+1)L(1,2)\Big\{\Big[\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^z t^{\frac{1}{\alpha}-1}\Big(\int_{|x|=1}\frac{1+(1-2\beta)tx}{1-tx}\mathrm{d}\eta(x)\Big)\mathrm{d}t\Big]\times\\ &\Big[\int_{|x|=1}\frac{1+(1-2\rho)zx}{1-zx}\mathrm{d}\mu(x)\Big]\Big\},\end{aligned} \tag{2.2}$$

when $\lambda=0$,

$$\begin{aligned}f(z)=&L(1,2)\Big\{\Big[\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^z t^{\frac{1}{\alpha}-1}\Big(\int_{|x|=1}\frac{1+(1-2\beta)tx}{1-tx}\mathrm{d}\eta(x)\Big)\mathrm{d}t\Big]\times\\ &\Big[\int_{|x|=1}\frac{1+(1-2\rho)zx}{1-zx}\mathrm{d}\mu(x)\Big]\Big\}.\end{aligned} \tag{2.3}$$

For fixed parameters $\lambda,\alpha,\beta,\rho$, $Q_{k,\lambda}(\alpha,\beta,\rho)$ and left continuous probability measure points $\{(\eta(x),\mu(x))\}$ on $X\times X$ are one-to-one correspondence through the relation expressed by (2.2).

Proof Let $f(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)$. Then there exists $g(z)\in V_{k,\lambda}(\alpha,\beta)$ such that

$$\mathrm{Re}\frac{z(L(\lambda+1,1)f(z))'}{L(\lambda+1,1)g(z)}>\rho,\quad z\in U.$$

By Theorem 2.1, we have

$$g(z)=L(1,\lambda+1)\Big\{\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^z t^{\frac{1}{\alpha}-1}\Big[\int_{|x|=1}\frac{1+(1-2\beta)tx}{1-tx}\mathrm{d}\eta(x)\Big]\mathrm{d}t\Big\},\tag{2.4}$$

where $\eta(x)$ is left continuous probability measure on X. By Herglots formula[5] for the functions in P class, we get

$$\frac{z(L(\lambda+1,1)f(z))'}{L(\lambda+1,1)g(z)}=\int_{|x|=1}\frac{1+(1-2\rho)xz}{1-xz}\mathrm{d}\mu(x),\tag{2.5}$$

where $\mu(x)$ is left continuous probability measure on X. From (2.4) and (2.5), we deduce that

$$\begin{aligned}L(2,1)L(\lambda+1,1)f(z)=&\Big\{\Big[\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^z t^{\frac{1}{\alpha}-1}\Big(\int_{|x|=1}\frac{1+(1-2\beta)tx}{1-tx}\mathrm{d}\eta(x)\Big)\mathrm{d}t\Big]\times\\&\Big[\int_{|x|=1}\frac{1+(1-2\rho)zx}{1-zx}\mathrm{d}\mu(x)\Big]\Big\}.\end{aligned}$$

By using the property of $L(\lambda+1,1)$, we get (2.2). Conversely it is true too. When $\lambda=0$, (2.2) reduce to (2.3). For fixed parameters $\lambda,\alpha,\beta,\rho$, since $\{(\eta(x),\mu(x))\}$ and $P_k(\beta)\times P_k(\rho)$ are one-to-one correspondence, $P_k(\beta)\times P_k(\rho)$ and $Q_{k,\lambda}(\alpha,\beta,\rho)$ are one-to-one correspondence too, so the last result is true. This completes the proof of Theorem 2.2. □

3. Distortion theorems

Lemma 3.1[6] *Let $p(z)=1+p_kz^k+\cdots\in P_k(0)$ $(z\in U,k\geq 1)$. Then for $|z|=r<1$, we have*

$$\frac{1-r^k}{1+r^k}\leq\mathrm{Re}p(z)\leq\frac{1+r^k}{1-r^k}.$$

The result is sharp.

