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A Generalization of Semiregular Rings

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Abstract In this paper, the concept of generalized semiregular rings is extended to generalized weak semiregular rings. Some properties of these rings are studied and some results about semiregular rings and generalized semiregular rings are extended. We also give some equivalent characterizations of I-weak semiregular rings.

Keywords weak semiregular rings; generalized semiregular rings; generalized weak semiregular rings; I-weak semiregular rings.

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1. Introduction

Throughout this paper, the ring R is always associative with identity and all modules are unitary. A ring R is called semiregular[1], if for any $a\in R$, there exists an idempotent $g\in Ra$ such that $a(1-g)\in J(R)$. A ring R is called right generalized semiregular[2] , if for any $a\in R$, there exist two left ideals P, L of R such that $lr(a) = P \oplus L$, where $P \subseteq Ra$ and $Ra\bigcap L$ is small in R. From [2], we know that generalized semiregular rings are non-trivial generalizations of semiregular rings.

In this paper, we call a ring R weak semiregular if, for any $a \in R$, there exist $0 \neq b \in R$ and an idempotent $g\in abR$ such that $(1-g)ab \in J(R)$. An example shows that these rings are non-trivial generalizations of semiregular rings. We call a ring R right generalized weak semiregular, if for any $a\in R$, there exist $0 \neq b \in R$ and two left ideals P, L of R such that $ab \neq 0$ and $lr(ab) = P \oplus L$, where $P \subseteq Rab$ and $Rab\bigcap L$ is small in R. This concept is a generalization of generalized semiregular ring. The notion of weak semiregular rings is left-right symmetric, but we do not know whether this is true for generalized weak semiregular rings.

In Section 2, we mainly study the properties of generalized weak semiregular rings. Some known results about generalized semiregular rings are extended. Also we consider corner subrings of generalized weak semiregular rings, and prove that if R is a right generalized weak semiregular ring with $e^2 = e \in R$ right semicentral, then eRe is a right generalized weak semiregular ring.

A ring R is called I-weak semiregular if, for any $a \in R$, there exist $0 \neq b \in R$ and $e^2 = e \in abR$ such that $(1-e)ab \in I$. These rings are non-trivial generalizations of I-semiregular rings. We give

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some equivalent conditions of I-weak semiregular rings and consider the I-weak semiregularity of related rings in Section 3.

In what follows, for a non-empty subset X of R, the right (resp. left) annihilator of X in R will be denoted by $r(X)$ (resp. $l(X)$). Also $J(R)$ and $Z(R_R) = Z_r$ will denote the *Jacobson radical* and the right singular ideal of R, respectively. We write the Goldie torsion right ideal of R as Z_2^r, and write $I \lhd R$ to indicate that I is a two-sided ideal of R. See [3–6] for the other undefined concepts and notations.

2. Generalized weak semiregular rings

An element $a \in R$ is von Neumann regular if $a = aca$ for some $c \in R$, and R itself is called von Neumann regular if every element is regular. A ring R is called strongly regular if, for any $a \in R$, there exists $b \in R$ such that $a = a^2b$.

A submodule K of a module M is said to be small in M[3], if $K + N \neq M$ for every submodule $N \neq M$. A submodule N of a module M lies over a summand of M if there exists a direct decomposition $M = P \oplus Q$ with $P \subseteq N$ and $Q \bigcap N$ is small in M. A ring R is said to be semiregular, if for any $a \in R$, there exists an idempotent $g \in Ra$ such that $a(1-g) \in J(R)$. We start with the following lemma.

