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Almost Fixed Point, Fixed Point and Quasi-Variational Inequality on Generalized Convex Spaces

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Abstract The definitions of S-KKM property and Γ-invariable property for multi-valued mapping are established, and by which, a new almost fixed point theorem and several fixed point theorems on Haudorff locally G-convex uniform space are obtained, and a quasi-variational inequality theorem for acyclic map on Hausdorff Φ-space is proved. Our results improve and generalize the corresponding results in recent literatures.

Keywords generalized convex space; Γ-convex; Φ-map; Φ-space; better admissible multimap; acyclic multimap; the almost fixed point property.

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1. Introduction

We first give some definitions and notations.

A generalized convex space or a G-convex space $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that, for each $A = \{a_0, a_1, \ldots, a_n\} \in \langle D\rangle$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset \{0, 1, \ldots, n\}$ implies $\phi_A(\Delta_J) \subset \Gamma(\{a_j : j \in J\})$, where, $\langle D\rangle$ denotes the set of all nonempty finite subset of D, Δ_n an n-simplex with vertices $v_0, v_1, \ldots, v_n$, and $\Delta_J = \mathrm{co}\,\{v_j : j \in J\}$, the face of Δ_n corresponding to J. Let $\Gamma_A = \Gamma(A)$ for each $A \in \langle D\rangle$.

There are a lot of examples of G-convex spaces[1]. The typical example of G-convex space is any nonempty convex subset of a topological vector space.

In this paper, we assume that $D \subset X$, and $(X, D; \Gamma)$ will be denoted by $(X; \Gamma)$ if $D = X$.

For a G-convex space $(X, D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-convex if each $N \in \langle D\rangle, N \subset Y$ implies $\Gamma_N \subset Y$.

Let X and Y be two topological spaces. A multimap (simply, a map) $T : X \multimap Y$ is a function from X into the power set 2^Y of Y. Denote $T(A) = \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

A map $T : X \multimap Y$ is called upper [resp. lower] semicontinuous (simply, u.s.c. [resp. l.s.c.]) if for each closed[resp. open] subset C of Y, $T^-(C) = \{x \in X : T(x) \cap C \neq \emptyset\}$ is closed [resp. open] in X; and T is called compact if $T(X) = \{y \in Y : y \in T(x), x \in X\}$ is contained in a

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compact subset of Y; T is said to be closed if the graph $Gr(T)$ of T is closed in $X \times Y$.

Definition 1 *Let X be a nonempty set, $(Y, D; \Gamma)$ a G-convex space, Z a topological space. If $S : X \multimap D$ is a multimap such that $S(x) \in \langle D \rangle$ for each $x \in X$, $T : Y \multimap Z$ and $F : X \multimap Z$ are two multimaps satisfying $T(\Gamma_{S(N)}) \subset F(N)$ for each $N \in \langle X \rangle$, then F is called a gengeralized S-KKM mapping with respect to T. If a multimap $T : Y \multimap Z$ satisfies that for each generalized S-KKM mapping F with respect to T the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The set $\{T : Y \multimap Z | T$ has the S-KKM property$\}$ is denoted by the class S-KKM(X,Y,D,Z), and S-KKM(X, Y, D, Z) is denoted by S-KKM(X, Y, Z) if $D = Y$.*

Definition 2[2] *A locally G-convex uniform space is a G-convex space $(X, D; \Gamma, \mathbb{U})$ satisfying the following conditions:*

(i) *X is a uniform space with the basis ν for the uniform structure $\mathbb{U}$;*

(ii) *D is a dense subset of X;*

(iii) *For each $V \in \nu$ and each $x \in X$, $V[x] = \{x' \in X : (x, x') \in V\}$ is Γ-convex.*

Definition 3 *Let Y be a topological space, $(X, D; \Gamma)$ a G-convex space. A map $T : Y \multimap X$ is called a Φ-map if there exists a map $S : Y \multimap D$ such that*

(i) *For each $y \in Y$, $M \in \langle S(y) \rangle$ implies $\Gamma_M \subset T(y)$;*

(ii) *$Y = \{\mathrm{Int} S^{-}(x) : x \in D\}$.*

Definition 4 *G-convex space $(X, D; \Gamma)$ is called a Φ-space if X is a uniform space and for each entourge V, there is a Φ-map $T : X \multimap X$ such that $\mathrm{Gr}(T) \subset V$.*

Definition 5 *Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. We define the better admissible class $\mathfrak{B}$ of multimaps from X into Y as follows:*

$F \in \mathfrak{B}(X, Y) \Longleftrightarrow F : X \multimap Y$ is a multimap such that for any $N \in \langle D \rangle$ with $|N| = n + 1$ and any continuous map $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point.

