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Asymptotic Behavior of Global Classical Solutions to Quasilinear Hyperbolic Systems of Diagonal Form

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Abstract This paper deals with the asymptotic behavior of global classical solutions to quasilinear hyperbolic systems of diagonal form with weakly linearly degenerate characteristic fields. On the basis of global existence and uniqueness of C^1 solution, we prove that the solution to the Cauchy problem approaches a combination of C^1 traveling wave solutions when t tends to the infinity.

Keywords quasilinear hyperbolic systems of diagonal form; weak linear degeneracy; global classical solution; rich system; traveling wave.

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1. Introduction and main result

In this paper, we consider the following Cauchy problem for quasilinear hyperbolic systems of diagonal form:

$$\begin{cases} \dfrac{\partial u_i}{\partial t}+\lambda_i(u)\dfrac{\partial u_i}{\partial x}=0, \quad i=1,\dots,n, \\ t=0: u=f(x), \end{cases} \tag{1}$$

where $u=(u_1,\dots,u_n)^{\mathrm{T}}$ is the unknown vector-valued function of (t,x), $\lambda_i(u)$ $(i=1,\dots,n)$ are supposed to be suitably smooth and $f(x)=(f_1(x),\dots,f_n(x))^{\mathrm{T}}\in(C_b^1(\mathbf{R})\cap W^{1,1}(\mathbf{R}))^n$, $C_b^1(\mathbf{R})$ is the space of C^1 functions with bounded $C^1(\mathbf{R})$ norm. We suppose that system (1) is strictly hyperbolic such that

$$\lambda_{i+1}(u)-\lambda_i(v)\geqslant\delta_0, \quad i=1,\dots,n-1 \tag{2}$$

for any given u and v on the domain under consideration, where δ_0 is a positive constant. We suppose furthermore that for each $i=1,\dots,n$, the i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate, i.e.,

$$\lambda_i(0,\dots,0,u_i,0,\dots,0)=\lambda_i(0). \tag{3}$$

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We say that system (1) is rich, if there exist n positive functions $N_i(u) > 0$ ($i = 1, \ldots, n$) such that on the domain under consideration we have

$$(\lambda_j(u) - \lambda_i(u)) \frac{\partial N_i(u)}{\partial u_j} = N_i(u) \frac{\partial \lambda_i(u)}{\partial u_j}, \quad \forall j \neq i.$$

The notion of rich system was introduced by Serre [1]. If system (1) possesses a form of conservation laws, then it must be rich. There are many results about the existence of global classical solutions to the Cauchy problem for quasilinear hyperbolic systems [2–6]. Based on these results, the asymptotic behavior of global classical solutions was studied in [7]–[9].

On the basis of [5], we prove the following theorem.

Theorem 1 *Let $f(x) \in (C_b^1(\mathbf{R}) \cap W^{1,1}(\mathbf{R}))^n$. Assume that $\lambda_i(u)$ ($i = 1, \ldots, n$) are C^2 functions, system (1) is weakly linearly degenerate and (2) holds. Then there exists a constant $\delta > 0$ depending only on $\|f\|_{C^0(\mathbf{R})}$ and $\|f'\|_{L^1(\mathbf{R})}$, such that if*

$$\|f'\|_{C^0(\mathbf{R})} \|f\|_{L^1(\mathbf{R})} \leqslant \delta, \tag{4}$$

then Cauchy problem (1) admits a unique global classical solution $u = u(t, x)$ for all $t \in \mathbf{R}$. Moreover, there is a unique C^1 vector-valued function $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))^{\mathrm{T}}$, such that $u(t, x)$ converges uniformly to $\sum_{i=1}^n \phi_i(x - \lambda_i(0)t) e_i$ as $t \to \infty$, where $e_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)^{\mathrm{T}}$. $\phi(x)$ is global Lipschitz continuous, i.e., there exists a positive constant K depending only on $\|f\|_{C^1(\mathbf{R})}$ and $\|f\|_{W^{1,1}(\mathbf{R})}$ such that

$$|\phi(\alpha) - \phi(\beta)| \leqslant K |\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbf{R}.$$

Furthermore, if system (1) is rich, $\lim_{|x| \to +\infty} f'(x) = 0$ and $f'(x)$ is global ρ-Hölder continuous ($0 < \rho \leqslant 1$), i.e.,

$$|f'(\alpha) - f'(\beta)| \leqslant \kappa |\alpha - \beta|^\rho, \quad \forall \alpha, \beta \in \mathbf{R}, \tag{5}$$

where κ is a positive constant, then $\phi'(x)$ satisfies

$$|\phi'(\alpha) - \phi'(\beta)| \leqslant K_1 |\alpha - \beta|^\rho + K_2 |\alpha - \beta|,$$

where K_1, K_2 are positive constants depending on κ, $\|f\|_{C^1(\mathbf{R})}$ and $\|f\|_{W^{1,1}(\mathbf{R})}$.