Lemma 2.1 *The followings are equivalent for an element a of a ring R:*

(1) *There exist $0 \neq b \in R$ and $e^2 = e \in abR$ such that $(1-e)ab \in J(R)$;*

(2) *There exist $0 \neq b \in R$ and $e^2 = e \in Rab$ such that $ab(1-e) \in J(R)$;*

(3) *There exist $0 \neq b \in R$ and a von Neumann regular element $c \in R$ with $ab - c \in J(R)$;*

(4) *There exists $0 \neq b \in R$ such that abR lies over a summand of R_R;*

(5) *There exists $0 \neq b \in R$ such that Rab lies over a summand of ${}_RR$.*

Proof We only need to prove the equivalences of (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

(1)$\Rightarrow$(3). Let $e = abx$ where $x \in R$, $c = abxab$. Then $cxc = c$ and $ab - c = (1 - abx)ab = (1-e)ab \in J(R)$.

(3)$\Rightarrow$(1). Let $ab - c \in J(R)$ where $c \in R$ is regular. If $c = cxc$ where $x \in R$, let $e = cx$. Then $e^2 = e \in R$ and $ec = c$. Hence $ab - eab = (1-e)(ab-c) \in J(R)$ and $e - abx = (c-ab)x \in J(R)$. So $1-(e-abx)$ is a unit. Let $(1-e+abx)t = 1$ where $t \in R$. Then $eabxt = e$ and $(eab)xt(eab) = eab$. This shows that eab is regular. So there exists $d \in R$ such that $(eab)d(eab) = eab$ and $d(eab)d = d$. Let $f = abde$. Then $f^2 = f \in abR$ and $ab - fab = (ab - eab)(1 - deab) \in J(R)$.

(1)$\Rightarrow$(4). Given (1), it is clear that $R = eR \oplus (1-e)R$ and $eR \subseteq abR$. Moreover, $(1-e)R \bigcap abR = (1-e)abR \subseteq J(R)$ is small in R.

(4)$\Rightarrow$(1). Let $R = P \oplus Q$ with $P \subseteq abR$ and $abR \bigcap Q$ is small in R. Then there exists $e^2 = e \in R$ such that $P = eR$ and $Q = (1-e)R$. So $e \in abR$. Since $abR \bigcap Q = abR \bigcap (1-e)R = (1-e)abR$ is small in R, $(1-e)ab \in J(R)$.

Similarly, we have the equivalences of (2)$\Leftrightarrow$(3)$\Leftrightarrow$(5). $\square$

Definition 2.2 *A ring R is called weak semiregular if every element of R satisfies the conditions* *in Lemma 2.1.*

A ring R is called semi-π-regular[2], if for any $a \in R$, there exists $e^2 = e \in a^n R$ for some positive integer n such that $(1-e)a^n \in J(R)$. Clearly, both semiregular rings and semi-π-regular rings are weak semiregular rings.

The following example shows that weak semiregular rings are non-trivial generalizations of semiregular rings.

Example 2.3 Let $M = M_2(Z_2)$, where Z_2 is the ring of integers module 2, and $I = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix}$. Put $R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | x_1, x_2, \ldots, x_n \in M, x \in I\}$. By [2], R is semi-π-regular but not semiregular. Thus R is weak semiregular ring which is not semiregular.

It is known that a ring R is semiregular if and only if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. But for weak semiregular rings, we only have

Proposition 2.4 *If $R/J(R)$ is weak semiregular and idempotents can be lifted modulo $J(R)$, then R is weak semiregular.*

Proof Let $a \in R$. Write $\bar{R} = R/J(R)$ and $\bar{a} = a + J(R)$. Then there exist $\bar{b} \in \bar{R}$ and $\bar{e}^2 = \bar{e} \in \bar{a}\bar{b}\bar{R}$ such that $(\bar{1} - \bar{e})\bar{a}\bar{b} \in J(R/J(R)) = 0$. So $\bar{a}\bar{b} = \bar{e}\bar{a}\bar{b}$. Choose $f^2 = f \in R$ such that $\bar{f} = \bar{e}$. Then $u = 1 - f + e$ is a unit in R. So $g = ufu^{-1} = efu^{-1}$ satisfies $g^2 = g \in abR$. Since $\bar{f} = \bar{e} = \bar{g}$, it follows that $(1-g)ab \in J(R)$. Thus R is weak semiregular by Lemma 2.1. $\square$

Definition 2.5 *A ring R is called right generalized weak semiregular if, for any $a \in R$, there exist $0 \neq b \in R$ and two left ideals P, L of R such that $ab \neq 0$ and $lr(ab) = P \oplus L$, where $P \subseteq Rab$ and $Rab \bigcap L$ is small in R.*

Similarly, we may define left generalized weak semiregular rings. Clearly, every right generalized semiregular ring is right generalized weak semiregular ring.