And we define the following two important multimaps:

$F \in \mathbb{V}(X, Y) \Longleftrightarrow F : X \multimap Y$ is an acyclic map; that is, a u.s.c multimap with compact acyclic values;

$F \in \mathbb{V}_c(X, Y) \Longleftrightarrow F : X \multimap Y$ is a finite composition of acyclic maps, where the intermediate spaces are topological.

Remark It is known that $\mathbb{V}_c(X, Y) \subset \mathfrak{B}(X, Y)$, and that any map in $\mathbb{V}_c(X, Y)$ is closed[3].

Definition 6 *Let Y be a real Hausdorff topological vector space with a convex cone K such that $\mathrm{Int} K \neq \emptyset$ and $K \neq Y$, and C a nonempty subset of Y.*

(1) *A point $\overline{y} \in C$ is called a vector minimal point of C if for any $y \in C$, $y - \overline{y} \notin K \backslash \{0\}$.*

The set of all the vector minimal points of C is denoted by $\mathrm{Min}_K C$.

(2) A point $\overline{y} \in C$ is called a weakly vector minimal point of C if for any $y \in C$, $y - \overline{y} \notin \mathrm{Int}K$. The set of all the weakly vector minimal points of C is denoted by $W\mathrm{Min}_K C$.

Definition 7 *Let X and Y be two topological spaces, $T : X \multimap Y$ a multimap, $f : X \to Y$ a single valued continuous map. If $f(x) \in T(x)$ for all $x \in X$, then f is called a continuous selection of T.*

Definition 8 *Let X be a nonempty set, $(Y, D; \Gamma)$ a G-convex space. The map $S : X \multimap D$ is said to have Γ-invariable property, if for each $x \in X$, $S(x) \in \langle D\rangle$ and for each $A \in \langle X\rangle$, $\Gamma_{S(A)} = \Gamma_{\{\omega_a : a\in A\}}$ for any $\omega_a \in S(a)$.*

Obviously, if $S : X \to D$ is a single valued map, then S has Γ-invariable property.

2. Almost fixed point theorem and fixed point theorems

Theorem 1 *Let $(X, D; \Gamma, \mathbb{U})$ be a locally G-convex uniform space, ν a basis of the uniform structure $\mathbb{U}$, I a nonempty set, and $S : I \multimap D$ have Γ-invariable property. If $T \in S$-KKM(I, X, D, X) is a compact map and $T(X) \subset \overline{S(I)}$, then $T : X \multimap X$ has the almost fixed point property; that is, for each $V \in \nu$, there exists an $x_V \in X$ such that $V[x_V] \bigcap T(x_V) \neq \emptyset$.*

Proof We may assume that each $V \in \nu$ is an open symmetric element. Define a map $F : I \multimap X$ by $F(z) = \overline{T(X)} \backslash \bigcup_{\omega\in S(z)} V[\omega]$ for each $z \in I$.

For each $y \in \overline{T(X)}$, since $T(X) \subset \overline{S(I)}$, $y \in \overline{S(I)}$ and $V[y]$ is open neighborhood of y, $V[y] \cap S(I) \neq \emptyset$, which implies that there exist a $z \in I$ and $x \in S(z)$ such that $x \in V[y]$. Hence $y \in V[x] \subset \bigcup_{\widehat{x}\in S(z)} V[\widehat{x}]$, and therefore $\overline{T(X)} \subset \bigcup_{z\in I} \bigcup_{x\in S(z)} V[x]$.

Since T is compact, of course, $\overline{T(X)}$ is compact. Therefore there exist $N = \{z_1, z_2, \ldots, z_n\} \in \langle I\rangle$ and $\{\omega_{i,j} \in S(z_i) : j = 1, 2, \ldots, k_i\}_{i=1}^n$ such that

$$\overline{T(X)} \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} V[\omega_{i,j}] \subset \bigcup_{z\in N} \bigcup_{\omega\in S(z)} V[\omega].$$