2. Uniform a priori estimate

In the following sections, we consider the global C^1 solutions for $t \geqslant 0$. The result for $t \leqslant 0$ follows easily by changing the variable from t to $-t$ in system (1). For convenience, we introduce

$$M = \sup_{x \in \mathbf{R}} |f'(x)| = \|f'(x)\|_{C^0(\mathbf{R})}, \quad M_0 = \sup_{x \in \mathbf{R}} |f(x)| = \|f(x)\|_{C^0(\mathbf{R})},$$

$$N_1 = \int_{-\infty}^{+\infty} |f(x)| \mathrm{d}x = \|f(x)\|_{L^1(\mathbf{R})}, \quad N_2 = \int_{-\infty}^{+\infty} |f'(x)| = \|f'(x)\|_{L^1(\mathbf{R})}.$$

For any fixed $T \geqslant 0$, we introduce

$$w_i(t, x) = \frac{\partial u_i(t, x)}{\partial x}, \ i = 1, \ldots, n, \quad W_1(T) = \sup_{0 \leqslant t \leqslant T} \int_{-\infty}^{+\infty} |w(t, x)| \mathrm{d}x,$$

$$\tilde{W}_1(T) = \max_{i\neq j} \sup_{x\in\mathbf{R}} \int_{\tilde{C}_j} |w_i(t,x)|\mathrm{d}t, \quad \tilde{U}_1(T) = \max_{i\neq j} \sup_{x\in\mathbf{R}} \int_{\tilde{C}_j} |u_i(t,x)|\mathrm{d}t,$$

$$\bar{W}_1(T) = \max_{i\neq j} \sup_{x\in\mathbf{R}} \int_{L_j} |w_i(t,x)|\mathrm{d}t, \quad \bar{U}_1(T) = \max_{i\neq j} \sup_{x\in\mathbf{R}} \int_{L_j} |u_i(t,x)|\mathrm{d}t,$$

$$W_\infty(T) = \sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbf{R}} |w(t,x)|, \quad U_\infty(T) = \sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbf{R}} |u(t,x)|,$$

where $\tilde{C}_j$ stands for any given j-th characteristic in the domain $[0,T]\times\mathbf{R}$ and L_j stands for the segment of any given straight line with the slope $\lambda_j(0)$ in the domain $[0,T]\times\mathbf{R}$.

Lemma 1 *Under the assumptions of Theorem 1, there exists a positive constant C depending only on M_0 and δ, such that the following estimates hold:*

$$\tilde{W}_1(T), \bar{W}_1(T), W_1(T) \leqslant CN_2, \quad \tilde{U}_1(T), \bar{U}_1(T) \leqslant CN_1 e^{CN_2},$$

$$W_\infty(T) \leqslant CMe^{CN_2}, \quad U_\infty(T) \leqslant C.$$

Proof For any fixed $\alpha\in\mathbf{R}$ and any $i=1,\ldots,n$, $u_i(t,x_i(t,\alpha))$ is a constant $f_i(\alpha)$ along the i-th characteristic. So we can get

$$U_\infty(T) \leqslant \sup_{\alpha\in\mathbf{R}} |f(\alpha)| = M_0 \leqslant C. \tag{6}$$

Differentiating system (1) with respect to x, we get

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x} = 0. \tag{7}$$

Multiplying (7) by $\mathrm{sgn}(w_i)$, we have

$$\mathrm{d}(|w_i(t,x)|(\mathrm{d}x - \lambda_i(u)\mathrm{d}t)) = 0. \tag{8}$$

For any fixed $\alpha\in\mathbf{R}$, let $\tilde{C}_j : x = x_j(t,\alpha)$ stand for any given j-th characteristic passing through any point $A:(0,\alpha)$ on the initial axis $t=0$ and intersecting $t=T$ at point P. We draw an i-th characteristic $\tilde{C}_i : x = x_i(t,\beta)$ from P downwards to the point $B:(0,\beta)$ on $t=0$. Without loss of generality, we assume $\alpha<\beta$ and integrate (8) in the region APB to get

$$\int_{\tilde{C}_j} |w_i(t,x)|(\lambda_j(u)-\lambda_i(u))\mathrm{d}t = \int_\alpha^\beta |w_i(0,x)|\mathrm{d}x. \tag{9}$$

Noting (2), we get

$$\int_{\tilde{C}_j} |w_i(t,x)|\mathrm{d}t \leqslant \frac{1}{\delta_0}\int_{-\infty}^{+\infty} |f'(x)|\mathrm{d}x \leqslant CN_2. \tag{10}$$

Then

$$\tilde{W}_1(T) \leqslant CN_2. \tag{11}$$

Similarly, we can get

$$\bar{W}_1(T) \leqslant CN_2. \tag{12}$$

To estimate $W_1(T)$, we need only to estimate $\int_{-l}^{l} |w(t,x)|\mathrm{d}x$ for any given $l>0$ and then let $l\to+\infty$. From point $M:(t,l)$, we draw an i-th characteristic downwards to the point $P:(0,\alpha_1)$

on $t=0$; From point $N:(t,-l)$, we draw an i-th characteristic downwards to the point $Q:(0,\beta_1)$ on $t=0$. Integrating (8) in the region MNQP, we easily get

$$\int_{-l}^{l}|w(t,x)|\mathrm{d}x \leqslant CN_2.$$

Thus

$$W_1(T) \leqslant CN_2. \tag{13}$$

The estimates of $W_\infty(T)$ and $\tilde{U}_1(T)$ have been given by Li and Peng in [5]. We recite them as follows for consistency.

We rewrite equation (7) and the corresponding initial data to get

$$\begin{cases} \dfrac{\partial w_i}{\partial t} + \lambda_i(u)\dfrac{\partial w_i}{\partial x} = -\dfrac{\partial \lambda_i(u)}{\partial u_i} w_i^2 - \sum\limits_{l\neq i} \dfrac{\partial \lambda_i(u)}{\partial u_l} w_l w_i, \\ t=0: w_i = f_i'(x). \end{cases} \tag{14}$$

