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Asymptotic Behavior of Global Classical Solutions to the Cauchy Problem on a Semi-Bounded Initial Axis for Quasilinear Hyperbolic Systems

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Abstract In this paper we study the asymptotic behavior of global classical solutions to the Cauchy problem with initial data given on a semi-bounded axis for quasilinear hyperbolic systems. Based on the existence result on the global classical solution, we prove that, when t tends to the infinity, the solution approaches a combination of C^1 travelling wave solutions with the algebraic rate $(1+t)^{-\mu}$, provided that the initial data decay with the rate $(1+x)^{-(1+\mu)}$ (resp. $(1-x)^{-(1+\mu)}$) as x tends to $+\infty$ (resp. $-\infty$), where μ is a positive constant.

Keywords quasilinear hyperbolic system; Cauchy problem on a semi-bounded initial axis; global classical solution; weak linear degeneracy; matching condition; travelling wave.

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1. Introduction and main result

Consider the following first order inhomogeneous quasilinear hyperbolic system

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=F(u), \tag{1.1}$$

where $u=(u_1,\ldots,u_n)^{\mathrm{T}}$ is the unknown vector function of (t,x), $A(u)$ is an $n\times n$ matrix with C^2 elements $a_{ij}(u)$ $(i,j=1,\ldots,n)$, $F(u)=(F_1(u),\ldots,F_n(u))^{\mathrm{T}}$ is a given vector function of u with C^2 elements $F_i(u)$ $(i=1,\ldots,n)$ and

$$F(0)=0,\quad \nabla F(0)=0. \tag{1.2}$$

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u),\ldots,\lambda_n(u)$ and a complete set of left (resp. right) eigenvectors. For $i=1,\ldots,n$, let $l_i(u)=(l_{i1}(u),\ldots,l_{in}(u))$ (resp. $r_i(u)=(r_{i1}(u),\ldots,r_{in}(u))^{\mathrm{T}}$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u)=\lambda_i(u)l_i(u) \tag{1.3}$$

and

$$A(u)r_i(u)=\lambda_i(u)r_i(u). \tag{1.4}$$

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We have

$$\det|l_{ij}(u)| \neq 0, \quad \text{resp. } \det|r_{ij}(u)| \neq 0. \tag{1.5}$$

Without loss of generality, we assume that on the domain under consideration

$$l_i(u) r_j(u) \equiv \delta_{ij}, \quad i,j = 1,\dots,n, \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

We suppose that all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i,j = 1,\dots,n)$ have the same regularity as $a_{ij}(u)$ $(i,j = 1,\dots,n)$.

In particular, if, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{1.7}$$

system (1.1) is called strictly hyperbolic.

First, let us recall the definition of weak linear degeneracy [10, 11] and matching condition related to $\lambda_n(u)$ [1].

Definition 1.1 *The i-th characteristic $\lambda_i(u)$ is called weakly linearly degenerate (WLD), if along the i-th characteristic trajectory $u = u^i(s)$ passing through $u = 0$, defined by*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s} = r_i(u), \\ s = 0 : u = 0, \end{cases} \tag{1.8}$$

we have

$$\nabla\lambda_i(u) r_i(u) \equiv 0, \quad \forall\ |u| \text{ small}, \tag{1.9}$$

i.e.,

$$\lambda_i(u^i(s)) \equiv \lambda_i(0), \quad \forall\ |s| \text{ small}. \tag{1.10}$$

If all characteristics are WLD, then system (1.1) is said to be WLD.

Definition 1.2 *$F(u)$ is called satisfying the matching condition related to $\lambda_n(u)$, if in a neighbourhood of $u = 0$, along the n-th characteristic trajectory $u = u^n(s)$ passing through the origin in the u-space, we have*

$$F(u^n(s)) \equiv 0, \quad \forall\ |s| \text{ small}. \tag{1.11}$$

When system (1.1) is strictly hyperbolic and WLD and $F(u) \in C^2$ satisfies the matching condition [4], for the Cauchy problem (1.1) with the initial data

$$t = 0 : u = \phi(x), \quad x \in \mathbb{R}, \tag{1.12}$$

where $\phi(x)$ is a C^1 vector function with the following decaying property: there exists a positive constant μ such that

$$\theta \triangleq \sup_{x\in R}\{(1+|x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} < \infty, \tag{1.13}$$

based on the global existence result of the classical solution obtained in [11], it was proved in [5] that there exists a unique C^1 vector function $\Phi(x) = (\Phi_1(x),\dots,\Phi_n(x))^{\mathrm{T}}$ such that on $t \geq 0$, in normalized coordinates [10, 11]

$$\left|u(t,x)-\sum_{i=1}^{n}\Phi_i(x-\lambda_i(0)t)e_i\right|\le K\theta^2(1+t)^{-\mu}, \tag{1.14}$$

where K is a positive constant independent of (t,x) and θ, and

$$e_i=(0,\dots,0,\overset{(i)}{\to}1,0,\dots,0)^{\mathrm{T}}. \tag{1.15}$$

Notice that each $\Phi_i(x-\lambda_i(0)t)e_i$ is a solution to system (1.1).