Note that $F(z)$ is closed for each $z \in I$ and

$$\bigcap_{z\in N} F(z) = \overline{T(X)} \backslash \bigcup_{z\in N} \bigcup_{\omega\in S(z)} V[\omega] \subset \overline{T(X)} \backslash \overline{T(X)} = \emptyset,$$

hence $\{F(z)\}_{z\in I}$ does not have the finite intersection property. Since $T \in S$-KKM(I, X, D, X), there exists $M \in \langle I\rangle$ such that $T(\Gamma_{S(M)}) \not\subseteq F(M)$. Hence there exist $x_V \in \Gamma_{S(M)}$ and $p \in T(x_V)$ such that $p \notin F(M) = \bigcup_{m\in M} F(m) = \bigcup_{m\in M} \left(\overline{T(X)} \backslash \bigcup_{\omega\in S(m)} V[\omega]\right)$. But $p \in T(x_V) \subset T(X) \subset \overline{T(X)}$, hence $p \in \bigcup_{\omega\in S(m)} V[\omega]$ for all $m \in M$, which implies that for any $m \in M$ there exists $\omega_m \in S(m)$ such that $p \in V[\omega_m]$, that is, $\omega_m \in V[p]$, hence $\{\omega_m \in S(m) : m \in M\} \subset D \bigcap V[p]$. By Definitions 2 and 8, we have that $x_V \in \Gamma_{S(M)} = \Gamma_{\{\omega_m\in S(m) : m\in M\}} \subset V[p]$, and hence $p \in V[x_V]$. This implies that $T(x_V) \bigcap V[x_V] \neq \emptyset$.

Remarks 1) Note that D is assumed to be a dense subset of X in Definition 2. But from the proof of Theorem 1, we can find that this condition is superfluous in Theorem 1.

2) S having Γ-invariable property can be replaced by S being a single valued map.

3) The condition $T(X) \subset \overline{S(I)}$ can be replaced by one of the following conditions: (i) $\overline{T(X)} \subset S(I)$; (ii) $T(X) \subset S(I)$; (iii) there exists a subset $X_0 \subset S(I)$ such that $T(X) \subset \overline{X_0}$.

4) The compactness of T can be replaced by the following weaker condition: there exists an $N \in \langle I \rangle$ such that $T(X) \subset \overline{S(N)}$. In fact, it is easy to prove that for each $V \in \nu$, $\overline{T(X)} \subset \bigcup_{z\in N} \bigcup_{\omega \in S(z)} V[\omega]$.

From Theorem 1, we can obtain the following fixed point theorem for multimap having the S-KKM property on Hausdorff locally G-convex uniform space.

Theorem 2 *Let $(X, D; \Gamma, \mathbb{U})$ be a Hausdorff locally G-convex uniform space, ν a basis of the uniform structure $\mathbb{U}$, I a nonempty set, and $S : I \multimap D$ have Γ-invariable property. If $T \in S$-KKM(I, X, D, X) is a compact closed map and $T(X) \subset \overline{S(I)}$, then $T : X \multimap X$ has a fixed point.*

Proof For each $V \in \nu$, there exists an $x_V \in X$ such that $T(x_V) \bigcap V[x_V] \neq \emptyset$ by Theorem 1. Take $y_V \in T(x_V) \bigcap V[x_V]$, then $(x_V, y_V) \in \mathrm{Gr}(T)$ and $(x_V, y_V) \in V$. Obviously, $\{y_V\}_{V\in\nu}$ is a net in the compact set $\overline{T(X)}$, so $\{y_V\}_{V\in\nu}$ has a convergent subnet. We may assume that $\{y_V\}_{V\in\nu}$ itself converges and $\{y_V\} \to x_0 \in \overline{T(X)}$. On the other hand, X is Hausdorff and $(x_V, y_V) \in V$ for all $V \in \nu$, hence $x_V \to x_0$. But $\mathrm{Gr}(T)$ is closed in $X \times X$, therefore $(x_0, x_0) \in \mathrm{Gr}(T)$. This implies that $x_0 \in T(x_0)$. □

Remarks 1) $S : X \multimap D$ having Γ-invariable property can be replaced by S being a single valued map.

2) The condition $T(X) \subset \overline{S(I)}$ can be replaced by one of the following conditions: (i) $\overline{T(X)} \subset S(I)$; (ii) $T(X) \subset S(I)$; (iii) there exists a subset $X_0 \subset S(I)$ such that $T(X) \subset \overline{X_0}$.

3) The compactness of T can be replaced by the compactness of X.

4) The closedness of T can be replaced by the upper semi-continuity of T with closed values.