First we study the relations between weak semiregular rings and generalized weak semiregular rings.

Proposition 2.6 *If R is a weak semiregular ring , then R is right generalized weak semiregular ring.*

Proof Let R be a weak semiregular ring and $a \in R$. Then there exist $0 \neq b \in R$ and $e^2 = e \in Rab$ such that $ab(1-e) \in J(R)$. Thus $R = Re \oplus R(1-e)$, where $Re \subseteq Rab$ and $Rab(1-e) \subseteq J(R)$ is small in R. Since $Rab \subseteq lr(ab)$, it follows by the modular law that $lr(ab) = lr(ab) \bigcap (Re \oplus R(1-e)) = Re \oplus (lr(ab) \bigcap R(1-e))$ and $Rab \bigcap (lr(ab) \bigcap R(1-e)) = Rab \bigcap R(1-e) = Rab(1-e)$ is small in R. Hence R is right generalized weak semiregular. $\square$

Now we show that right generalized weak semiregular rings are weak semiregular under some sufficient conditions.

Theorem 2.7 *Let R be a right generalized weak semiregular ring. If for any $a \in R$ there exists $e^2 = e \in R$ such that $r(a) = r(e)$, then R is weak semiregular.*

Proof Let $a \in R$. Then there exist $0 \neq b \in R$ and two left ideals P, L of R such that $ab \neq 0$ and $lr(ab) = P \oplus L$, where $P \subseteq Rab$ and $Rab \bigcap L$ is small in R. Since $r(ab) = r(e)$ where $e^2 = e \in R$, we have $lr(ab) = lr(e) = Re = P \oplus L$. So $ab = abe$. Take $e = g + t$, where $g = rab \in P \subseteq Rab$ and $t \in L$. Then $ab = abe = abrab + abt$ and $rab = rabrab + rabt$. Thus $rab - rabrab = rabt \in P \bigcap L = 0$ and $ab - abrab = abt \in Rab \bigcap L \subseteq J(R)$. It follows from the fact that $g^2 = g \in Rab$ and $ab(1-g) \in J(R)$, a is a weak semiregular element. Hence R is weak semiregular. □

A ring R is called right (resp. left) PP[7] if every principal right (resp. left) ideal of R is projective, equivalently, for any $a \in R$ there exists an idempotent $e \in R$ such that $r(a) = eR$.

Corollary 2.8 *Let R be a right PP-ring. If R is right generalized weak semiregular, then R is weak semiregular.*

Lemma 2.9[8] *Let $a \in R$. If $aR \cong eR$ where $e^2 = e \in R$, then there exists an idempotent $f \in R$ such that $af = f$ and $r(a) = r(f)$.*

Corollary 2.10 *Let $a \in R$ be a right generalized weak semiregular element. If $aR \cong eR$ where $e^2 = e \in R$, then a is weak semiregular.*

Proof The conclusion follows from Lemma 2.9 and from the proof of Theorem 2.8.