In this paper we suppose that in a neighbourhood of $u=0$,

$$\lambda_1(u),\dots,\lambda_{n-1}(u)<\lambda_n(u). \tag{1.16}$$

Consider the Cauchy problem for system (1.1) with the initial data

$$t=0:u=\phi(x),\quad x\ge 0, \tag{1.17}$$

where $\phi(x)$ is a C^1 vector function with the following decaying property: there exists a positive constant μ such that

$$\theta\triangleq\sup_{x\ge 0}\{(1+x)^{1+\mu}(|\phi(x)|+|\phi'(x)|)\}<\infty. \tag{1.18}$$

The following existence theorem was proved in [1]:

Theorem A *Under the hypotheses mentioned above, suppose furthermore that in a neighbourhood of $u=0$, $\lambda_n(u)$ is WLD and $F(u)$ satisfies the matching condition related to $\lambda_n(u)$. Then there exists $\theta_0>0$ so small that for any $\theta\in[0,\theta_0]$, Cauchy problem (1.1) and (1.17) admits a unique global C^1 solution $u=u(t,x)$ with small C^1 norm on the domain $D=\{(t,x)|\ t\ge 0, x\ge x_n(t)\}$, where $x=x_n(t)$ is the n-th characteristic passing through the origin $O(0,0)$:*

$$\begin{cases}\dfrac{\mathrm{d}x_n(t)}{\mathrm{d}t}=\lambda_n(u(t,x_n(t))),\\ x_n(0)=0.\end{cases} \tag{1.19}$$

In this paper, based on Theorem A, we will prove the following result:

Theorem 1.1 *Under the assumptions of Theorem A, there exists a unique C^1 function $\Phi_n(x)$ such that in the generalized normalized coordinates [12], on the existence domain D of the global classical solution $u=u(t,x)$ to Cauchy problem (1.1) and (1.17), the following estimate holds:*

$$\left|u(t,x)-\Phi_n(x-\lambda_n(0)t)e_n\right|\le K\theta(1+t)^{-\mu}, \tag{1.20}$$

where K is a positive constant independent of (t,x) and θ.

Remark 1.1 If $F(u)\equiv 0$, the conclusion of Theorem 1.1 is valid.

Remark 1.2 Suppose that in a neighbourhood of $u=0$,

$$\lambda_1(u),\dots,\lambda_p(u)<\lambda_{p+1}(u)\equiv\cdots\equiv\lambda_n(u), \tag{1.21}$$

where $\lambda(u)\triangleq\lambda_{p+1}(u)\equiv\cdots\equiv\lambda_n(u)$ is a characteristic with constant multiplicity $n-p$. Suppose furthermore that $\lambda_{p+1}(u),\dots,\lambda_n(u)$ are WLD [7]. Thus, the conclusion of Theorem 1.1 will be replaced as follows: there exists a unique C^1 vector function $\Phi(x) = (0, \ldots, 0, \Phi_{p+1}(x), \ldots, \Phi_n(x))^{\mathrm{T}}$ such that

$$\Big|u(t,x) - \sum_{i=p+1}^{n} \Phi_i(x - \lambda_i(0)t)e_i\Big| \le K\theta(1+t)^{-\mu}, \tag{1.22}$$

where K is a positive constant independent of (t,x) and θ.

Remark 1.3 When, in a neighbourhood of $u = 0$,

$$\lambda_1(u) < \lambda_2(u), \ldots, \lambda_n(u) \tag{1.23}$$

or

$$\lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u), \ldots, \lambda_n(u), \tag{1.24}$$

for the initial data

$$t = 0 : u = \phi(x), \quad x \le 0 \tag{1.25}$$

such that

$$\theta \triangleq \sup_{x \le 0}\{(1+|x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} < +\infty, \tag{1.26}$$

similar results hold as in Theorem 1.1 and Remarks 1.1–1.2.

In Section 2 we give some preliminaries, then, the main result is proved in Section 3.

2. Preliminaries

By the proof of Lemma 2.5 in [10], for any given complete system of right eigenvectors $r_1(u), \ldots, r_n(u)$ of $A(u)$ (without assuming the strict hyperbolicity), there exists a suitably smooth invertible transformation $u = u(\widetilde{u})$ $(u(0) = 0)$ such that in the $\widetilde{u}$-space, for each $i = 1, \ldots, n$, the i-th characteristic trajectory passing through $\widetilde{u} = 0$ coincides with the $\widetilde{u}_i$-axis at least for $|\widetilde{u}_i|$ small, namely,

$$\widetilde{r}_i(\widetilde{u}_i e_i) // e_i, \quad \forall\ |\widetilde{u}_i| \text{ small}, \quad i = 1, \ldots, n, \tag{2.1}$$

where $\widetilde{r}_i(\widetilde{u})$ denotes the i-th right eigenvector corresponding to $r_i(u)$ and e_i is given by (1.15).