5) If $I = X = D$ is a nonempty convex subset of a topological vector space, S is a single valued map and $\overline{T(X)} \subset S(X)$ instead of $T(X) \subset \overline{S(X)}$, then Theorem 2 becomes the corresponding result in [4]; If $I = X = D$ is an H-space, S is a single valued map and $\overline{T(X)} \subset S(X)$ instead of $T(X) \subset \overline{S(X)}$, then Theorem 2 becomes the corresponding result in [5]; If $I = X = D$ is a G-convex space, S is a single valued map and $\overline{T(X)} \subset S(X)$ instead of $T(X) \subset \overline{S(X)}$, then Theorem 2 becomes the corresponding result in [6]. And the method of our proof is completely different from those in [4], [5] and [6]. Using their method, there must be $\overline{T(X)} \subset S(X)$ instead of $T(X) \subset \overline{S(X)}$ even if $I = D = X$ and S is a single valued map.

From now on, we only consider the case that $S : X \multimap D$ is a single valued map, and S is denoted by s.

Theorem 3 *Let X be a nonempty set, (Y, Γ) a G-convex space, Z and W two topological spaces, $s : X \to Y$ a single valued map. If $T \in s$-KKM(X, Y, Z) and $f \in \mathbb{C}(\mathbb{Z}, \mathbb{W})$, then $fT \in s$-KKK(X, Y, W).*

Proof Let $F : X \multimap W$ be a generalized s-KKM map with respect to fT, and assume that for each $x \in X$, $F(x)$ is closed. If each $N \in \langle X \rangle$ satisfies $fT(\Gamma_{s(N)}) \subset F(N)$, then $T(\Gamma_{s(N)}) \subset f^{-1}F(N) = \bigcup_{x \in N} f^{-1}F(x)$, which implies that $f^{-1}F$ is a generalized s-KKM map with respect to T and for each $x \in X$, $f^{-1}F(x)$ is closed. Since $T \in s$-KKM(X, Y, Z), $\{f^{-1}F(x) : x \in X\}$ has the finite intersection property, and so does the family $\{F(x) : x \in X\}$, we have $fT \in s$-KKK(X, Y, W). □

Remark Theorem 3 improves the corresponding result in [4] and [6].

Lemma 1[7] *Let Y be a Hausdorff space, $(X, D; \Gamma)$ a G-convex space, and $T : Y \multimap X$ a Φ-map. Then for any nonempty compact subset K of Y, $T|_K$ has a continuous selection $f : K \to X$ such that $F(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$. More precisely, there exist two continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ such that $f = \phi_N \circ p$ for some $N \in \langle D \rangle$ with $|N| = n + 1$.*

From Theorem 2, Theorem 3 and Lemma 1, we can obtain a coincident point theorem for two multimaps or a fixed point theorem for composition of two multimaps.

Theorem 4 *Let $(X, \Gamma, \mathbb{U})$ be a Hausdorff locally G-convex uniform space, ν a basis of the uniform structure $\mathbb{U}$, Y a compact Hausdorff space, and $s : X \to X$ a map such that $s(X)$ is dense in X. If $T \in s$-KKM(X, X, Y) is a closed map, then for any Φ-map $F : Y \multimap X$, FT and TF have a fixed point in X and Y, respectively.*

Proof In view of Lemma 1, F has a continuous selection $f : Y \to X$; and by Theorem 3, $fT \in s$-KKK(X, X, X). Since f is continuous and Y is compact, fT is a compact map. And since T is a closed map and f is continuous, fT is also a closed map. On the other hand, $fT(X) \subset X = \overline{s(X)}$, then by Theorem 2 with $I = D = X$, fT has a fixed point $x_0 \in X$, that is, $x_0 \in fT(x_0)$. So there exists a $y_0 \in T(x_0)$ such that $x_0 = f(y_0) \in F(y_0)$, which implies that $x_0 \in FT(x_0)$ and $y_0 \in TF(y_0)$. □

From Theorem 4, we can obtain the following three fixed point corollaries:

Corollary 1 *Let $(X, \Gamma, \mathbb{U})$ be a Hausdorff locally G-convex uniform space, ν a basis of the uniform structure $\mathbb{U}$, Y a compact Hausdorff space. If $T \in \mathrm{id}_X$-KKM(X, X, Y) is a closed map, then for any Φ-map $F : Y \multimap X$, FT and TF have a fixed point in X and Y, respectively.*

