Journal of Mathematical Research & Exposition
Nov., 2009, Vol. 29, No. 6, pp. 1117–1123
DOI:10.3770/j.issn:1000-341X.2009.06.023
Http://jmre.dlut.edu.cn

Generalized Macaulay-Northcott Modules and Tor-Groups

LIU Zhong Kui, QIAO Hu Sheng
(Department of Mathematics, Northwest Normal University, Gansu 730070, China)
(E-mail: liuzk@nwnu.edu.cn; qiaohs@nwnu.edu.cn)

Abstract Let $(S,\leq)$ be a strictly totally ordered monoid which is also artinian, and R a right noetherian ring. Assume that M is a finitely generated right R-module and N is a left R-module. Denote by $[[M^{S,\leq}]]$ and $[N^{S,\leq}]$ the module of generalized power series over M, and the generalized Macaulay-Northcott module over N, respectively. Then we show that there exists an isomorphism of Abelian groups:

$$\mathrm{Tor}_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]],[N^{S,\leq}])\cong\bigoplus_{s\in S}\mathrm{Tor}_i^R(M,N).$$

Keywords generalized Macaulay-Northcott module; ring of generalized power series; Tor-group.

Document code A
MR(2000) Subject Classification 16W60; 16E30
Chinese Library Classification O153.3

1. Introduction

Let R be a ring and $(S,\leq)$ a strictly totally ordered monoid. Assume that $[[R^{S,\leq}]]$ is the ring of generalized power series with coefficients in R and exponents in S. The generalized Macaulay-Northcott modules $[M^{S,\leq}]$ and the generalized power series modules $[[M^{S,\leq}]]$ play an important role in the theory of category of $[[R^{S,\leq}]]$-modules. For polynomial rings $R[x]$, it was shown in ([1], Theorems 1.2 and 2.1) that there are isomorphisms of Abelian groups $\mathrm{Ext}^i_{R[x]}(M[x^{-1}],\ N[x^{-1}])\cong\mathrm{Ext}^i_R(M,N)[[x]]$ for left Noetherian rings R and R-modules $_RM$, $_RN$, and $\mathrm{Tor}_i^{R[x]}(M[x^{-1}],\ N[x^{-1}])\cong\mathrm{Tor}^R_{i-1}(M,N)[x^{-1}]$ for any rings R and R-modules M_R and $_RN$. It was shown in ([2], Lemma 2.3) and ([3], Lemma 3.3) that there exists a natural isomorphism of Abelian groups $\mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],\ [N^{S,\leq}])\cong[[\mathrm{Hom}_R(M,N)^{S,\leq}]]$. More generally, under some additional conditions, it was shown that there exist isomorphisms of Abelian groups $\mathrm{Ext}^i_{[[R^{S,\leq}]]}([M^{S,\leq}],[N^{S,\leq}])\cong\prod_{s\in S}\mathrm{Ext}^i_R(M,N)$[3] and $\mathrm{Tor}_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]],[[N^{S,\leq}]])\cong[[\mathrm{Tor}_i^R(M,N)^{S,\leq}]]$[4]. In this paper we will consider the Tor-group determined by a generalized power series module $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ and a generalized Macaulay-Northcott module $_{[[R^{S,\leq}]]}[N^{S,\leq}]$ under some additional conditions.

Received date: 2007-06-03; **Accepted date**: 2008-01-03
Foundation item: the National Natural Science Foundation of China (No. 10961021); the Teaching and Research Award Program for Outsanding Young Teachers in Higher Education Institutions of Ministry of Education (No. NCET-02-080).

All rings considered here are associative with identity. Any concept and notation not defined here can be found in [5]–[7].

Let $(S,\leq)$ be an odered set. Recall that $(S,\leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S,\leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to [8].

Definition 1.1 *Let $(S,\leq)$ be a strictly ordered monoid (that is, $(S,\leq)$ is an ordered monoid satisfying the condition that, if $s,s',t\in S$ and $s<s'$, then $s+t<s'+t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f:S\longrightarrow R$ such that* $\mathrm{supp}(f)=\{s\in S|f(s)\neq 0\}$ *is artinian and narrow. With pointwise addition, and the operation of convolution*

$$(fg)(s)=\sum_{(u,v)\in X_s(f,g)} f(u)g(v),$$

where $X_s(f,g)=\{(u,v)\in S\times S|s=u+v,\ f(u)\neq 0,\ g(v)\neq 0\}$ is a finite set by [8, 4.1] for every $s\in S$ and $f,g\in[[R^{S,\leq}]]$, $[[R^{S,\leq}]]$ becomes a ring, which is called the ring of generalized power series. The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in R and exponents in S.