This transformation is called a generalized normalized transformation, and the unknown variables $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^{\mathrm{T}}$ are called generalized normalized variables or generalized normalized coordinates [12]; for the normalized transformation and the normalized coordinates, also see [10], [11]. Without loss of generality, we assume that

$$\widetilde{r}_i^{\mathrm{T}}(\widetilde{u})\widetilde{r}_i(\widetilde{u}) \equiv 1, \tag{2.2}$$

then, (2.1) can be written as

$$\widetilde{r}_i(\widetilde{u}_i e_i) \equiv e_i, \quad \forall\ |\widetilde{u}_i| \text{ small}, \quad i = 1, \ldots, n. \tag{2.3}$$

Let

$$w_i = l_i(u)u_x, \quad i = 1, \ldots, n. \tag{2.4}$$

By (1.5), we have

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.5}$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.6}$$

denote the directional derivative with respect to t along the i-th characteristic.

We have [2, 8]

$$\frac{\mathrm{d}u}{\mathrm{d}_i t} = \sum_{k=1}^{n}(\lambda_i(u) - \lambda_k(u))w_k(u)r_k(u) + F(u), \quad i = 1, \dots, n. \tag{2.7}$$

Then, in the corresponding generalized normalized coordinates, it is easy to see that [2, 8]

$$\frac{\mathrm{d}u_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} \Big(\rho_{ijk}(u)u_j w_k + f_{ijk}(u)u_j u_k\Big), \quad i = 1, \dots, n, \tag{2.8}$$

where

$$\rho_{ijj}(u) \equiv 0, \quad \forall i, j, \tag{2.9}$$

$$f_{ink}(u) \equiv 0, \quad \forall i, k, \tag{2.10}$$

$$\rho_{ijk}(u) = (\lambda_i(u) - \lambda_k(u)) \int_0^1 \frac{\partial r_{ki}}{\partial u_j}(\tau u_1, \dots, \tau u_{k-1}, u_k, \tau u_{k+1}, \dots, \tau u_n)\mathrm{d}\tau, \quad \forall\ j \neq k \tag{2.11}$$

and

$$f_{ijk}(u) = \int_0^1 \theta_k(\tau) \int_0^1 \frac{\partial^2 F_i(\sigma\tau u_1, \dots, \sigma\tau u_{n-1}, \sigma\tau u_n)}{\partial u_j \partial u_k} \mathrm{d}\sigma\mathrm{d}\tau, \quad \forall\ j \neq n, \tag{2.12}$$

in which

$$\theta_k(\tau) = \begin{cases} \tau, & k = 1, \dots, n-1, \\ 1, & k = n. \end{cases} \tag{2.13}$$

Obviously,

$$\rho_{iji}(u) \equiv 0, \quad \forall i, j. \tag{2.14}$$

On the other hand, we have [3, 6, 9]

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = \sum_{j=1}^{n}\big(\sum_{k=1}^{n} B_{ijk}(u)\rho_k(u) + \nu_{ij}(u)\big)w_j + \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k, \quad i = 1, \dots, n, \tag{2.15}$$

where

$$B_{ijk}(u) = -l_i(u)\nabla r_j(u)r_k(u), \tag{2.16}$$

$$\rho_k(u) = l_k(u)F(u), \tag{2.17}$$

$$\nu_{ij}(u) = l_i(u)\nabla F(u)r_j(u), \tag{2.18}$$

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u))l_i(u)\nabla r_k(u)r_j(u) - \nabla\lambda_k(u)r_j(u)\delta_{ik} + (j|k)\}, \tag{2.19}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. It is easy to see that

$$\gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i \tag{2.20}$$

and

$$\gamma_{iii}(u) = -\nabla\lambda_i(u)r_i(u), \quad \forall i. \tag{2.21}$$

Moreover, as $\lambda_n(u)$ is WLD, in the corresponding generalized normalized coordinates, we have

$$\gamma_{nnn}(u_n e_n) \equiv 0,\ \ \lambda_n(u_n e_n) \equiv \lambda_n(0), \quad \forall |u_n| \text{ small.} \tag{2.22}$$

According to Lemma 2.1 in [2], if in a neighbourhood of $u = 0$, $F(u) \in C^2$ with (1.2) satisfying the matching condition related to $\lambda_n(u)$, i.e., (1.10) holds, then in the corresponding generalized normalized coordinates, we can rewrite (2.15) as

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} (Q_{ijk}(u)u_k w_j + \gamma_{ijk}(u)w_j w_k), \tag{2.23}$$

where $Q_{ijk}(u)$ is continuous in a neighbourhood of $u = 0$, and

$$Q_{inn}(u) \equiv 0, \quad \forall |u| \text{ small}, \quad \forall i. \tag{2.24}$$

3. Proof of Theorem 1.1

It is easy to see that under any smooth transformation $u = u(\tilde{u})$ $(u(0) = 0)$, the weak linear degeneracy of $\lambda_n(u)$ is invariant. Then, without loss of generality, we may assume that system (1.1) is written in the corresponding generalized normalized coordinates. In what follows, we always assume that $\theta > 0$ is suitably small.