By Hadamard's formula, we get:

$$\begin{aligned} \frac{\partial \lambda_i(u)}{\partial u_i} =& \frac{\partial \lambda_i(u)}{\partial u_i} - \frac{\partial \lambda_i(0,\dots,0,u_i,0,\dots,0)}{\partial u_i} \\ =& \sum_{l\neq i}\Big(\int_0^1 \frac{\partial \lambda_i^2(su_1,\dots,su_{i-1},u_i,su_{i+1},\dots,su_n)}{\partial u_l \partial u_i}\mathrm{d}s\Big)u_l \\ \overset{\text{def}}{=}& \sum_{l\neq i} b_{il}(u)u_l, \end{aligned}$$

where $b_{il}(u)$ $(l\neq i)$ are continuous functions of u. Along the i-th characteristic $x=x_i(s,\beta)$, w_i can be expressed as

$$w_i(t,x_i(t,\beta)) = \frac{f_i'(\beta)e^{-A_i(t,\beta)}}{1+f_i'(\beta)\int_0^t(\sum_{l\neq i} b_{il}(u)u_l)(s,x_i(s,\beta))e^{-A_i(s,\beta)}\mathrm{d}s}, \tag{15}$$

where

$$A_i(s,\beta) = \int_0^s \Big(\sum_{l\neq i}\frac{\partial \lambda_i(u)}{\partial u_l}w_l\Big)(\tau,x_i(\tau,\beta))\mathrm{d}\tau.$$

Noting (11) and $\lambda_i(u)\in C^2$, we have

$$|A_i(s,\beta)| \leqslant CN_2. \tag{16}$$

For any given $l\neq i$, the l-th characteristic passing through point $(t,x_i(t,\beta))$ on the i-th characteristic must intersect $t=0$ at a point denoted by $(0,y_{il}(t,\beta))$. Let $x=x_l(t,y_{il}(t,\beta))$ be this l-th characteristic. We have $x_i(t,\beta)=x_l(t,y_{il}(t,\beta))$. Differentiating it with respect to t, we get

$$\lambda_i(u(t,x_i(t,\beta))) = \lambda_l(u(t,x_i(t,\beta))) + \frac{\partial x_l(t,y_{il}(t,\beta))}{\partial y_{il}}\frac{\partial y_{il}(t,\beta)}{\partial t}. \tag{17}$$

From (2), we know that $\partial y_{il}(t,\beta)/\partial t$ is always different from zero for all $l\neq i$, so $t\to y_{il}(t,\beta)$ is a strictly monotone function. Therefore, (17) can be rewritten as

$$\frac{1}{(\lambda_i-\lambda_l)(u(t,x_i(t,\beta)))}\frac{\partial x_l(t,y_{il}(t,\beta))}{\partial y_{il}}\frac{\partial y_{il}(t,\beta)}{\partial t} = 1. \tag{18}$$

Since u_l is a constant along the l-th characteristic, we have

$$u_l(t, x_i(t, \beta)) = u_l(t, x_l(t, y_{il}(t, \beta))) = f_l(y_{il}(t, \beta)). \tag{19}$$

According to (18)–(19), we have

$$\begin{aligned}\tilde{U}_{il}(t, \beta) &\overset{\text{def}}{=} \int_0^t |u_l(s, x_i(s, \beta))| \mathrm{d}s \\ &= \int_0^t |f_l(y_{il}(s, \beta))| |\frac{1}{(\lambda_i - \lambda_l)(u(s, x_i(s, \beta)))}| |\frac{\partial x_l(s, y_{il}(s, \beta))}{\partial y}| |\frac{\partial y_{il}(s, \beta)}{\partial s}| \mathrm{d}s. \end{aligned}\tag{20}$$

Differentiating $u_i(t, x_i(t, \beta)) = f_i(\beta)$ with respect to β gives

$$w_i(t, x_i(t, \beta)) \frac{\partial x_i(t, \beta)}{\partial \beta} = f_i'(\beta). \tag{21}$$

Comparing with (15), we get

$$\begin{aligned} |\frac{\partial x_i(t, \beta)}{\partial \beta}| &= |e^{A_i(t, \beta)}(1 + f_i'(\beta) \int_0^t (\sum_{l \neq i} b_{il}(u) u_l)(s, x_i(s, \beta)) e^{-A_i(s, \beta)} \mathrm{d}s)| \\ &\leqslant Ce^{CN_2}(1 + M \sum_{l \neq i} \int_0^t |u_l(s, x_i(s, \beta))| \mathrm{d}s). \end{aligned}\tag{22}$$

Therefore, for any given $\beta \in \mathbf{R}$ and $i, l = 1, \ldots, n$ $(l \neq i)$, we get

$$\begin{aligned} \tilde{U}_{il}(t, \beta) \leqslant & Ce^{CN_2} \int_0^t |f_l(y_{il}(s, \beta))| \cdot \\ & (1 + M \sum_{j \neq l} \int_0^s |u_j(\tau, x_l(\tau, y_{il}(s, \beta)))| \mathrm{d}\tau) |\frac{\partial y_{il}(s, \beta)}{\partial s}| \mathrm{d}s \\ \leqslant & Ce^{CN_2} \int_{-\infty}^{+\infty} |f_l(y)| (1 + M \sum_{j \neq l} \int_0^{g_{il}(y, \beta)} |u_j(\tau, x_l(\tau, y))| \mathrm{d}\tau) \mathrm{d}y \\ \leqslant & Ce^{CN_2} \int_{-\infty}^{+\infty} |f_l(y)| (1 + M \sum_{j \neq l} \int_0^t |u_j(\tau, x_l(\tau, y))| \mathrm{d}\tau) \mathrm{d}y \\ \leqslant & CN_1 e^{CN_2} + C(n-1) MN_1 e^{CN_2} \tilde{U}_1(T), \end{aligned}$$

where $s = g_{il}(y, \beta)$ is the inverse function of $y = y_{il}(s, \beta)$. Thus

$$\tilde{U}_1(T) \leqslant CN_1 e^{CN_2} + C(n-1) MN_1 e^{CN_2} \tilde{U}_1(T).$$

Taking $\delta > 0$ so small that $2C(n-1)MN_1 e^{CN_2} \leqslant 1$, we get

$$\tilde{U}_1(T) \leqslant 2CN_1 e^{CN_2}. \tag{23}$$

We can estimate $\bar{U}_1(T)$ in a similar way.