Many examples and results of rings of generalized power series are given in [5]–[11].

2. Modules of generalized power series

Let M be a right R-module over a ring R and $(S,\leq)$ a strictly ordered monoid. Denote by $[[M^{S,\leq}]]$ the set of all maps $\phi:S\longrightarrow M$ such that $\mathrm{supp}(\phi)=\{s\in S|\phi(s)\neq 0\}$ is artinian and narrow. With pointwise addition, $[[M^{S,\leq}]]$ is an abelian additive group. For each $f\in[[R^{S,\leq}]]$, each $\phi\in[[M^{S,\leq}]]$, and $s\in S$, denote

$$X_s(\phi,f)=\{(u,v)\in S\times S|s=u+v,\phi(u)\neq 0,f(v)\neq 0\}.$$

Then, by [12, Lemma 1], $X_s(\phi,f)$ is finite. Now $[[M^{S,\leq}]]$ can be turned into a right $[[R^{S,\leq}]]$-module by the scalar multiplication defined as follows

$$(\phi f)(s)=\sum_{(u,v)\in X_s(\phi,f)} \phi(u)f(v)$$

for each $f\in[[R^{S,\leq}]]$ and each $\phi\in[[M^{S,\leq}]]$. $[[M^{S,\leq}]]$ is called the module of generalized power series over a right R-module M. The elements of $[[M^{S,\leq}]]$ are called generalized power series with coefficients in M and exponents in S.

Similarly, if M is a left R-module, then $[[M^{S,\leq}]]$ is a left $[[R^{S,\leq}]]$-module. Examples and results of modules of generalized power series are given in [12].

Let M, N be right R-modules and $\alpha:M\longrightarrow N$ an R-homomorphism. Define a mapping

$[[\alpha^{S,\leq}]] : [[M^{S,\leq}]] \longrightarrow [[N^{S,\leq}]]$ via

$$[[\alpha^{S,\leq}]](g) : \qquad \begin{aligned} S &\longrightarrow N \\ s &\longrightarrow \alpha(g(s)) \end{aligned}$$

for any $g \in [[M^{S,\leq}]]$. Clearly $\mathrm{supp}([[\alpha^{S,\leq}]](g)) \subseteq \mathrm{supp}(g)$. Thus it follows that $\mathrm{supp}([[\alpha^{S,\leq}]](g))$ is artinian and narrow. Hence $[[\alpha^{S,\leq}]](g) \in [[N^{S,\leq}]]$. This means that $[[\alpha^{S,\leq}]]$ is well-defined. The following results appeared in [3].

Lemma 2.1 *(1) $[[\alpha^{S,\leq}]]$ is an $[[R^{S,\leq}]]$-homomorphism.*

(2) If $M \xrightarrow{\alpha} N \xrightarrow{\beta} L$ is a complex, then so is

$$[[M^{S,\leq}]] \xrightarrow{[[\alpha^{S,\leq}]]} [[N^{S,\leq}]] \xrightarrow{[[\beta^{S,\leq}]]} [[L^{S,\leq}]].$$

(3) The functor $[[(-)^{S,\leq}]] : \text{Mod-}R \longrightarrow \text{Mod-}[[R^{S,\leq}]]$ is exact.