By (1.16), there exist positive constants δ_0 and δ so small that

$$\lambda_n(u) - \lambda_i(u') \geq 2\delta_0, \quad \forall |u|, |u'| \leq \delta, \quad i = 1, \ldots, n-1 \tag{3.1}$$

and

$$|\lambda_i(u) - \lambda_i(u')| \leq \frac{\delta_0}{2}, \quad \forall |u|, |u'| \leq \delta, \quad i = 1, \ldots, n. \tag{3.2}$$

Without loss of generality, we suppose that

$$\lambda_i(0) \geq \delta_0, \quad i = 1, \ldots, n. \tag{3.3}$$

The following lemma comes from [1].

Lemma 3.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, $F(u) \in C^2$ with (1.2) satisfying the matching condition related to $\lambda_n(u)$, system (1.1) is hyperbolic and (1.16) holds. Suppose furthermore that $\lambda_n(u)$ is WLD and $\phi(x)$ is a C^1 vector function satisfying (1.18). Then there exists $\theta_0 > 0$ so small that for any $\theta \in [0, \theta_0]$, Cauchy problem (1.1) and (1.17) admits a unique global C^1 solution $u = u(t, x)$ with small C^1 norm on the domain $D = \{(t, x) | \ t \geq 0, x \geq x_n(t)\}$. Moreover, when $\theta_0 > 0$ is suitably small, the solution $u = u(t, x)$ satisfies*

$$|u(t, x)| \leq \delta. \tag{3.4}$$

By (3.2) and (3.3), it is easy to see that

$$x_n(t) \geq (\lambda_n(0) - \frac{\delta_0}{2})t \geq \frac{\delta_0}{2}t. \tag{3.5}$$

Let

$$U_\infty^c = \max_{i=1,\dots,n-1} \sup_{(t,x)\in D} \{(1+x)^{1+\mu}|u_i(t,x)|\}, \tag{3.6}$$

$$W_\infty^c = \max_{i=1,\dots,n-1} \sup_{(t,x)\in D} \{(1+x)^{1+\mu}|w_i(t,x)|\}, \tag{3.7}$$

$$U_\infty = \max_{i=1,\dots,n} \sup_{(t,x)\in D} |u_i(t,x)| \tag{3.8}$$

and

$$W_\infty = \max_{i=1,\dots,n} \sup_{(t,x)\in D} |w_i(t,x)|. \tag{3.9}$$

Combining Lemma 3.1 and Theorem 1.1 in [1] gives

Lemma 3.2 *Under the assumptions of Lemma 3.1, there exists a positive constant K_0 independent of θ and (t,x), such that on the existence domain D, in the corresponding generalized normalized coordinates, we have the following uniform a priori estimates:*

$$U_\infty^c,\ W_\infty^c,\ U_\infty, W_\infty \le K_0\theta. \tag{3.10}$$

Throughout this section, we will use the generalized normalized coordinates. For simplicity, we still denote the generalized normalized coordinates as $u = (u_1, \dots, u_n)^{\mathrm{T}}$ and the global C^1 solution as $u = u(t,x)$.

First, noting (3.5), by Lemma 3.2 we can easily get

$$|u_i(t,x)| \le K_0\theta(1+x)^{-(1+\mu)} \le C\theta(1+t)^{-\mu}, \quad i = 1, \dots, n-1, \tag{3.11}$$

here and hereafter, C denotes a positive constant independent of (t,x) and θ.

Let

$$\frac{D}{D_n t} = \frac{\partial}{\partial t} + \lambda_n(0)\frac{\partial}{\partial x}. \tag{3.12}$$

Clearly,

$$\frac{D}{D_n t} = \frac{\mathrm{d}}{\mathrm{d}_n t} + (\lambda_n(0) - \lambda_n(u))\frac{\partial}{\partial x}. \tag{3.13}$$

Thus, using (2.8) and (2.22), we have

$$\begin{aligned}\frac{Du_n}{D_n t} &= \frac{\mathrm{d}u_n}{\mathrm{d}_n t} + (\lambda_n(u_n e_n) - \lambda_n(u))\frac{\partial u_n}{\partial x}\\ &= \sum_{j,k=1}^{n} \Big(\rho_{njk}(u)u_j w_k + f_{njk}(u)u_j u_k\Big) + (\lambda_n(u_n e_n) - \lambda_n(u))\frac{\partial u_n}{\partial x}.\end{aligned} \tag{3.14}$$