Proof Put $s = \mathrm{id}_X : X \to X$ to be an identity map in Theorem 4. □

Corollary 2 *Let $(X, \Gamma, \mathbb{U})$ be a compact Hausdorff locally G-convex uniform space, ν a basis of the uniform structure $\mathbb{U}$, $s : X \to X$ a surjective map. If $\mathrm{id}_X \in s$-KKM(X, X, X), then any Φ-map $F : X \multimap X$ has a fixed point in X.*

Proof Put $T = \mathrm{id}_X : X \to X$ to be an identity map and let $Y = X$ in Theorem 4. □

Corollary 3 *Let $(X, \Gamma, \mathbb{U})$ be a compact Hausdorff locally G-convex uniform space, ν a basis of the uniform structure $\mathbb{U}$. If $\mathrm{id}_X \in \mathrm{id}_X$-KKM$(X, X, X)$, then any Φ-map $F : X \multimap X$ has a fixed point in X.*

Proof Put $s = T = \mathrm{id}_X : X \to X$ to be an identity map and let $Y = X$ in Theorem 4. $\square$

3. Quasi-variational inequality on Φ-spaces

In this part, we use the well-known fixed point theorem for acyclic map on Φ-space to establish quasi-variational inequality theorem. First, we introduce some well-known results.

Lemma 2[8] *Let C be a nonempty compact subset of a real Hausdorff topological vector space Y with a closed convex cone K such that $K \neq Y$, then $\mathrm{Min}_K C \neq \emptyset$.*

Lemma 3[3] *Let $(X, D; \Gamma)$ be a Hausdorff Φ-space and $F \in \mathfrak{B}(X, X)$. If F is closed and compact, then F has a fixed point.*

In view of Lemma 3 and Remark after Definition 5, we have the following lemma.

Lemma 4 *Let $(X, D; \Gamma)$ be a Hausdorff Φ-space. Then any compact map $F \in \mathbb{V}_c(X, X)$ has a fixed point.*

Lemma 5[9] *Let $(X, D; \Gamma)$ be a G-convex space, Y a Γ-convex subset of X with $Y \bigcap D \neq \emptyset$. Then $(Y, Y \bigcap D, \Gamma)$ is also a G-convex space.*

Now, we give a quasi-variational inequality theorem on Φ-space.

Theorem 5 *Let $(Z, D; \Gamma_1)$ be a G-convex space, (X, Γ_2) a Hausdorff Φ-space, Y a Hausdorff topological vector space with a closed convex cone K such that $K \neq Y$ and $\mathrm{Int}K \neq \emptyset$. Let $S : X \multimap X$ be a continuous compact multimap with nonempty compact values such that $\overline{S(X)}$ is a Γ-convex subset of X, $T : X \multimap Z$ a Φ-map, C a subset of Z such that $T(X) \subset C$. If $\Psi : X \times C \times X \to Y$ is a continuous mapping such that for each $(x, z) \in X \times C$, the set $G(x, z) = \{u \in S(x) : \Psi(x, z, u) \in W\mathrm{Min}_K \Psi(x, z, S(x))\}$ is acyclic, then there exist $\overline{x} \in S(\overline{x})$ and $\overline{z} \in T(\overline{x})$ such that $\Psi(\overline{x}, \overline{z}, x) - \Psi(\overline{x}, \overline{z}, \overline{x}) \notin$-$\mathrm{Int}K$ for all $x \in S(\overline{x})$.*

Proof Since $S : X \multimap X$ is a compact map, $\overline{S(X)} := X_0$ is compact. By Lemma 1, $T|_{X_0}$ has a continuous selection f, that is, there exists a continuous map $f : X_0 \to X$ such that $f(x) \in T(x)$ for all $x \in X_0 \subset X$. Obviously, $S|_{X_0} : X_0 \multimap X_0$ and $\Psi|_{X_0 \times C \times X_0} : X_0 \times C \times X_0 \to Y$ are still continuous maps.

Define two multimaps as follows

$H : X_0 \multimap X_0$ by $H(x) = \{u \in S(x) : \Psi(x, f(x), u) \in W\mathrm{Min}_K \Psi\,(x, f(x), S(x))\}$ for each $x \in X_0$; and

$M : X_0 \multimap Y$ by $M(x) = W\mathrm{Min}_K \Psi\,(x, f(x), S(x))$ for each $x \in X_0$.

Since S is a continuous map with nonempty compact values, and Ψ and f are both continuous, $\Psi(x, f(x), S(x))$ is a nonempty compact subset of Y. It follows from Lemma 2 that $M(x) \neq \emptyset$ for all $x \in X_0$.