Now we consider *corner* subrings of generalized weak semiregular rings. Recall that an idempotent element $e \in R$ is left (resp. right) semicentral[9], if $Re = eRe$ (resp. $eR = eRe$). In general we have

Theorem 2.11 *Let R be a right generalized weak semiregular ring. If $e^2 = e \in R$ is right semicentral, then eRe is right generalized weak semiregular.*

Proof Let $a \in eRe$. Then there exist $0 \neq b \in R$ and two left ideals P, L of R such that $ab \neq 0$ and $lr(ab) = P \oplus L$, where $P \subseteq Rab$ and $Rab \bigcap L$ is small in R. Since e is right semicentral, we have $eb \in eRe$, and $aeb = ab$ since $a \in eRe$. Now we claim that $l_{eRe}r_{eRe}(ab) = eP \oplus eL$. In fact $eP \bigcap eL \subseteq P \bigcap L = 0$. Take any $y \in eP \subseteq ePe$, $y = ey_1$, $y_1 \in P \subseteq lr(ab)$. Then for any $x \in r_{eRe}(ab) \subseteq r(ab)$, $y_1 x = 0$. So $yx = ey_1 x = 0$. Hence $y \in l_{eRe}r_{eRe}(ab)$, $eP \subseteq l_{eRe}r_{eRe}(ab)$. Similarly, $eL \subseteq l_{eRe}r_{eRe}(a)$. On the other hand, take $x \in l_{eRe}r_{eRe}(ab)$. Then for any $y \in r(ab)$, we have $abeye = abye = 0$ since $ab = aeb \in eRe$. So $xeye = 0$, which implies $xy = xey = xeye = 0$. Thus $l_{eRe}r_{eRe}(ab) \subseteq lr(ab)$. Let $x = s+t$, where $s \in P, t \in L$. Then $x = ex = es+et \in eP \oplus eL$. This shows that $l_{eRe}r_{eRe}(ab) = eP \oplus eL$. Since $eP \subseteq eRab \subseteq eReab$, it remains to prove that $eReab \bigcap eL$ is small in eRe. Since e is right semicentral, we have $eReab \bigcap eL \subseteq e(eReab \bigcap eL)e$. Since $eReab \bigcap eL \subseteq Rab \bigcap L \subseteq J(R)$, we have $eReab \bigcap eL \subseteq eJ(R)e = J(eRe)$. Thus eRe is a right generalized weak semiregular ring. □

Proposition 2.12 *Let e, f be orthogonal central idempotents of R. If eR and fR are right generalized weak semiregular rings, then $gR = eR \oplus fR$ is right generalized weak semiregular.*

Proof Let $a \in gR$. Then $ea \in eR, fa \in fR$. By assumption, there exists $0 \neq b \in eR$ such that $eab \neq 0$ and $l_{eR}r_{eR}(eab) = P_e \oplus L_e$, where $P_e \subseteq eReab = eRab$ and $eRab \bigcap L_e \subseteq J(eRe)$. Similarly, there exists $0 \neq t \in fR$ such that $fat \neq 0$ and $l_{fR}r_{fR}(fat) = P_f \oplus L_f$, where $P_f \subseteq fRat$ and $fRat \bigcap L_f \subseteq J(fRf)$. Then $eab + fat = a(eb + ft) \neq 0$. Let $h = eb + ft \in gR$. Then $ah \neq 0$. We claim that $l_{gR}r_{gR}(ah) = P_e \oplus P_f \oplus L_e \oplus L_f$. Take any $x \in l_{gR}r_{gR}(ah)$. Then for any $y \in r_{eR}(eab)$, we have $eaby = 0$. Hence $ahy = a(eb + ft)y = atfy = atfey = 0$, which gives $ahgy = ahyg = 0$ and $gy \in r_{gR}(ah)$. Thus $xy = xgy = 0$ and $exy = xye = 0$. So $ex \in l_{eR}r_{eR}(eab) = P_e \oplus L_e$. Similarly, $fx \in l_{fR}r_{fR(fat)} = P_f \oplus L_f$. Then $x = gx = ex + fx \in P_e \oplus P_f \oplus L_e \oplus L_f$. On the other hand, $P_e \oplus P_f \subseteq eRab \oplus fRat \subseteq gRah$. Let $x \in L_e$. For any $y \in r_{gR}(ah)$, we have $ahy = 0$, which implies $eahey = 0$. So $eabey = 0$. Since $L_e \subseteq l_{eR}r_{eR}(eab)$, $xey = 0$. Note that $L_e \subseteq eR \subseteq gR$ and $x = ex = xe$, so $xy = 0$ and $L_e \subseteq l_{gR}r_{gR}(ah)$. Similarly, $L_f \subseteq l_{gR}r_{gR}(ah)$. This shows that $l_{gR}r_{gR}(ah) = P_e \oplus P_f \oplus L_e \oplus L_f$. Since gR is a ring with identity, $J(gR)$ is small in gR, we have $gRah \bigcap (L_e \oplus L_f) \subseteq J(eR) \oplus J(fR) = J(gR)$ is small in gR. Thus gR is right generalized weak semiregular. □