Then, by Hadamard's formula, we have

$$\frac{Du_n}{D_n t} = \sum_{j,k=1}^{n} \Big(\tilde\rho_{njk}(u)u_j w_k + f_{njk}(u)u_j u_k\Big), \tag{3.15}$$

where

$$\tilde\rho_{njk}(u) = \begin{cases} \rho_{njk}(u) - r_{kn}(u)\displaystyle\int_0^1 \frac{\partial\lambda_n}{\partial u_j}(\tau u_1, \dots, \tau u_{n-1}, u_n)\mathrm{d}\tau, \ j = 1, \dots, n-1; k = 1, \dots, n,\\ \rho_{nnk}(u), \quad j = n; k = 1, \dots, n, \end{cases} \tag{3.16}$$

in which $r_{nn}(u)$ denotes the n-th component of $r_n(u)$.

By (2.9), we have

$$\tilde{\rho}_{nnn}(u) \equiv 0. \tag{3.17}$$

On the existence domain D, we now study the asymptotic behavior of $u_n(t,x)$ as $t \to +\infty$. For any fixed $(t,x) \in D$, let $\alpha = x - \lambda_n(0)t$. Noting the definition of D and (3.5), either $\alpha \geq 0$ or $\alpha \leq 0$.

(1) When $\alpha \geq 0$, it follows from (3.15) that

$$\begin{aligned} u_n(t,x) &= u_n(t, \alpha + \lambda_n(0)t) \\ &= u_n(0,\alpha) + \int_0^t \sum_{j,k=1}^n \Big(\tilde{\rho}_{njk}(u)u_j w_k + f_{njk}(u)u_j u_k\Big)(s, \alpha + \lambda_n(0)s)\mathrm{d}s. \end{aligned} \tag{3.18}$$

Noting (2.10), (3.17) and (3.5), by Lemma 3.2, for any fixed $\alpha \geq 0$ we have

$$\int_0^t \sum_{j,k=1}^n \Big(|\tilde{\rho}_{njk}(u)u_j w_k| + |f_{njk}(u)u_j u_k|\Big)(s, \alpha + \lambda_n(0)s)\mathrm{d}s \leq C\theta^2 \int_0^t \frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s, \tag{3.19}$$

which implies that when $t \to +\infty$, the integral

$$\int_0^t \sum_{j,k=1}^n \Big(\tilde{\rho}_{njk}(u)u_j w_k + f_{njk}(u)u_j u_k\Big)(s, \alpha + \lambda_n(0)s)\mathrm{d}s$$

converges uniformly with respect to $\alpha \geq 0$. Moreover, noting that all functions on the righthand side of (3.18) are continuous with respect to $\alpha \geq 0$, we obtain that there exists a unique $\tilde{\Phi}_n(\alpha) \in C^0[0,+\infty)$ such that

$$u_n(t,x) - \tilde{\Phi}_n(x - \lambda_n(0)t) \to 0 \ \text{ as } \ t \to +\infty. \tag{3.20}$$

Moreover, from (3.18)–(3.20), we have

$$\begin{aligned} &|u_n(t,x) - \tilde{\Phi}_n(x - \lambda_n(0)t)| \\ &\leq \int_t^\infty \sum_{j,k=1}^n \Big(|\tilde{\rho}_{njk}(u)u_j w_k| + |f_{njk}(u)u_j u_k|\Big)(s, \alpha + \lambda_n(0)s)\mathrm{d}s \\ &\leq C\theta^2 \int_t^{+\infty} \frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s \leq C\theta^2(1+t)^{-\mu}. \end{aligned} \tag{3.21}$$

(2) When $\alpha \leq 0$, let (t_0, x_0) be the intersection point of line $x = \lambda_n(0)t + \alpha$ with $x = x_n(t)$. Obviously, $t_0 \leq t$, by (3.15), we have

$$\begin{aligned} u_n(t,x) &= u_n(t, \alpha + \lambda_n(0)t) \\ &= u_n(t_0, x_0) + \int_{t_0}^t \sum_{j,k=1}^n \Big(\tilde{\rho}_{njk}(u)u_j w_k + f_{njk}(u)u_j u_k\Big)(s, \alpha + \lambda_n(0)s)\mathrm{d}s. \end{aligned} \tag{3.22}$$

Then, using (2.8), we have

$$u_n(t,x) = u_n(0,0) + \int_0^{t_0} \sum_{j,k=1}^n \Big(\rho_{njk}(u)u_j w_k + f_{njk}(u)u_j u_k\Big)(s, x_n(s))\mathrm{d}s +$$

$$\int_{t_0}^{t}\sum_{j,k=1}^{n}\Big(\tilde{\rho}_{njk}(u)u_jw_k+f_{njk}(u)u_ju_k\Big)(s,\alpha+\lambda_n(0)s)\mathrm{d}s$$
$$\triangleq u_n(0,0)+I(t,\alpha). \tag{3.23}$$