For any given $l \neq i$, the l-th characteristic passing through point $(t, \bar{x}_i(t, \beta))$ on the line $\bar{x}_i(t, \beta) = \lambda_i(0)t + \beta$ must intersect $t = 0$ at a point denoted by $(0, \alpha_{il}(t, \beta))$. So we have $\bar{x}_i(t, \beta) = x_l(t, \alpha_{il}(t, \beta))$. Differentiating it with respect to t, we get

$$\lambda_i(0) = \lambda_l(u(t, \bar{x}_i(t, \beta))) + \frac{\partial x_l(t, \alpha_{il}(t, \beta))}{\partial \alpha_{il}} \frac{\partial \alpha_{il}(t, \beta)}{\partial t}. \tag{24}$$

Noting (2), $\partial\alpha_{il}(t,\beta)/\partial t$ is always different from zero for any given $l \neq i$. Therefore, $t \to \alpha_{il}(t,\beta)$ is a strict monotone function and equation (24) can be rewritten as

$$\frac{1}{\lambda_i(0)-\lambda_l(u(t,\bar{x}_i(t,\beta)))}\frac{\partial x_l(t,\alpha_{il}(t,\beta))}{\partial\alpha_{il}}\frac{\partial\alpha_{il}(t,\beta)}{\partial t}=1. \tag{25}$$

Similarly to the estimate of $\tilde{U}_1(T)$, substituting (25) for (18), we easily get

$$\bar{U}_1(T) \leqslant 2CN_1e^{CN_2}. \tag{26}$$

On the other hand, taking $\delta > 0$ so small that $2C\delta e^{CN_2} \leqslant 1$, we get

$$1+f_i'(\beta)\int_0^t(\sum_{l\neq i}b_{il}(u)u_l)(s,x_i(s,\beta))e^{-A_i(s,\beta)}\mathrm{d}s \geqslant 1-CMN_1e^{CN_2} \geqslant \frac{1}{2}.$$

So, for any given $0 \leqslant t \leqslant T$ and $x \in \mathbf{R}$, we get

$$W_\infty(T) \leqslant 2Me^{CN_2}. \tag{27}$$

The proof of Lemma 1 is completed. □

We can get the existence and uniqueness of global classical solutions of system (1) according to Lemma 1.

Lemma 2 *Suppose that system (1) is rich. Then under the assumptions of Theorem 1, there exists a constant C depending only on M_0 and δ such that there holds*

$$\tilde{U}_1(T) \leqslant CN_1,\ \bar{U}_1(T) \leqslant CN_1,\ W_\infty(T) \leqslant CM.$$

Proof Since system (1) is rich, we introduce the Lax transformation

$$v_i = N_i^{-1}(u)\frac{\partial u_i}{\partial x}, \quad i=1,\ldots,n.$$

From the boundness of u, we have $C_1 \leqslant N_i(u) \leqslant C_2$. Moreover, (14) can be rewritten as

$$\begin{cases} \dfrac{\partial v_i}{\partial t}+\lambda_i(u)\dfrac{\partial v_i}{\partial x} = -\dfrac{\partial\lambda_i(u)}{\partial u_i}N_i(u)v_i^2, \\ t=0: v_i = N_i^{-1}(f(x))f_i'(x). \end{cases} \tag{28}$$

Integrating v_i along the i-th characteristic, we get

$$v_i(t,x_i(t,\beta)) = \frac{v_i(0,\beta)}{1+v_i(0,\beta)\int_0^t(N_i(u)\sum_{l\neq i}b_{il}(u)u_l)(s,x_i(s,\beta))\mathrm{d}s}.$$

So

$$w_i(t,x_i(t,\beta)) = \frac{N_i(u(t,x_i(t,\beta)))N_i^{-1}(f(\beta))f_i'(\beta)}{1+N_i^{-1}(f(\beta))f_i'(\beta)\int_0^t(N_i(u)\sum_{l\neq i}b_{il}(u)u_l)(s,x_i(s,\beta))\mathrm{d}s}. \tag{29}$$

Noting (21), (29) and the boundness of $N_i(u)$, we get

$$\begin{aligned} |\frac{\partial x_i(t,\beta)}{\partial\beta}| &= |\frac{N_i(f(\beta))+f_i'(\beta)\int_0^t(N_i(u)\sum_{l\neq i}b_{il}(u)u_l)(s,x_i(s,\beta))\mathrm{d}s)}{N_i(u(t,x_i(t,\beta)))}| \\ &\leqslant C(1+M\sum_{l\neq i}\int_0^t|u_i(s,x(s,\beta))|\mathrm{d}s). \end{aligned} \tag{30}$$

Similarly to the estimate of $\tilde{U}_1(T)$ in Lemma 1, we get

$$\tilde{U}_1(T) \leqslant 2CN_1,\ \bar{U}_1(T) \leqslant CN_1. \tag{31}$$

So, taking $\delta > 0$ so small that $2C\delta \leqslant 1$, we get the following estimate on the denominator in (29)

$$1 + N_i^{-1}(f(\beta))f_i'(\beta)\int_0^t (N_i(u)\sum_{l\neq i} b_{il}(u)u_l)(s, x_i(s,\beta))\mathrm{d}s \geqslant 1 - CMN_1 \geqslant \frac{1}{2}. \tag{32}$$

Thus, it is easy to get that

$$W_\infty(T) \leqslant CM. \tag{33}$$

This ends the proof of Lemma 2. □

The i-th characteristic passing through point $(t, \alpha + \lambda_i(0)t)$ must intersect t=0 at a point denoted by $(0, \theta_i(t,\alpha))$.