Let M be a right R-module. Define a mapping $\alpha : M \otimes_R [[R^{S,\leq}]] \longrightarrow [[M^{S,\leq}]]$ via

$$\alpha(\sum(m_i \otimes f_i))(s) = \sum m_i f_i(s), \quad \forall m_i \in M,\ \forall f_i \in [[R^{S,\leq}]],\ \forall s \in S.$$

Lemma 2.2 *If M is a finitely presented right R-module, then α is an isomorphism of right $[[R^{S,\leq}]]$-modules.*

Proof The conclusion follows from [4, Lemma 5]. □

Example 2.3 The converse of Lemma 2.2 is not true in general. Let R be a ring. Suppose that the monoid S is trivially ordered. Then the artinian and narrow subsets are the finite subsets. Thus for every right R-module M, there exists an isomorphism of right R-modules $[[M^{S,\leq}]] \cong \oplus_{s\in S} M$. Similarly, there exists an isomorphism of left R-modules $[[R^{S,\leq}]] \cong \oplus_{s\in S} R$. Thus there exists an isomorphism of Abelian groups $\beta : M \otimes_R [[R^{S,\leq}]] \cong M \otimes_R (\oplus_{s\in S} R) \cong \oplus_{s\in S}(M \otimes_R R) \cong \oplus_{s\in S} M \cong [[M^{S,\leq}]]$. It is easy to see that $\alpha = \beta$. Thus, by Lemma 1 of [4], α is an isomorphism of right $[[R^{S,\leq}]]$-modules. But we can take M such that it is not finitely presented.

The following result appeared in [4, Lemma 7].

Lemma 2.4 *If P_R is finitely generated projective, then $[[P^{S,\leq}]]$ is a projective right $[[R^{S,\leq}]]$-module.*

3. Generalized Macaulay-Northcott modules

If M is a left R-module, we let $[M^{S,\leq}]$ be the set of all maps $\phi : S \longrightarrow M$ such that the set $\mathrm{supp}(\phi) = \{s \in S | \phi(s) \neq 0\}$ is finite. Now $[M^{S,\leq}]$ can be turned into a left $[[R^{S,\leq}]]$-module under some additional conditions. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows

$$(f\phi)(s) = \sum_{t\in S} f(t)\phi(s+t), \quad \text{for every} \ \ s \in S,$$

where $f \in [[R^{S,\leq}]]$, and $\phi \in [M^{S,\leq}]$. Since the set $\mathrm{supp}(\phi)$ is finite, this multiplication is well-defined. If $(S,\leq)$ is a strictly totally ordered monoid which is also artinian, then, from [2], $[M^{S,\leq}]$ becomes a left $[[R^{S,\leq}]]$-module, which we call the generalized Macaulay-Northcott module.

For example, if $S=\mathbb{N}$ and $\leq$ is the usual order, then $[M^{\mathbb{N},\leq}] \cong M[x^{-1}]$, the usual left $R[[x]]$-module introduced in [13] and [4], which is called the Macaulay-Northcott module in [14] and [1].

We shall henceforth assume that $(S,\leq)$ is a strictly totally ordered monoid which is also artinian. Then it is easy to see that $(S,\leq)$ satisfies the condition that $0 \leq s$ for every $s \in S$[15].

For any abelian additive group G, we denote by $[[G^{S,\leq}]]$ the set of all maps $h: S \longrightarrow G$. With pointwise addition, $[[G^{S,\leq}]]$ is an abelian additive group.

For any R-homomorphism $\alpha: M \longrightarrow N$, define $f \in [[\mathrm{Hom}_R(M,N)^{S,\leq}]]$ via $f(0)=\alpha$ and $f(x)=0$ for all $0 \neq x \in S$. By ([2], Lemma 2.3) and its proof, there exists $[\alpha^{S,\leq}] \in \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],[N^{S,\leq}])$ such that for any $\phi \in [M^{S,\leq}]$ and any $s \in S$,

$$[\alpha^{S,\leq}](\phi)(s) = \sum_{u\in S} f(u)(\phi(s+u)) = \alpha(\phi(s)).$$

The following result appeared in [3, Lemma 3.2].

Lemma 3.1 *The functor* $[(-)^{S,\leq}]: R\text{-Mod} \longrightarrow [[R^{S,\leq}]]\text{-Mod}$ *defined as* $[(-)^{S,\leq}](M)=[M^{S,\leq}]$, $[(-)^{S,\leq}](\alpha)=[\alpha^{S,\leq}]$, *is exact.*

Lemma 3.2 *Let $N \leq M$ be left R-modules. Then*

$$[M^{S,\leq}]/[N^{S,\leq}] \cong [(M/N)^{S,\leq}]$$

as left $[[R^{S,\leq}]]$-modules.