Noting (2.9)–(2.12), (3.16)–(3.17) and (3.5), by Lemma 3.2, for any fixed $\alpha\le 0$ we have

$$\sum_{j,k=1}^{n}\Big[\int_0^{t_0}\Big(|\rho_{njk}(u)u_jw_k|+|f_{njk}(u)u_ju_k|\Big)(s,x_n(s))\mathrm{d}s+$$
$$\int_{t_0}^{t}\Big(|\tilde{\rho}_{njk}(u)u_jw_k|+|f_{njk}(u)u_ju_k|\Big)(s,\alpha+\lambda_n(0)s)\mathrm{d}s\Big]$$
$$\le C\theta^2\Big[\int_0^{t_0}\frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s+\int_{t_0}^{t}\frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s\Big]$$
$$=C\theta^2\int_0^{t}\frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s. \tag{3.24}$$

Hence, when $t\to+\infty$, the integral $I(t,\alpha)$ converges uniformly with respect to $\alpha\le 0$. On the other hand, noting that all functions on the righthand side of (3.23) are continuous with respect to α, we obtain that there exists a unique $\tilde{\tilde{\Phi}}_n(\alpha)\in C^0(-\infty,0]$ such that

$$u_n(t,x)-\tilde{\tilde{\Phi}}_n(x-\lambda_n(0)t)\to 0\quad\text{as}\quad t\to+\infty \tag{3.25}$$

and

$$|u_n(t,x)-\tilde{\tilde{\Phi}}_n(x-\lambda_n(0)t)|$$
$$\le\int_t^{\infty}\sum_{j,k=1}^{n}\Big(|\tilde{\rho}_{njk}(u)u_jw_k|+|f_{njk}(u)u_ju_k|\Big)(s,\alpha+\lambda_n(0)s)\mathrm{d}s$$
$$\le C\theta^2\int_t^{+\infty}\frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s\le C\theta^2(1+t)^{-\mu}. \tag{3.26}$$

By (3.18) and (3.23), we get

$$\tilde{\Phi}_n(0)=\tilde{\tilde{\Phi}}_n(0). \tag{3.27}$$

Combining (1) and (2), there exists a unique C^0 function

$$\Phi_n(\alpha)\triangleq\begin{cases}\tilde{\Phi}_n(\alpha), & \text{if } \alpha\ge 0,\\ \tilde{\tilde{\Phi}}_n(\alpha), & \text{if } \alpha\le 0,\end{cases}$$

such that

$$u_n(t,x)-\Phi_n(x-\lambda_n(0)t)\to 0\quad\text{as}\quad t\to+\infty \tag{3.28}$$

and

$$|u_n(t,x)-\Phi_n(x-\lambda_n(0)t)|\le C\theta^2(1+t)^{-\mu}. \tag{3.29}$$

By (3.11) and (3.28)–(3.29), in order to prove Theorem 1.1, it is only necessary to prove $\Phi_n(\alpha)\in C^1(\mathbb{R})$. Noting that $\Phi_n(\alpha)\in C^0(R)$, we only need to show that $\frac{\mathrm{d}\Phi_n(\alpha)}{\mathrm{d}\alpha}\in C^0(R)$.

To this end, we calculate

$$\frac{\mathrm{d}\Phi_n(\alpha)}{\mathrm{d}\alpha}=\lim_{\triangle\alpha\to 0}\frac{\Phi_n(\alpha+\triangle\alpha)-\Phi_n(\alpha)}{\triangle\alpha}$$

$$= \lim_{\triangle\alpha\to 0}\lim_{t\to+\infty}\frac{u_n(t,\alpha+\triangle\alpha+\lambda_n(0)t)-u_n(t,\alpha+\lambda_n(0)t)}{\triangle\alpha}$$
$$= \lim_{t\to+\infty}\lim_{\triangle\alpha\to 0}\frac{u_n(t,\alpha+\triangle\alpha+\lambda_n(0)t)-u_n(t,\alpha+\lambda_n(0)t)}{\triangle\alpha}$$
$$= \lim_{t\to+\infty}\frac{\partial u_n}{\partial x}(t,\alpha+\lambda_n(0)t). \tag{3.30}$$

Noting (2.3) and (2.5) and using Hadamard's formula, we have

$$\begin{aligned}\frac{\partial u_n}{\partial x} &= \sum_{k=1}^{n} w_k r_k(u)\cdot e_n = \sum_{k=1}^{n-1} w_k r_k(u)\cdot e_n + w_n r_n(u)\cdot e_n \\ &= \sum_{k=1}^{n} w_k(r_k(u)-r_k(u_k e_k))\cdot e_n + w_n \\ &= w_n + \sum_{j\neq k} I_{jk}(u)u_j w_k,\end{aligned} \tag{3.31}$$

where

$$I_{jk}(u)=\int_0^1 \frac{\partial r_{kn}}{\partial u_j}(\tau u_1,\ldots,\tau u_{k-1},u_k,\tau u_{k+1},\ldots,\tau u_n)\mathrm{d}\tau. \tag{3.32}$$