Lemma 3 *Under the assumptions of Theorem 1, for any fixed α, there exists a unique $\vartheta_i(\alpha)$ such that*

$$\theta_i(t,\alpha) \to \vartheta_i(\alpha),\ \text{as } t \to +\infty.$$

Moveover, $\vartheta_i(\alpha)$ is global Lipschitz continuous with respect to α, i.e., for any given $\alpha, \beta \in \mathbf{R}$, we have

$$|\vartheta_i(\alpha) - \alpha| \leqslant CN_1 e^{CN_2},\ |\vartheta_i(\alpha) - \vartheta_i(\beta)| \leqslant e^{C(N_2+1)}|\alpha - \beta|.$$

Proof First we prove $\theta_i(t,\alpha)$ converges uniformly when $t \to +\infty$. Noting that the i-th characteristic passes through $(t, \alpha + \lambda_i(0)t)$, we get

$$\alpha + \lambda_i(0)t = x_i(t, \theta_i(t,\alpha)) = \theta_i(t,\alpha) + \int_0^t \lambda_i(u(s, x_i(s, \theta_i(t,\alpha))))\mathrm{d}s.$$

Therefore

$$\begin{aligned} \theta_i(t,\alpha) - \alpha &= \int_0^t (\lambda_i(0) - \lambda_i(u(s, x_i(s, \theta_i(t,\alpha)))))\mathrm{d}s \\ &= -\int_0^t \sum_{j\neq i} \Lambda_{ij} u_j(s, x_i(s, \theta(t,\alpha)))\mathrm{d}s, \end{aligned} \tag{34}$$

where

$$\Lambda_{ij}(u) \overset{\text{def}}{=} \int_0^1 \frac{\partial \lambda_i(\tau u_1, \ldots, \tau u_{i-1}, u_i, \tau u_{i+1}, \ldots, \tau u_n)}{\partial u_j}\mathrm{d}\tau. \tag{35}$$

By Lemma 1, we get

$$\begin{aligned} |\int_0^t \sum_{j\neq i} \Lambda_{ij} u_j(s, x_i(s, \theta_i(t,\alpha)))\mathrm{d}s| &\leqslant \int_0^t |\sum_{j\neq i} \Lambda_{ij} u_j(s, x_i(s, \theta_i(t,\alpha)))|\mathrm{d}s \\ &\leqslant C\int_0^t \sum_{j\neq i} |u_j(s, x_i(s, \theta_i(t,\alpha)))|\mathrm{d}s \leqslant C\tilde{U}_1(t) \leqslant CN_1 e^{CN_2}. \end{aligned} \tag{36}$$

This implies that $\int_0^t \sum_{j\neq i} \Lambda_{ij} u_j(s, x_i(s, \theta_i(t,\alpha)))\mathrm{d}s$ converges uniformly for any given $\alpha \in \mathbf{R}$ when $t \to +\infty$. Therefore, there exists a unique $\vartheta_i(\alpha)$ such that

$$\lim_{t\to+\infty} \theta_i(t,\alpha) = \vartheta_i(\alpha). \tag{37}$$

Noting (34)–(37), we get

$$|\vartheta_i(\alpha)-\alpha| \leqslant CN_1 e^{CN_2}. \tag{38}$$

In what follows, we prove that $\vartheta_i(\alpha)$ is Lipschitz continuous. For any given $\alpha,\beta\in\mathbf{R}$, we have

$$\vartheta_i(\alpha)-\vartheta_i(\beta)=\lim_{t\to+\infty}(\theta_i(t,\alpha)-\theta_i(t,\beta)). \tag{39}$$

There exists an α^* such that

$$|\theta_i(t,\alpha)-\theta_i(t,\beta)| \leqslant |\frac{\partial\theta_i(t,\alpha^*)}{\partial\alpha}||\alpha-\beta|. \tag{40}$$

By the definition of characteristic, we have

$$\begin{cases} \dfrac{\mathrm{d}x_i(s,\theta_i)}{\mathrm{d}s}=\lambda_i(u(s,x_i(s,\theta_i))),\\ s=0,\quad x_i=\theta_i. \end{cases}$$

Differentiating it with respect to θ_i, we get

$$\begin{cases} \dfrac{\mathrm{d}\frac{\partial x_i(s,\theta_i))}{\partial\theta_i}}{\mathrm{d}s}=\displaystyle\sum_l \frac{\partial\lambda_i(u(s,x_i(s,\theta_i)))}{\partial u_l}\frac{\partial u_l(s,x_i(s,\theta_i))}{\partial x}\frac{\partial x_i(s,\theta_i)}{\partial\theta_i},\\ s=0,\quad \frac{\partial x_i}{\partial\theta_i}=1. \end{cases}$$

Then

$$\frac{\partial x_i(s,\theta_i)}{\partial\theta_i}=\exp\{\int_0^s\sum_l \frac{\partial\lambda_i(u(\tau,x_i(\tau,\theta_i)))}{\partial u_l}\frac{\partial u_l(\tau,x_i(\tau,\theta_i))}{\partial x}\mathrm{d}\tau\}.$$

Differentiating $x_i(t,\theta_i(t,\alpha))=\lambda_i(0)t+\alpha$ with respect to α, we get $\frac{\partial x_i}{\partial\theta_i}\frac{\partial\theta_i}{\partial\alpha}=1$. Then

$$\frac{\partial\theta_i(t,\alpha)}{\partial\alpha}=(\frac{\partial x_i}{\partial\theta_i})^{-1}=\exp\{-\int_0^t\sum_l \frac{\partial\lambda_i(u(s,x_i(s,\theta_i)))}{\partial u_l}\frac{\partial u_l(s,x_i(s,\theta_i))}{\partial x}\mathrm{d}s\}.$$