Proof The conclusion follows from Lemma 3.1. □

Lemma 3.3 *Let M be a finitely presented right R-module and N a left R-module. Then there is a natural isomorphism* $[[M^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \cong [(M\otimes_R N)^{S,\leq}]$.

Proof It is easy to see that there exists an isomorphism of left R-modules $[N^{S,\leq}] \cong \oplus_{s\in S} N$. By Lemma 2.2, there exists a natural isomorphism of right $[[R^{S,\leq}]]$-modules $M\otimes_R [[R^{S,\leq}]] \cong [[M^{S,\leq}]]$ since M is finitely presented. Now, we have

$$\begin{aligned}
[[M^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] &\cong (M\otimes_R [[R^{S,\leq}]]) \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \\
&\cong M\otimes_R ([[R^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}]) \\
&\cong M\otimes_R [N^{S,\leq}] \cong M\otimes_R (\oplus_{s\in S} N) \\
&\cong \oplus_{s\in S}(M\otimes_R N) \\
&\cong [(M\otimes_R N)^{S,\leq}].
\end{aligned}$$

Clearly all isomorphisms mentioned above are natural. □

4. Tor-groups

Theorem 4.1 *Let S be a strictly totally ordered monoid which is also artinian and R a right noetherian ring. If M is a finitely generated right R-module and N is a left R-module, then there exist isomorphisms of Abelian groups:*

$$\mathrm{Tor}_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]],[N^{S,\leq}]) \cong [\mathrm{Tor}_i^R(M,N)^{S,\leq}] \cong \bigoplus_{s\in S} \mathrm{Tor}_i^R(M,N).$$

Proof Since R is right noetherian, there exists a projective resolution

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

of M such that $P_0, P_1, \ldots$ are finitely generated and projective. Then, by Lemmas 2.1 and 2.4,

$$\cdots \longrightarrow [[P_2^{S,\leq}]] \longrightarrow [[P_1^{S,\leq}]] \longrightarrow [[P_0^{S,\leq}]] \longrightarrow [[M^{S,\leq}]] \longrightarrow 0$$

is a projective resolution of right $[[R^{S,\leq}]]$-module $[[M^{S,\leq}]]$. Consider the deleted projective resolution

$$\cdots \longrightarrow [[P_2^{S,\leq}]] \xrightarrow{[[\delta_2^{S,\leq}]]} [[P_1^{S,\leq}]] \xrightarrow{[[\delta_1^{S,\leq}]]} [[P_0^{S,\leq}]] \longrightarrow 0.$$

We have the complex

$$\cdots \longrightarrow [[P_2^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \xrightarrow{[[\delta_2^{S,\leq}]](*)} [[P_1^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}]$$
$$\xrightarrow{[[\delta_1^{S,\leq}]](*)} [[P_0^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \xrightarrow{[[\delta_0^{S,\leq}]](*)} 0,$$

where $[[\delta_i^{S,\leq}]](*) = [[\delta_i^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} 1_{[N^{S,\leq}]}$ for every $i = 0, 1, \ldots$. On the other hand, we have the complex

$$\cdots \longrightarrow P_2 \otimes_R N \xrightarrow{\delta_2(*)} P_1 \otimes_R N \xrightarrow{\delta_1(*)} P_0 \otimes_R N \xrightarrow{\delta_0(*)} 0,$$

where $\delta_i(*) = \delta_i \otimes_R 1_N$ for every $i = 0, 1, \ldots$. Thus, by Lemma 3.1, we have the complex

$$\cdots \longrightarrow [(P_2 \otimes_R N)^{S,\leq}] \xrightarrow{[\delta_2(*)^{S,\leq}]} [(P_1 \otimes_R N)^{S,\leq}]$$
$$\xrightarrow{[\delta_1(*)^{S,\leq}]} [(P_0 \otimes_R N)^{S,\leq}] \xrightarrow{[\delta_0(*)^{S,\leq}]} 0.$$

Clearly $P_0, P_1, \ldots$ are finitely presented. Thus by Lemma 3.3, there exists a natural isomorphism

$$[[P_i^{S,\leq}]] \otimes_{[[R^{S,\leq}]]} [N^{S,\leq}] \cong [(P_i \otimes_R N)^{S,\leq}].$$

Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}([\delta_i(*)^{S,\leq}]) \arrow[r] \arrow[d] & {[(P_i \otimes_R N)^{S,\leq}]} \arrow[r, "{[\delta_i(*)^{S,\leq}]}"] \arrow[d, "\simeq"] & {[(P_{i-1} \otimes_R N)^{S,\leq}]} \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathrm{Ker}([\delta_i(*)^{S,\leq}]) \arrow[r] & {[(P_i \otimes_R N)^{S,\leq}]} \arrow[r, "{[\delta_i(*)^{S,\leq}]}"] & {[(P_{i-1} \otimes_R N)^{S,\leq}]}
\end{tikzcd}$$

It follows that $\mathrm{Ker}([\delta_i(*)^{S,\leq}]) \simeq [\mathrm{Ker}(\delta_i(*)^{S,\leq})]$ by Five Lemma. Similarly, we have

$$\mathrm{Im}([\delta_{i+1}(*)^{S,\leq}]) \simeq [\mathrm{Im}(\delta_{i+1}(*))^{S,\leq}].$$

Thus, by Lammas 3.1, 3.2 and 3.3, we have

$$\begin{aligned}\mathrm{Tor}_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]],[N^{S,\leq}]) &= \mathrm{Ker}([[\delta_i^{S,\leq}]](*))/\mathrm{Im}([[\delta_{i+1}^{S,\leq}]](*)) \\ &\cong \mathrm{Ker}([\delta_i(*)^{S,\leq}])/\mathrm{Im}([\delta_{i+1}(*)^{S,\leq}]) \\ &\cong [\mathrm{Ker}(\delta_i(*))^{S,\leq}]/[\mathrm{Im}(\delta_{i+1}(*))^{S,\leq}] \\ &\cong [(\mathrm{Ker}(\delta_i(*))/\mathrm{Im}(\delta_{i+1}(*)))^{S,\leq}] \\ &= [\mathrm{Tor}_i^R(M,N)^{S,\leq}].\end{aligned}$$

The isomorphism $[\mathrm{Tor}_i^R(M,N)^{S,\leq}] \cong \oplus_{s\in S}\mathrm{Tor}_i^R(M,N)$ is clear. □

Corollary 4.2 *If R is a right noetherian ring, M is a finitely generated right R-module and N is a left R-module, then there exist isomorphisms of Abelian groups*

$$\mathrm{Tor}_i^{R[[x]]}(M[[x]],N[x^{-1}]) \cong \bigoplus_{n=0}^{\infty}\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(M,N)[x^{-1}].$$

Corollary 4.3 *Let S be a torsion-free and cancellative monoid, and $(S,\leq)$ be artinian and narrow. If R is a right noetherian ring, M is a finitely generated right R-module and N is a left R-module, then*

$$\mathrm{Tor}_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]],[N^{S,\leq}]) \cong \bigoplus_S \mathrm{Tor}_i^R(M,N).$$

Proof If $(S,\leq)$ is torsion-free and cancellative, then by [5, 3.3], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$, that is, for any $s,t\in S$, $s\leq t$ implies $s\leq' t$. Since $(S,\leq)$ is artinian and narrow, by [5, 2.5] it follows that $(S,\leq')$ is artinian and narrow. Thus, by Theorem 4.1, $\mathrm{Tor}_i^{[[R^{S,\leq'}]]}([[M^{S,\leq'}]],[N^{S,\leq'}]) \cong \oplus_S\mathrm{Tor}_i^R(M,N)$.

On the other hand, since $(S,\leq)$ is narrow, by [5, 4.4], $[[R^{S,\leq}]] = [[R^{S,\leq'}]]$. Clearly $[[M^{S,\leq}]] = [[M^{S,\leq'}]]$ and $[N^{S,\leq}] = [N^{S,\leq'}]$. Now the result follows. □

Any submonoid of the additive monoid $\mathbb{N}\cup\{0\}$ is called a numerical monoid. We have

