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Perturbed Proximal-Projection Methods for Nonlinear Mixed Variational-Like Inequalities

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Abstract In this paper we introduce a new perturbed proximal-projection algorithm for finding the common element of the set of fixed points of non-expansive mappings and the set of solutions of nonlinear mixed variational-like inequalities. The convergence criteria of the iterative sequences generated by the new iterative algorithm is also given. Our approach and results generalize many known results in this field.

Keywords nonlinear mixed variational-like inequalities; perturbed proximal-projection methods; common elements; nonexpansive mappings; relaxed (γ, r)-cocoercive mappings.

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1. Introduction

In recent years, variational inequality theory has become very effective and powerful tools for studying a wide class of nonlinear problems arising in many diverse fields of pure mathematics and applied sciences, such as mathematical programming, optimization theory etc [1–10].

Recently, Noor and Huang [3] considered the problem of finding the common element of the set of the fixed points of the nonexpansive mappings and the set of the solutions of variational inequalities. Noor [4] studied the problem of finding the common element of two different sets of the fixed points of the nonexpansive mappings and the set of solutions of the general variational inequalities. However, the methods of [3, 4] are limited to the study of variational inequalities and general variational inequalities. Furthermore, Algorithm 2.1 of [4] is not strict.

Motivated and inspired by the research work going on in this field, in this paper, we introduce a new perturbed proximal-projection algorithm for studing the problem of finding the common element of the set of the solutions of generalized mixed variational-like inequalities and the set of the fixed points of the nonexpansive mappings. By applying the novel method, we generalize the problem of [3]–[4] from variational inequalities and general variational inequalities to nonlinear mixed variational-like inequalities. Moreover, we take into account a possible inexact computation of the proximal-projection algorithm by using the perturbed errors. At the same time,

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our method modifies and complements Algorithm 2.1 of [4]. The convergence of the iterative sequences generated by the new algorithm is also discussed. Our result and method are new and different from those in the known literature.

2. Preliminaries

Let H be a Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed and convex set in H. Let $\eta : H \times H \to H$ and $T, A, g : H \to H$ be single-valued operators. Let $S : K \to K$ be a non-expansive operator and $\varphi : H \to R \cup \{+\infty\}$ be a proper function. Let P_K be the projection of H onto the convex set K.

We now consider the problem of finding $x \in \mathrm{dom}\varphi$ such that

$$\langle T(x) - A(x), \eta(y, g(x))\rangle + \varphi(y) - \varphi(g(x)) \geq 0, \quad \forall y \in H, \tag{2.1}$$

where $\mathrm{dom}\varphi := \{x \in H : \varphi(x) < \infty\} \neq \varnothing$. The inequality (2.1) is called the nonlinear mixed variational-like inequality.

Special cases

(I) If $\eta(x, y) = x - y$, for all $x, y \in H$, then problem (2.1) is equivalent to finding $x \in \mathrm{dom}\varphi$ such that

$$\langle T(x) - A(x), y - g(x)\rangle + \varphi(y) - \varphi(g(x)) \geq 0, \quad \forall y \in H. \tag{2.2}$$

Problem (2.2) is called the mixed variational inequality.

(II) If $\eta(x, y) = x - y$, $A(x) \equiv 0$, for all $x, y \in H$, then problem (2.1) reduces the problem of finding $x \in \mathrm{dom}\varphi$ such that

$$\langle T(x), y - g(x)\rangle + \varphi(y) - \varphi(g(x)) \geq 0, \quad \forall y \in H. \tag{2.3}$$

(III) If $A(x) \equiv 0$, $\eta(y, g(x)) = g(y) - g(x)$, for all $x, y \in H$, and φ is the indicator of the closed convex set K, that is,

$$\varphi(x) \equiv I_K(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K, \end{cases}$$

then problem (2.1) is equivalent to the one considered in [4]: To find $x \in H, g(x) \in K$ such that

$$\langle T(x), g(y) - g(x)\rangle \geq 0, \quad \forall g(y) \in K. \tag{2.4}$$

(IV) If $A(x) \equiv 0$, $\eta(x, y) = x - y$, for all $x, y \in H, g = I$, the identity operator, and φ is the indicator of the closed convex set K, then problem (2.1) becomes to find $x \in K$ such that

$$\langle T(x), y - x\rangle \geq 0, \quad \forall y \in K. \tag{2.5}$$

The problem was studied in [3].

In brief, for appropriate and suitable choices of $\eta(\cdot,\cdot), T, A, g$, we can obtain many known and new classes of generalized variational inequalities as special cases of the problem (2.1).