When $l\neq i$, we have

$$\int_0^t |\frac{\partial u_l(s,x_i(s,\theta_i(t,\alpha)))}{\partial x}|\mathrm{d}s \leqslant CN_2;$$

while, when $l=i$, it follows from Hadamard's formula

$$\begin{aligned} \int_0^t |\frac{\partial\lambda_i(u)}{\partial u_i}\frac{\partial u_i(s,x_i(s,\theta_i(t,\alpha)))}{\partial x}|\mathrm{d}s &= \int_0^t |\sum_{l\neq i} b_{il}u_l w_i|\mathrm{d}s\\ &\leqslant CW_\infty(T)\tilde{U}_1(T)\leqslant C. \end{aligned}$$

Thus, we get

$$|\frac{\partial\theta_i(t,\alpha)}{\partial\alpha}|=\exp\{-\int_0^t(\sum_{l\neq i}\frac{\partial\lambda_i}{\partial u_l}\frac{\partial u_l(s,x_i(s,\theta_i))}{\partial x}+\sum_{l\neq i}b_{il}u_l w_i)\mathrm{d}s\}\leqslant e^{C(N_2+1)}. \tag{41}$$

According to (39)–(41), we have

$$|\vartheta_i(\alpha)-\vartheta_i(\beta)|\leqslant e^{C(N_2+1)}|\alpha-\beta|. \tag{42}$$

3. Asymptotic behaviour of global classical solutions

Theorem 2 *The limit*

$$\lim_{t\to\infty} u_i(t,x) = \phi_i(\alpha) = \phi_i(x-\lambda_i(0)t)$$

exists and satisfies the following estimates

$$|\phi_i(\alpha)| \leqslant C, \quad |\phi_i(\alpha)-\phi_i(\beta)| \leqslant CMe^{CN_2}|\alpha-\beta|.$$

Proof We have

$$\begin{aligned}\frac{Du_i}{D_it} &\stackrel{\text{def}}{=} \frac{\partial u_i}{\partial t} + \lambda_i(0)\frac{\partial u_i}{\partial x} = \frac{\mathrm{d}u_i}{\mathrm{d}_it} + (\lambda_i(0)-\lambda_i(u))\frac{\partial u_i}{\partial x}\\ &= (\lambda_i(0)-\lambda_i(u))\frac{\partial u_i}{\partial x} = -\sum_{j\neq i}\{\Lambda_{ij}u_jw_i\}.\end{aligned}$$

Integrating it along $x=\alpha+\lambda_i(0)t$ gives

$$u_i(t,x) = u_i(t,\alpha+\lambda_i(0)t) = f_i(\alpha) - \int_0^t \sum_{j\neq i}(\Lambda_{ij}u_jw_i)(s,\alpha+\lambda_i(0)s)\mathrm{d}s, \tag{43}$$

where

$$\begin{aligned}&\Big|\int_0^t \sum_{j\neq i}(\Lambda_{ij}u_jw_i)(s,\alpha+\lambda_i(0)s)\mathrm{d}s\Big| \leqslant \int_0^t \sum_{j\neq i}|(\Lambda_{ij}u_jw_i)(s,\alpha+\lambda_i(0)s)|\mathrm{d}s\\ &\leqslant C\int_0^t \sum_{j\neq i}|(u_jw_i)(s,\alpha+\lambda_i(0)s)|\mathrm{d}s \leqslant CW_\infty(t)\bar U_1(t) \leqslant C. \end{aligned}\tag{44}$$

Hence, $\int_0^t \sum_{j\neq i}(\Lambda_{ij}u_jw_i)(s,\alpha+\lambda_i(0)s)\mathrm{d}s$ converges uniformly for any given $\alpha\in\mathbf{R}$ when $t\to+\infty$. Noting (43)–(44), we get

$$\lim_{t\to\infty} u_i(t,x) = \phi_i(\alpha). \tag{45}$$

According to (26) and (43)–(45), we have

$$|\phi_i(\alpha)| \leqslant |f_i(\alpha)| + CMN_1e^{CN_2} \leqslant C, \tag{46}$$

$$\begin{aligned}|\phi_i(\alpha)-\phi_i(\beta)| &= \lim_{t\to\infty}|(u_i(t,\alpha+\lambda_i(0)t)-u_i(t,\beta+\lambda_i(0)t))|\\ &= \lim_{t\to\infty}|w_i(t,\alpha^*+\lambda_i(0)t)||\alpha-\beta| \leqslant CMe^{CN_2}|\alpha-\beta|.\end{aligned}\tag{47}$$

This completes the proof of Theorem 2. □

Theorem 3 *Under the assumptions of Theorem 1, for any given $i\in\{1,\dots,n\}$ and $\alpha\in\mathbf{R}$, the limit*

$$\lim_{t\to\infty} w_i(t,x_i(t,\alpha)) = \psi_i(\alpha)$$

exists. $\psi_i(\alpha)$ is continuous with respect to $\alpha\in\mathbf{R}$ and satisfies

$$|\psi_i(\alpha)| \leqslant CMe^{CN_2}.$$

Moreover, denoting $\tilde w^*_{\alpha,\beta}(\infty) \stackrel{\text{def}}{=} \max_{i=1,\dots,n}\sup_{t\in[0,+\infty)}|w_i(t,x_i(t,\alpha))-w_i(t,x_i(t,\beta))|$, we have

$$\tilde w^*_{\alpha,\beta}(\infty) \leqslant C\kappa|\alpha-\beta|^\rho + CM^2(N_2+1)|\alpha-\beta|.$$