Corollary 4.4 *Let S be a numerical monoid and $\leq$ the usual natural order of $\mathbb{N}\cup\{0\}$. If R is a right noetherian ring, M is a finitely generated right R-module and N is a left R-module, then*

$$\mathrm{Tor}_i^{[[R^{S,\leq}]]}([[M^{S,\leq}]],[N^{S,\leq}]) \cong \bigoplus_S \mathrm{Tor}_i^R(M,N).$$

Corollary 4.5 *Suppose that $(S_1,\leq_1),\dots,(S_n,\leq_n)$ are strictly totally ordered monoids which are artinian. Denote by $(\mathrm{lex}\leq)$ and $(\mathrm{rev\,lex}\leq)$ the lexicographic order, the reverse lexicographic order, respectively, on the monoid $S_1\times\cdots\times S_n$. If R is a right noetherian ring, M is a finitely generated right R-module and N is a left R-module, then there exist isomorphisms of Abelian groups*

$$\begin{aligned}&\mathrm{Tor}_i^{[[R^{S_1\times\cdots\times S_n,(\mathrm{lex}\leq)}]]}([[M^{S_1\times\cdots\times S_n,(\mathrm{lex}\leq)}]],[N^{S_1\times\cdots\times S_n,(\mathrm{lex}\leq)}]) \\ &\cong \mathrm{Tor}_i^{[[R^{S_1\times\cdots\times S_n,(\mathrm{rev\,lex}\leq)}]]}([[M^{S_1\times\cdots\times S_n,(\mathrm{rev\,lex}\leq)}]],[N^{S_1\times\cdots\times S_n,(\mathrm{rev\,lex}\leq)}]) \\ &\cong \bigoplus_{S_1\times\cdots\times S_n}\mathrm{Tor}_i^R(M,N).\end{aligned}$$

Proof It is easy to see that $(S_1 \times \cdots \times S_n, (\text{lex} \leq))$ and $(S_1 \times \cdots \times S_n, (\text{rev lex} \leq))$ are strictly totally ordered monoids which are artinian. Thus the result follows from Theorem 4.1. □

References

[1] PARK S. *The Macaulay-Northcott functor* [J]. Arch. Math. (Basel), 1994, **63**(3): 225–230.
[2] LIU Zhongkui, CHENG Hui. *Quasi-duality for the rings of generalized power series* [J]. Comm. Algebra, 2000, **28**(3): 1175–1188.
[3] LIU Zhongkui. *The ext-groups of generalized Macaulay-Northcott modules* [J]. Vietnam J. Math., 2004, **32**(2): 161–169.
[4] LIU Zhongkui, AHSAN J. *The Tor-groups of modules of generalized power series* [J]. Algebra Colloq., 2005, **12**(3): 477–484.
[5] RIBENBOIM P. *Noetherian rings of generalized power series* [J]. J. Pure Appl. Algebra, 1992, **79**(3): 293–312.
[6] RIBENBOIM P. *Rings of generalized power series. II. Units and zero-divisors* [J]. J. Algebra, 1994, **168**(1): 71–89.
[7] RIBENBOIM P. *Special properties of generalized power series* [J]. J. Algebra, 1995, **173**(3): 566–586.
[8] RIBENBOIM P. *Semisimple rings and von Neumann regular rings of generalized power series* [J]. J. Algebra, 1997, **198**(2): 327–338.
[9] VARADARAJAN K. *Noetherian generalized power series rings and modules* [J]. Comm. Algebra, 2001, **29**(1): 245–251.
[10] LIU Zhongkui. *Injectivity of modules of generalized inverse polynomials* [J]. Comm. Algebra, 2001, **29**(2): 583–592.
[11] LIU Zhongkui. *Morita duality for the rings of generalized power series* [J]. Acta Math. Sin. (Engl. Ser.), 2002, **18**(2): 245–252.
[12] LIU Zhongkui. *A note on Hopfian modules* [J]. Comm. Algebra, 2000, **28**(6): 3031–3040.
[13] MCKERROW A S. *On the injective dimension of modules of power series* [J]. Quart. J. Math. Oxford, 1974, **25**(2): 359–368.
[14] NORTHCOTT D G. *Injective envelopes and inverse polynomials* [J]. J. London Math. Soc., 1974, **8**(2): 290–296.
[15] PARK S. *Inverse polynomials and injective covers* [J]. Comm. Algebra, 1993, **21**(12): 4599–4613.