If $\mathrm{Re}p(z)>\beta$, then by setting $q(z)=p(z)-\beta$, we have $\mathrm{Re}(p(z)-\beta)>0$. Hence it is easy to get from Lemma 3.1 and integral representation of positive real part functions[5] that

Lemma 3.2 *Let $q(z)=1+q_kz^k+\cdots\in P_k(\beta)$ $(z\in U,k\geq 1)$. Then for $|z|=r<1$, we have*

$$\frac{1-(1-2\beta)r^k}{1+r^k}\leq\mathrm{Re}q(z)\leq|q(z)|\leq\frac{1+(1-2\beta)r^k}{1-r^k}.$$

The result is sharp.

Theorem 3.1 *Let $\alpha>0$, $f(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)$. Then for $|z|=r<1$, we have*

$$\begin{aligned}&\frac{1-(1-2\rho)r^k}{r\alpha(1+r^k)}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1-(1-2\beta)(rt)^k}{1+(rt)^k}\mathrm{d}t\leq|(L(\lambda+1,1)f(z))'|\\&\leq\frac{1+(1-2\rho)r^k}{r\alpha(1-r^k)}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1+(1-2\beta)(rt)^k}{1-(rt)^k}\mathrm{d}t.\end{aligned}\tag{3.1}$$

The result is sharp.

Proof Let $f(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)$. Then there exists $g(z)\in V_{k,\lambda}(\alpha,\beta)$ such that

$$\mathrm{Re}\frac{z(L(\lambda+1,1)f(z))'}{L(\lambda+1,1)g(z)}>\rho,\quad z\in U.$$

Set $\frac{z(L(\lambda+1,1)f(z))'}{L(\lambda+1,1)g(z)}=q(z)$, $z\in U$. Then $\mathrm{Re}q(z)>\rho$. Firstly, we prove the distortion property of $|L(\lambda+1,1)g(z)|$. By Lemma 3.2 and Since $g(z)\in V_{k,\lambda}(\alpha,\beta)$, there exists $\mathrm{Re}p(z)>\beta$ such that

$$\begin{aligned}|L(\lambda+1,1)g(z)|\geq \mathrm{Re}(L(\lambda+1,1)g(z))&=\mathrm{Re}\Big\{\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1-(1-2\beta)(rt)^k}{1+(rt)^k}\mathrm{d}t\Big\}\\ &\geq\frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1-(1-2\beta)(rt)^k}{1+(rt)^k}\mathrm{d}t;\end{aligned}\tag{3.2}$$

$$\begin{aligned}|L(\lambda+1,1)g(z)|=\Big|\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^1 t^{\frac{1}{\alpha}-1}p(t)\mathrm{d}t\Big|&\leq\frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}|p(zt)|\mathrm{d}t\\ \leq\frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1+(1-2\beta)(rt)^k}{1-(rt)^k}\mathrm{d}t.\end{aligned}\tag{3.3}$$

Since $z(L(\lambda+1,1)f(z))'=L(2,1)L(\lambda+1,1)f(z)=q(z)L(\lambda+1,1)g(z)$, $z\in U$. By (3.2), (3.3) and Lemma 3.2, we have

$$\begin{aligned}&\frac{1-(1-2\rho)r^k}{\alpha(1+r^k)}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1-(1-2\beta)(rt)^k}{1+(rt)^k}\mathrm{d}t\leq|q(z)L(\lambda+1,1)g(z)|\\ &\leq\frac{1+(1-2\rho)r^k}{\alpha(1-r^k)}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1+(1-2\beta)(rt)^k}{1-(rt)^k}\mathrm{d}t.\end{aligned}$$

We get (3.1). Equality in (3.1) is obtained by function

$$f(z)=L(1,\lambda+1)L(1,2)\Big[\frac{1+(1-2\rho)z^k}{\alpha(1-z^k)z^{\frac{1}{\alpha}-1}}\int_0^z t^{\frac{1}{\alpha}-1}\frac{1+(1-2\beta)t^k}{1-t^k}\mathrm{d}t\Big]\tag{3.4}$$

at $z=re^{i\frac{\pi}{k}}$.