Proof First we prove that $w_i(t,x_i(t,\alpha))$ converges uniformly with respect to α when $t\to+\infty$. Denote

$$h(t,\beta)\overset{\text{def}}{=}\int_0^t(\sum_{l\neq i}b_{il}(u)u_l)(s,x_i(s,\beta))e^{-A_i(s,\beta)}\mathrm{d}s.$$

According to (15), we have

$$\begin{aligned}&|w_i(t_1,x_i(t_1,\beta))-w_i(t_2,x_i(t_2,\beta))|\\&\quad\leqslant|\frac{f_i'(\beta)(e^{-A_i(t_1,\beta)}-e^{-A_i(t_2,\beta)})+f_i'(\beta)^2h(t_2,\beta)(e^{-A_i(t_1,\beta)}-e^{-A_i(t_2,\beta)})}{(1+f_i'(\beta)h(t_1,\beta))(1+f_i'(\beta)h(t_2,\beta))}|+\\&\quad\quad|\frac{f_i'(\beta)^2e^{-A_i(t_2,\beta)}(h(t_2,\beta)-h(t_1,\beta))}{(1+f_i'(\beta)h(t_1,\beta))(1+f_i'(\beta)h(t_2,\beta))}|.\end{aligned}\tag{48}$$

Since $f\in(C_b^1(\mathbf{R})\cap W^{1,1}(\mathbf{R}))^n$ and $\lim\limits_{|x|\to+\infty}f'(x)=0$, there exists an M_1 such that

$$\sup_{|x|\geqslant M_1}|f(x)|\leqslant\varepsilon,\quad\sup_{|x|\geqslant M_1}|f'(x)|\leqslant\varepsilon,\quad\int_{|x|\geqslant M_1}|f(x)|\leqslant\varepsilon,\quad\int_{|x|\geqslant M_1}|f'(x)|\leqslant\varepsilon.$$

The l-th characteristic passing through point $(t_1,x_i(t_1,\beta))$ intersects $t=0$ at a point denoted by α_1, and the l-th characteristic passing through point $(t_2,x_i(t_2,\beta))$ intersects $t=0$ at a point denoted by α_2. Without loss of generality, we assume $l<i$ (the case $l>i$ can be proved similarly). When $\beta\geqslant-M_1$, taking $T_1\geqslant\frac{2M_1}{\delta_0}$ and $t_2>t_1>T_1$, we get $M_1<\alpha_1<\alpha_2$. Similarly to (9)–(10), we get

$$\int_{t_1}^{t_2}|w_l(\tau,x_i(\tau,\beta))|\mathrm{d}\tau\leqslant\frac{1}{\delta_0}\int_{\alpha_1}^{\alpha_2}|f_l'(x)|\mathrm{d}x\leqslant C\int_{|x|\geqslant M_1}|f_l'(x)|\mathrm{d}x\leqslant C\varepsilon.$$

Therefore, we can make estimates as follows

$$\begin{aligned}&|\exp(-A_i(t_1,\beta))-\exp(-A_i(t_2,\beta))|\\&\quad=|\exp(-A_i(t_2,\beta))||\exp(\int_{t_1}^{t_2}(\sum_{l\neq i}\frac{\partial\lambda_i(u)}{\partial u_l}w_l)(\tau,x_i(\tau,\beta))\mathrm{d}\tau)-1|\\&\quad\leqslant e^{CN_2}|e^{C\varepsilon}-1|\leqslant Ce^{CN_2}\varepsilon,\end{aligned}\tag{49}$$

$$\begin{aligned}&|h(t_2,\beta)-h(t_1,\beta)|=|\int_{t_1}^{t_2}(\sum_{l\neq i}b_{il}(u)u_l)(s,x_i(s,\beta))e^{-A_i(s,\beta)}\mathrm{d}s|\\&\quad\leqslant Ce^{CN_2}\int_{t_1}^{t_2}|u_l(s,x_i(s,\beta))|\mathrm{d}s\leqslant Ce^{CN_2}\int_{|x|\geqslant M_1}|f_l(x)|\mathrm{d}x\leqslant Ce^{CN_2}\varepsilon.\end{aligned}\tag{50}$$

According to (48)–(50), we have

$$|w_i(t_1,x_i(t_1,\beta))-w_i(t_2,x_i(t_2,\beta))|\leqslant C\varepsilon.\tag{51}$$

When $\beta\leqslant-M_1$, noting (15), we have $|w_i(t,x_i(t,\beta))|\leqslant C|f_i'(\beta)|\leqslant C\varepsilon$. So, for any given $\beta\in\mathbf{R}$, we get

$$|w_i(t_1,x_i(t_1,\beta))-w_i(t_2,x_i(t_2,\beta))|\leqslant C\varepsilon.\tag{52}$$

So, we conclude that $w_i(t,x_i(t,\alpha))$ converges uniformly, i.e.,

$$\lim_{t\to\infty}w_i(t,x_i(t,\alpha))=\psi_i(\alpha).\tag{53}$$

In what follows, we estimate $\tilde{w}^*_{\alpha,\beta}(\infty)$.