Thus,

$$\frac{\partial u_n}{\partial x}(t,\alpha+\lambda_n(0)t) = w_n(t,\alpha+\lambda_n(0)t) + \sum_{j\neq k}[I_{jk}(u)u_j w_k](t,\alpha+\lambda_n(0)t). \tag{3.33}$$

By (3.5) and Lemma 3.2, when $t\to+\infty$,

$$\sum_{j\neq k}[I_{jk}(u)u_j w_k](t,\alpha+\lambda_n(0)t)\to 0$$

uniformly with respect to $\alpha\in\mathbb{R}$ with the algebraic rate $(1+t)^{-(1+\mu)}$. Hence,

$$\lim_{t\to+\infty}\frac{\partial u_n}{\partial x}(t,\alpha+\lambda_n(0)t) = \lim_{t\to+\infty} w_n(t,\alpha+\lambda_n(0)t). \tag{3.34}$$

For any fixed point $(t,\alpha+\lambda_n(0)t)\in D$, there exists a unique $\beta(t,\alpha)\in[0,+\infty)$ such that

$$\beta(t,\alpha)+\int_0^t \lambda_n(u(s,x_n(s,\beta(t,\alpha))))\mathrm{d}s = \alpha+\lambda_n(0)t,$$

namely,

$$\beta(t,\alpha)=\alpha+\int_0^t(\lambda_n(0)-\lambda_n(u(s,x_n(s,\beta(t,\alpha)))))\mathrm{d}s, \tag{3.35}$$

where $x=x_n(s,\beta(t,\alpha))$ is the n-th characteristic passing through the point $(0,\beta(t,\alpha))$:

$$\begin{cases}\dfrac{\mathrm{d}x_n(s,\beta(t,\alpha))}{\mathrm{d}s} = \lambda_n(u(s,x_n(s,\beta(t,\alpha)))),\\ x_n(0,\beta(t,\alpha))=\beta(t,\alpha).\end{cases}$$

Clearly,

$$w_n(t,\alpha+\lambda_n(0)t) = w_n(t,x_n(t,\beta(t,\alpha))). \tag{3.36}$$

Lemma 3.3 *There exists a unique nonnegative $\beta(\alpha)\in C^0(R)$ such that, when $t\to+\infty$, $\beta(t,\alpha)$*

uniformly converges to $\beta(\alpha)$.

Proof Since $\lambda_n(u)$ is WLD, it follows from (3.35) that

$$\beta(t,\alpha)=\alpha+\int_0^t[\lambda_n(u_ne_n)-\lambda_n(u)](s,x_n(s,\beta(t,\alpha)))\mathrm{d}s. \tag{3.37}$$

By Hadamard's formula, (3.37) can be rewritten as

$$\beta(t,\alpha)=\alpha+\int_0^t\sum_{k=1}^{n-1}[J_{nk}(u)u_k](s,x_n(s,\beta(t,\alpha)))\mathrm{d}s, \tag{3.38}$$

where

$$J_{nk}(u)=-\int_0^1\frac{\partial\lambda_n}{\partial u_k}(\tau u_1,\dots,\tau u_{n-1},u_n)\mathrm{d},\quad k=1,\dots,n-1.$$

Noting (3.5) and $\beta(t,\alpha)\ge 0$, $(s,x_n(s,\beta(t,\alpha)))\in D$ and, by Lemma 3.2 we have

$$\int_0^t\sum_{k=1}^{n-1}|J_{nk}(u)u_k|(s,x_n(s,\beta(t,\alpha)))\mathrm{d}s\le C\theta\int_0^t\frac{1}{(1+s)^{1+\mu}}\mathrm{d}s, \tag{3.39}$$

which implies that when $t\to+\infty$, the integral

$$\int_0^t\sum_{k=1}^{n-1}[J_{nk}(u)u_k](s,x_n(s,\beta(t,\alpha)))\mathrm{d}s$$

converges uniformly with respect to α. We notice that all functions on the righthand side of (3.38) are continuous with respect to $\alpha\in\mathbb{R}$, so there exists a unique $\beta(\alpha)\in C^0(\mathbb{R})$ such that when $t\to+\infty$,

$$\beta(t,\alpha)\to\beta(\alpha) \tag{3.40}$$

uniformly with respect to $\alpha\in\mathbb{R}$ with the algebraic rate $(1+t)^{-\mu}$. Moreover, noting $\beta(t,\alpha)\ge 0$, $\beta(\alpha)$ is nonnegative. This completes the proof. □

By Lemma 3.3, noting that $w_n(t,x)$ is a continuous function of t and x, we have

$$\lim_{t\to+\infty}w_n(0,\beta(t,\alpha))=w_n(0,\beta(\alpha)). \tag{3.41}$$

Lemma 3.4 *There exists a unique* $\Psi_n(\alpha)\in C^0(\mathbb{R})$ *such that when* $t\to+\infty$,

$$w_n(t,x_n(t,\beta(t,\alpha)))\to\Psi_n(\alpha) \tag{3.42}$$

uniformly on $\mathbb{R}$ *with the algebraic rate* $(1+t)^{-\mu}$, *where* $x=x_n(t,\beta(t,\alpha))$ *stands for the* n*-th characteristic passing through the point* $(0,\beta(t,\alpha))$.