Corollary 2.13 *Let e be a nonzero central idempotent of a ring R. Then eRe and $(1-e)R(1-e)$ are right generalized weak semiregular if and only if R is right generalized weak semiregular.*

Theorem 2.14 *Let $1 = f_1 + f_2 + \cdots + f_n$ in R, where $f_i's$ are orthogonal central idempotents. Then R is a right generalized weak semiregular ring if and only if each f_iR is a right generalized weak semiregular ring.*

3. I-Weak semiregular rings

Let I be an ideal of R. A ring R is called right I-semiregular[10], if for any $a \in R$, there exists $e^2 = e \in aR$ such that $a - ea \in I$. In this section, the concept of I-semiregular rings is generalized to I-weak semiregular rings, and some properties of I-weak semiregular rings are studied. We begin with the following lemma:

Lemma 3.1 *Let I be an ideal of a ring R. The following are equivalent for $a \in R$:*

(1) There exist $0 \neq b \in R$ and $e^2 = e \in abR$ such that $(1-e)ab \in I$;

(2) There exist $0 \neq b \in R$ and $e^2 = e \in abR$ such that $abR \bigcap (1-e)R \subseteq I$;

(3) There exists $0 \neq b \in R$ such that $abR = eR \oplus S$, where $e^2 = e$ and $S \subseteq I$ is a right ideal.

Proof Suppose (1) holds. Then $abR = eR \oplus [abR \bigcap (1-e)R]$. If $x \in abR \bigcap (1-e)R$, then $x = (1-e)x \in (1-e)abR \subseteq I$. This proves (2). If (2) holds, then let $S = abR \bigcap (1-e)R$, proving (3). Finally, given (3), let $ab = er + s, r \in R, s \in S$. Then $(1-e)ab = s - es \in S \subseteq I$, proving (1). □

Definition 3.2 *Let I be an ideal of R. An element $a \in R$ is called right I-weak semiregular if the conditions in Lemma 3.1 are satisfied, and R is called a right I-weak semiregular ring if every element of R is right I-weak semiregular.*

Left I-weak semiregular elements and rings are defined analogously. By Lemma 2.1, a ring

R is weak semiregular if it is right (equivalently left) J-weak semiregular.

The following example shows that I-weak semiregular rings need not be I-semiregular.

Example 3.3 Let C be a commutative Von Neumann regular ring with no minimal ideal and M a maximal ideal. Let R_1 be the matrix ring of the form $\begin{pmatrix} C & C \\ M & C \end{pmatrix}$, $R_2 = Z_2[[X]]$ be the formal power series ring over the ring Z_2, and $R = R_1 \oplus R_2$. By [2], R is a semi-π-regular ring but not semiregular. Thus R is a $J(R)$-weak semiregular ring but not $J(R)$-semiregular.

If $I \lhd R$, we say that I *respects* a right ideal $T \subseteq R$[11], if there exists $e^2 = e \in T$ such that $(1-e)T \subseteq I$.

Lemma 3.4 *Let $I \lhd R$, $a, b \in R$. Then I respects abR if and only if I respects Rab.*

Proof If I respects abR, then there exists $e^2 = e \in abR$ such that $(1-e)abR \subseteq I$. If $e = abr$ where $r \in R$, and let $f = reab$. Then $f^2 = f$ and $f \in Rab$. Since $ab(1-f) = ab - abreab = (1-e)ab \in I$, $Rab(1-f) \subseteq I$. Thus I respects Rab. □

Now we give some equivalent characterizations of I-weak semiregular rings.