If the system is rich, we apply Lax transformation and get (28)–(29). Denote

$$g_1(t,\beta) \stackrel{\text{def}}{=} N_i(u(t,x_i(t,\beta)))N_i^{-1}(f(\beta)),$$
$$g_2(t,\beta) \stackrel{\text{def}}{=} N_i^{-1}(f(\beta))f_i'(\beta)\int_0^t (N_i(u)\sum_{l\neq i} b_{il}(u)u_l)(s,x_i(s,\beta))\mathrm{d}s.$$

By Lemma 2, we get

$$|g_1(t,\beta)| \leqslant C, \quad |g_2(t,\beta)| \leqslant \frac{1}{2}, \quad |\frac{\partial g_1(t,\beta)}{\partial \beta}| \leqslant CM.$$

Thus

$$\begin{aligned}
&|g_1(t,\beta)f_i'(\beta) - g_1(t,\alpha)f_i'(\alpha)| \leqslant |g_1(t,\beta)(f_i'(\beta) - f_i'(\alpha))| + |g_1(t,\beta) - g_1(t,\alpha)||f_i'(\alpha)| \\
&\quad \leqslant C|f_i'(\beta) - f_i'(\alpha)| + |\frac{\partial g_1(t,\alpha^*)}{\partial \alpha}||f_i'(\alpha)||\alpha - \beta| \\
&\quad \leqslant C|f_i'(\beta) - f_i'(\alpha)| + CM^2|\alpha - \beta|, \\
&|g_1(t,\beta)g_2(t,\alpha)f_i'(\beta) - g_1(t,\alpha)g_2(t,\beta)f_i'(\alpha)| \\
&\quad \leqslant |g_1(t,\alpha)g_2(t,\beta)(f_i'(\beta) - f_i'(\alpha))| + |g_1(t,\alpha)||f_i'(\beta)||g_2(t,\alpha) - g_2(t,\beta)| + \\
&\qquad |g_2(t,\beta)||f_i'(\beta)||g_1(t,\alpha) - g_1(t,\beta)| \\
&\quad \leqslant C|f_i'(\beta) - f_i'(\alpha)| + CM^2(N_2+1)|\alpha - \beta|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|w_i(t,x_i(t,\beta)) - w_i(t,x_i(t,\alpha))| \\
&\quad = |\frac{g_1(t,\beta)f_i'(\beta) - g_1(t,\alpha)f_i'(\alpha) + g_1(t,\beta)g_2(t,\alpha)f_i'(\beta) - g_1(t,\alpha)g_2(t,\beta)f_i'(\alpha)}{(1+g_2(t,\alpha))(1+g_2(t,\beta))}| \\
&\quad \leqslant C|f_i'(\alpha) - f_i'(\beta)| + CM^2(N_2+1)|\alpha - \beta|.
\end{aligned}$$

So, we conclude that

$$\tilde{w}^*_{\alpha,\beta}(\infty) \leqslant C|f_i'(\alpha) - f_i'(\beta)| + CM^2(N_2+1)|\alpha - \beta|.$$

When the initial data satisfies (5), we get

$$\tilde{w}^*_{\alpha,\beta} \leqslant C\kappa|\alpha - \beta|^\rho + CM^2(N_2+1)|\alpha - \beta|. \tag{54}$$

Moreover, we can get the continuity of $\psi_i(\alpha)$ from the continuity of w_i with respect to α and the following estimate

$$|\psi_i(\alpha)| = \lim_{t\to\infty} |w_i(t,x_i(t,\alpha))| \leqslant CMe^{CN_2}. \tag{55}$$

This completes the proof of Theorem 3. □

Theorem 4 *The limit*

$$\lim_{t\to\infty} w_i(t,\alpha + \lambda_i(0)t) = \psi_i(\vartheta_i(\alpha)) = \phi_i'(\alpha)$$

exists and $\phi_i'(\alpha)$ is continuous with respect to $\alpha \in \mathbf{R}$. Moreover, for any given $\alpha, \beta \in \mathbf{R}$, we have *the following estimate*

$$|\phi_i'(\alpha)-\phi_i'(\beta)|\leqslant C\kappa e^{C\rho(N_2+1)}|\alpha-\beta|^\rho+CM^2(N_2+1)e^{C(N_2+1)}|\alpha-\beta|.$$

Proof By Theorem 3, we have

$$\lim_{t\to\infty} w_i(t,\alpha+\lambda_i(0)t)=\lim_{t\to\infty} w_i(t,x_i(t,\theta_i(t,\alpha)))=\lim_{t\to\infty} w_i(t,x_i(t,\vartheta_i(\alpha)))=\psi_i(\vartheta(\alpha)).$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}\phi_i(\alpha)}{\mathrm{d}\alpha}&=\lim_{\triangle\alpha\to 0}\frac{\phi_i(\alpha+\triangle\alpha)-\phi_i(\alpha)}{\triangle\alpha}\\&=\lim_{\triangle\alpha\to 0}\lim_{t\to\infty}\frac{u_i(t,\alpha+\lambda_i(0)t+\triangle\alpha)-u_i(t,\alpha+\lambda_i(0)t)}{\triangle\alpha}\\&=\lim_{t\to\infty} w_i(t,\alpha+\lambda_i(0)t)=\psi_i(\vartheta_i(\alpha)).\end{aligned}$$

We obtain the continuity of $\psi_i(\vartheta_i(\alpha))$ from the continuity of $\psi_i(x)$ and $\vartheta_i(\alpha)$ as well as the following estimate

$$\begin{aligned}&|\phi_i'(\alpha)-\phi_i'(\beta)|=|\psi_i(\vartheta_i(\alpha))-\psi_i(\vartheta_i(\beta))|\\&\quad=\lim_{t\to\infty}|w_i(t,x_i(t,\vartheta_i(\alpha)))-w_i(t,x_i(t,\vartheta_i(\beta)))|\\&\quad\leqslant C|f_i'(\vartheta_i(\alpha))-f_i'(\vartheta_i(\beta))|+CM^2(N_2+1)|\vartheta_i(\alpha)-\vartheta_i(\beta)|\\&\quad\leqslant C\kappa e^{C\rho(N_2+1)}|\alpha-\beta|^\rho+CM^2(N_2+1)e^{C(N_2+1)}|\alpha-\beta|.\end{aligned}$$

Finally, according to Theorems 2–4, we get Theorem 1. □

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