Proof Noting (2.23), we have

$$\begin{aligned}&w_n(t,x_n(t,\beta(t,\alpha)))\\&\quad=w_n(0,\beta(t,\alpha))+\int_0^t\sum_{j,k=1}^{n}(Q_{njk}(u)u_kw_j+\gamma_{njk}(u)w_jw_k)(s,x_n(s,\beta(t,\alpha)))\mathrm{d}s.\end{aligned} \tag{3.43}$$

Noting (2.20), (2.24) and (3.5) and using Lemma 3.2, we have

$$\int_0^t\sum_{j,k=1}^{n}(|Q_{njk}(u)u_kw_j|+|\gamma_{njk}(u)w_jw_k|)(s,x_n(s,\beta(t,\alpha)))\mathrm{d}s$$

$$\leq \int_0^t \Big[\sum_{j=1}^{n-1}\sum_{k=1}^{n}(|Q_{njk}(u)u_jw_k|+|\gamma_{njk}(u)w_jw_k|)+\sum_{k=1}^{n-1}(|Q_{nnk}(u)u_kw_n|+|\gamma_{nnk}(u)w_nw_k|)+$$
$$|(\gamma_{nnn}(u)-\gamma_{nnn}(u_ne_n))w_n^2|\Big](s,x_n(s,\beta(t,\alpha)))\mathrm{d}s$$
$$\leq C\theta^2\int_0^t \frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s+\int_0^t |(\gamma_{nnn}(u)-\gamma_{nnn}(u_ne_n))w_n^2|(s,x_n(s,\beta(t,\alpha)))\mathrm{d}s. \tag{3.44}$$

Then, noting (2.22) and using Hadamard's formula and Lemma 3.2 again, we have

$$\int_0^t |(\gamma_{nnn}(u)-\gamma_{nnn}(u_ne_n))w_n^2|(s,x_n(s,\beta(t,\alpha)))\mathrm{d}s$$
$$\leq \int_0^t |\sum_{j=1}^{n-1}\gamma_{nnn}^j(u)u_jw_n^2|(s,x_n(s,\beta(t,\alpha)))\mathrm{d}s$$
$$\leq C\theta^2\int_0^t \frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s, \tag{3.45}$$

where

$$\gamma_{nnn}^j(u)=\int_0^1 \frac{\partial\gamma_{nnn}}{\partial u_j}(\tau u_1,\ldots,\tau u_{n-1},u_n),\quad j=1,\ldots,n-1. \tag{3.46}$$

Thus,

$$\int_0^t \sum_{j,k=1}^{n}(|Q_{njk}(u)u_kw_j|+|\gamma_{njk}(u)w_jw_k|)(s,x_n(s,\beta(t,\alpha)))\mathrm{d}s\leq C\theta^2\int_0^t \frac{1}{(1+s)^{(1+\mu)}}\mathrm{d}s, \tag{3.47}$$

which implies that when $t\to+\infty$, the integral

$$\int_0^t \sum_{j,k=1}^{n}(Q_{njk}(u)u_kw_j+\gamma_{njk}(u)w_jw_k)(s,x_n(s,\beta(t,\alpha)))\mathrm{d}s$$

converges uniformly with respect to $\alpha\in\mathbb{R}$ with the algebraic rate $(1+t)^{-\mu}$. Noting that all functions on the righthand side of (3.43) are continuous with respect to α, by (3.41) there exists a unique $\Psi_n(\alpha)\in C^0(\mathbb{R})$ such that when $t\to+\infty$, (3.42) holds uniformly on $\mathbb{R}$ with the algebraic rate $(1+t)^{-\mu}$. The proof of Lemma 3.4 is completed. □

Proof of Theorem 1.1 Clearly, it follows from (3.36) and Lemma 3.4 that

$$\lim_{t\to+\infty} w_n(t,\alpha+\lambda_n(0)t)=\lim_{t\to+\infty} w_n(t,x_n(t,\beta(t,\alpha)))=\Psi_n(\alpha)\in C^0(\mathbb{R}). \tag{3.48}$$

Then, by (3.30), (3.34) and (3.48), we have

$$\frac{\mathrm{d}\Phi_n(\alpha)}{\mathrm{d}\alpha}=\Psi_n(\alpha)\in C^0(\mathbb{R}). \tag{3.49}$$

Thus, the proof of Theorem 1.1 is completed. □

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