4. Radius of close-to-convexity

Lemma 4.1[7] *If $q(z)=1+q_kz^k+\cdots\in P_k(\beta)$ $(z\in U,k\geq1)$, then for $|z|=r<1$, we have*

$$\Big|\frac{zq'(z)}{q(z)}\Big|\leq\frac{2k(1-2\beta)r^k}{(1-r^k)[1+(1-2\beta)r^k]}.$$

The result is sharp.

Theorem 4.1 *Let $\alpha>0$, $f(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)$. Then $D^\lambda f(z)$ is close-to-convex in disk $|z|<r_1$, where r_1 is the minimum positive root of the following equation:*

$$1-2[m+k(1-m)]r^k-(1-2m)r^{2k}=0\tag{4.1}$$

and

$$m=\frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1-(1-2\beta)t^k}{1+t^k}\mathrm{d}t<1.\tag{4.2}$$

Proof Let $f(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)$. It suffices to prove that $D^\lambda g(z)$ is starlike function. Let $F(z)=\frac{D^\lambda g(z)}{z}$. Then $F(z)$ is analytic in U. By Theorem 2.1, Lemma 3.2 and since $g(z)\in V_{k,\lambda}(\alpha,\beta)$, there exists $p(z) \in P(\beta)$ such that

$$\mathrm{Re}\frac{L(\lambda+1,1)g(z)}{z} = \mathrm{Re}\Big\{\frac{1}{\alpha z^{\frac{1}{\alpha}}}\int_0^1 t^{\frac{1}{\alpha}-1}p(t)\mathrm{d}t\Big\} > m = \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1-(1-2\beta)t^k}{1+t^k}\mathrm{d}t.$$

Noticing the definition of $F(z)$ and by making use of Lemma 4.1, we have

$$\mathrm{Re}\Big\{\frac{z(D^\lambda g(z))'}{D^\lambda g(z)}\Big\} = 1+\mathrm{Re}\frac{zF'(z)}{F(z)} \geq 1-\Big|\frac{zF'(z)}{F(z)}\Big| \geq \frac{1-2[m+k(1-m)]r^k-(1-2m)r^{2k}}{(1-r^k)[1+(1-2m)r^k]}.$$

Let $\varphi(r) = 1-2[m+k(1-m)]r^k-(1-2m)r^{2k}$. Then $\varphi(r)$ is continuous on $[0,1]$ and $\varphi(0) = 1 > 0$, $\varphi(1) = -2k(1-m) < 0$. So (4.1) has minimum positive root in $(0,1)$ denoted by r_1. For $|z| < r_1$, we have $\mathrm{Re}\{\frac{z(D^\lambda g(z))'}{D^\lambda g(z)}\} > 0$. So $D^\lambda g(z)$ is starlike function, namely, $D^\lambda f(z)$ is close-to-convex function in disk $|z| < r_1$.

Corollary 4.1 *Let $\alpha > 0$, $f(z) \in Q_{k,0}(\alpha,\beta,\rho)$. Then f(z) is close-to-convex in disk $|z| < r_1$, where r_1 is the minimum positive root of (4.1).*

5. Hadamard product

Lemma 5.1[8] *Let $\varphi(z)$ and $h(z)$ be analytic in U and satisfy $\varphi(0) = h(0) = 0$, $\varphi'(0) \neq 0$, $h'(0) \neq 0$ and suppose for all complex numbers σ, τ satisfying $|\sigma| = |\tau| = 1$, there holds*

$$\varphi(z) * \frac{1+\tau\sigma z}{1-\sigma z}h(z) \neq 0 \ (0 < |z| < 1).$$

Let $F(z)$ be analytic in U and satisfy $\mathrm{Re}F(z) > 0$ $(0 < |z| < 1)$. Then

$$\mathrm{Re}\Big\{\frac{\varphi(z) * (F(z)h(z))}{\varphi(z) * h(z)}\Big\} > 0, \quad 0 < |z| < 1.$$