Theorem 3.5 *The following conditions are equivalent for $I \lhd R$:*

(1) R is I-weak semiregular;

(2) For any $a \in R$, I respects abR for some $b \in R$;

(3) For any $a \in R$, there exist $b, c \in R$ such that $cabc = c$ and $ab - abcab \in I$;

(4) For any $a \in R$, there exist $0 \neq b \in R$ and a regular element $d \in abR$ such that $ab - d \in I$.

Proof (1)⇒(2). Clear.

(2)⇒(3). Since I respects abR, choose $e^2 = e \in abR$ such that $(1-e)abR \subseteq I$. Let $e = abd$ where $d \in R$. Then $cabc = ce = c$ and $ab - abcab = ab - eab \in I$.

(3)⇒(4). Let b, c as in (3). If $d = abcab$, then $dcd = d$ and $ab - d \in I$.

(4)⇒(1). Let $ab - d \in I$ where $d \in abR$ is regular. If $dcd = d$ with $c \in R$, let $f = dc$. Then $f^2 = f \in abR$ and $fd = d$. So $ab - fab = (1-f)(ab-d) \in I$.

If $I \lhd R$, the ring R is called I-semiperfect[12]. If R/I is semisimple and idempotents can be lifted modulo I, equivalently, I *respects* every right ideals of R. So I-semiperfect rings are I-weak semiregular.

There is an artinian ring R that is both Z_l- and Z_2^l-weak semiregular but neither Z_r- nor Z_2^r-semiregular.

Example 3.6 Let $R = \begin{pmatrix} Z_4 & Z_2 \\ 0 & Z_2 \end{pmatrix}$. Then $Z_r = \begin{pmatrix} 2Z_4 & 0 \\ 0 & 0 \end{pmatrix}$ and $Z_l = \begin{pmatrix} 2Z_4 & Z_2 \\ 0 & 0 \end{pmatrix}$. Moreover, $Z_2^l = R$ and $Z_2^r = Z_r$. From [11], we know that R is both Z_l- and Z_2^l-semiperfect but neither Z_r- nor Z_2^r-semiregular. Hence R is a Z_l- and Z_2^l-weak semiregular ring.

Finally, we show that I-weak semiregularity is inherited by related rings in a natural way.

Proposition 3.7 *Let R be an I-weak semiregular ring, where $I \lhd R$.*

(1) If $\varphi : R \longrightarrow S$ is an onto ring morphism, then S is $\varphi(I)$-weak semiregular.

(2) If $e^2 = e \in R$, then eRe is eIe-weak semiregular.

Proof (1) For any $b \in S$, write $b = \varphi(a)$ where $a \in R$. By assumption, there exist $0 \neq b \in R$ and $e^2 = e \in abR$ such that $(1-e)ab \in I$. Let $f = \varphi(e)$. Then $f^2 = f \in btS$ where $t = \varphi(b) \neq 0 \in S$. Moreover, $(1-f)bt \in \varphi(I)$. Hence S is $\varphi(I)$-weak semiregular.

(2) For any $a \in eRe$, choose $f^2 = f \in abR$ for some $0 \neq b \in R$ such that $(1-f)ab \in I$. Then $ef = f$. Let $f = abr$ where $r \in R$, $g = fe$. Then $g^2 = g \in at(eRe)$ where $t = ebre \in eRe$. Moreover, $(1-g)at = (1-f)at \in I \bigcap eRe = eIe$. Hence eRe is eIe-weak semiregular. $\square$

Corollary 3.8 *Let $R = \prod_{x \in X} R_x$, $I = \prod_{x \in X} I_x$, where $I_x \lhd R_x$ for each x. Then R is I-weak semiregular if and only if each R_x is I_x-weak semiregular.*

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