First, we prove that M is a closed map.

Let $\{(x_j, y_j)\}_{j \in J}$ be a net in $\mathrm{Gr}(M) \subset X_0 \times Y$ such that $(x_j, y_j) \to (x_0, y_0) \in X_0 \times Y$. Then $y_j \in M(x_j)$ for each $j \in J$, hence there exists an $s_j \in S(x_j)$ such that $y_j = \Psi(x_j, f(x_j), s_j)$.

Since S is continuous, and X_0 and $S(x)$ are both compact for each $x \in X$, $S(X_0)$ is a compact subset of X and S is closed map on X_0. And since $s_j \in S(x_j) \subset S(X_0)$ for each $j \in J$, we assume that $s_j \to s_0$ for some $s_0 \in s(X_0)$. Since $s_j \in S(x_j)$ and S is closed map, $s_0 \in S(x_0)$. Hence $y_0 = \Psi(x_0, f(x_0), s_0)$ by the continuty of Ψ and f. Of course, $y_0 \in \Psi(x_0, f(x_0), S(x_0))$.

Suppose that $y_0 \notin M(x_0)$, then by the definition of $W\text{Min}_K$, there exists $s^* \in S(x_0)$ such that $\Psi(x_0, f(x_0), s^*) - y_0 \in$-Int$K$. Let $y^* = \Psi(x_0, f(x_0), s^*)$. Then $y^* - y_0 \in$-Int$K$. Since $x_j \to x_0$, $s^* \in S(x_0)$, S is lower semicontinuous on X_0, there exists a net $\{s_j^*\}$ such that $s_j^* \in S(x_j)$ and $s_j^* \to s^*$. Let $y_j^* = \Psi(x_j, f(x_j), s_j^*)$. Then $y_j^* \to \Psi(x_0, f(x_0), s^*) = y^*$ and $y_j^* - y_j \to y^* - y_0$ by the continuty of Ψ and f. But $y^* - y_0 \in$-Int$K$, hence for j large enough, $y_j^* - y_j \in$-Int$K$, which contradicts $y_j \in M(x_j)$. Thus $y_0 \in M(x_0)$, which means that M is a closed map.

Next, we prove that $H : X_0 \to X_0$ is a closed valued map.

Let $\{(x_j, u_j)\}_{j \in J}$ be a net in $\text{Gr}(H) \subset X_0 \times X_0$ such that $(x_j, u_j) \to (x_0, u_0) \in X_0 \times X_0$. Then $u_j \in H(x_j)$ for all $j \in J$, which implies that $u_j \in S(x_j)$ and $\Psi(x_j, f(x_j), u_j) \in M(x_j)$ for all $j \in J$. Since S is closed, $u_0 \in S(x_0)$. On the other hand, since f and Ψ are continuous, and M is a closed map, $\Psi(x_j, f(x_j), u_j) \to \Psi(x_0, f(x_0), u_0) \in M(x_0)$, so that $u_0 \in H(x_0)$, that is, $(x_0, u_0) \in \text{Gr}(H)$. This means that H is a closed map on X_0. And since X_0 is compact, H is upper semicontinuous map. Notice that X_0 is Haudorff space, therefore H is a closed valued map.

Since (X, Γ_2) is a Hausdorff Φ-space and X_0 is a Γ-convex subset of X, (X_0, Γ_2) is also a Hausdorff Φ-space by Lemma 5 and the definition of Φ-space. In view of given condition, $H(x) = G(x, f(x))$ is acyclic, therefore $H : X_0 \multimap X_0$ satisfies all conditions in Lemma 4, so that there exists an $\overline{x} \in X_0$ such that $\overline{x} \in H(\overline{x})$, that is, $\overline{x} \in \{u \in S(\overline{x}) : \Psi(\overline{x}, f(\overline{x}), u) \in W\text{Min}_K \Psi(\overline{x}, f(\overline{x}), S(\overline{x}))\}$. Let $\overline{z} = f(\overline{x}) \in T(\overline{x})$. Then $\overline{x} \in S(\overline{x}) \subset X$, $\overline{z} \in T(\overline{x})$ and $\Psi(\overline{x}, \overline{z}, \overline{x}) \in WMin_K \Psi(\overline{x}, \overline{z}, S(\overline{x}))$. Therefore, $\Psi(\overline{x}, \overline{z}, x) - \Psi(\overline{x}, \overline{z}, \overline{x}) \notin$-Int$K$ for all $x \in S(\overline{x})$. □

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