Theorem 5.1 *Let σ, τ satisfy $|\sigma| = |\tau| = 1$, $\alpha \geq 0$, $f(z) \in Q_{k,\lambda}(\alpha,\beta,\rho)$, $\varphi(z) = z + \sum_{n=k}^{\infty} a_{k+1}z^{k+1}$ be analytic in U and*

$$\varphi(z) * \frac{1+\tau\sigma z}{1-\sigma z}z \neq 0, \quad 0 < |z| < 1.$$

Then

$$f(z) * \varphi(z) \in Q_{k,\lambda}(\alpha,\beta,\rho).$$

Proof (i) Firstly, we prove that $g * \varphi(z) \in V_{k,\lambda}(\alpha,\beta)$. Let

$$F(z) = (1-\alpha)\frac{D^\lambda g(z)}{z} + \alpha(D^\lambda g(z))' - \beta, \quad h(z) = z.$$

Then $F(z)$ is analytic in U and $\mathrm{Re}F(z) > 0$ and $\varphi * h(z) = z$. Since

$$\begin{aligned}\varphi(z) * (F(z)h(z)) &= \varphi * [(1-\alpha)D^\lambda g(z) + \alpha z(D^\lambda g(z))' - \beta z]\\ &= (1-\alpha)\varphi * D^\lambda g(z) + \alpha\varphi * z(D^\lambda g(z))' - \beta z\\ &= (1-\alpha)D^\lambda(\varphi * g)(z) + \alpha z(D^\lambda(\varphi * g))'(z) - \beta z, \qquad (5.1)\end{aligned}$$

by Lemma 5.1, we get

$$\mathrm{Re}\Big\{\frac{\varphi(z) * (F(z)h(z))}{\varphi(z) * h(z)}\Big\} = \mathrm{Re}\Big\{(1-\alpha)\frac{D^\lambda(\varphi * g)(z)}{z} + \alpha(D^\lambda(\varphi * g))'(z)\Big\} - \beta > 0, \qquad (5.2)$$

that is,

$$\mathrm{Re}\Big\{(1-\alpha)\frac{D^\lambda(\varphi*g)(z)}{z}+\alpha(D^\lambda(\varphi*g))'(z)\Big\}>\beta,\quad z\in U.$$

So $g*\varphi(z)\in V_{k,\lambda}(\alpha,\beta)$.

(ii) Next we prove that $f*\varphi(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)$. Let $f(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)$, $p(z)=\frac{z(D^\lambda f(z))'}{D^\lambda g(z)}-\rho$ and $h(z)=z$. Then $p(z)$ is analytic in U and $\mathrm{Re}p(z)>0\ \ (z\in U)$ and $\varphi*h(z)=z$. Since

$$\varphi*D^\lambda g(z)\cdot p(z)=\varphi*z(D^\lambda f(z))'-\rho\varphi*D^\lambda g(z),\tag{5.3}$$

noticing that

$$\varphi*D^\lambda g(z)=D^\lambda(\varphi*g)(z);\quad \varphi*z(D^\lambda f(z))'=z(D^\lambda(\varphi*f))'(z),$$

by (5.3), we get

$$\mathrm{Re}p(z)=\mathrm{Re}\Big\{\frac{z(D^\lambda(\varphi*f))'(z)}{D^\lambda(\varphi*g)(z)}\Big\}>\rho.$$

From (i), $\varphi*g(z)\in V_{k,\lambda}(\alpha,\beta)$. So $f*\varphi(z)\in Q_{k,\lambda}(\alpha,\beta,\rho)$. This completes the proof of Theorem 5.1. □

Remark 5.1 Setting $\lambda=0,\alpha=0,\beta=\frac{1}{2}$ and $\rho=0$ in Theorem 5.1, respectively, we get the corresponding product properties of functions in $Q_{1,0}(0,\frac{1}{2},0)$.

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