Definition 2.1 ([6]) *Let $\eta : H \times H \to H$ and $\varphi : H \to R \cup \{+\infty\}$. A vector $\omega \in H$ is called an*

η-subgradient of φ at $x \in \mathrm{dom}\varphi$ if

$$\langle \omega, \eta(y,x)\rangle \le \varphi(y) - \varphi(x), \text{ for all } y \in H.$$

We can associate with each φ the η-subdifferential map $\partial_\eta \varphi(x)$ defined by

$$\partial_\eta \varphi(x) = \begin{cases} \{\omega \in H : \langle \omega, \eta(y,x)\rangle \le \varphi(y) - \varphi(x), \forall\, y \in H\}, & x \in \mathrm{dom}\varphi, \\ \varnothing, & x \notin \mathrm{dom}\varphi. \end{cases}$$

For $x \in \mathrm{dom}\varphi$, $\partial_\eta\varphi(x)$ is called the η-subdifferential of φ at x.

Definition 2.2 ([6]) *Let $\eta : H \times H \to H$ be a given map. Then a multivalued map $Q : H \to 2^H$ is called η-monotone, if for all $x, y \in H$,*

$$\langle u - v, \eta(x,y)\rangle \ge 0, \quad \forall u \in Q(x),\ v \in Q(y).$$

Q is called maximal η-monotone if and only if it is η-monotone and there is no other η-monotone multivalued map whose graph strictly contains the Graph(Q), *where*

$$\mathrm{Graph}(Q) = \{(x,y) \in H \times H : y \in Q(x)\}.$$

Definition 2.3 *An operator $\eta : H \times H \to H$ is called:*

(i) Monotone, if

$$\langle x - y, \eta(x,y)\rangle \ge 0, \quad \forall x, y \in H. \tag{2.6}$$

(ii) Strictly monotone, if the equality holds in (2.6) only when $x = y$.

(iii) Stongly monotone, if there exists a constant $\sigma > 0$ such that

$$\langle x - y, \eta(x,y)\rangle \ge \sigma \|x - y\|^2, \quad \forall x, y \in H.$$

(iv) Lipschitz continuous, if there exists a constant $\delta > 0$ such that

$$\|\eta(x,y)\| \le \delta \|x - y\|, \quad \forall x, y \in H.$$

Remark 2.1 From (iii) and (iv) of Definition 2.3, we have $\sigma \le \delta$.

Definition 2.4 *An operator $g : H \to H$ is called:*

(i) μ-Lipschitz continuous, if there exists a constant $\mu > 0$ such that

$$\|g(x) - g(y)\| \le \mu \|x - y\|, \quad \forall x, y \in H.$$

(ii) α-stongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle g(x) - g(y), x - y\rangle \ge \alpha \|x - y\|^2, \quad \forall x, y \in H.$$

Definition 2.5 ([7]) *An operator $g : H \to H$ is called relaxed (γ, r)-cocoercive, if there exist constants $\gamma > 0$, $r > 0$ such that*

$$\langle g(x) - g(y), x - y\rangle \ge -\gamma \|g(x) - g(y)\|^2 + r\|x - y\|^2, \quad \forall x, y \in H.$$

Assumption 2.1

(i) $\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in H$,

(ii) $\eta : H \times H \to H$ is strictly monotone,

(iii) the range of $(I+\rho\partial_\eta\varphi)$, $R(I+\rho\partial_\eta\varphi)=H$, where $\rho>0$ and I is the identity operator.

Lemma 2.1 ([6]) *Let $\eta : H\times H\to H$ satisfy $\eta(x,y)+\eta(y,x)=0$, $\forall x,y\in H$, and $\varphi : H\to R\cup\{+\infty\}$. Then the multivalued map $\partial_\eta\varphi : H\to 2^H$ is η-monotone.*

Lemma 2.2 ([6]) *Let $\eta : H\times H\to H$ be strictly monotone and $Q : H\to 2^H$ be an η-monotone multivalued map. If the range of $(I+\rho Q)$, $R(I+\rho Q)=H$, for $\rho>0$ where I is the identity operator, then Q is maximal η-monotone. Furthermore, the inverse operator $(I+\rho Q)^{-1} : H\to H$ is single-valued.*

Lemma 2.3 *Let $\eta : H\times H\to H$ and $\varphi : H\to R\cup\{+\infty\}$ satisfy Assumption 1. Then the mapping*

$$J_\rho^\varphi(x) := (I+\rho\partial_\eta\varphi)^{-1}(x), \text{ for all } x\in H$$

is single-valued.

Proof From Lemmas 2.1 and 2.2, we know that the conclusion of Lemma 2.3 is correct. □

Lemma 2.4 *$x\in \mathrm{dom}\varphi$ is a solution of (2.1) if and only if it satisfies the relation*

$$g(x)=J_\rho^\varphi(g(x)-\rho((T(x)-A(x))), \tag{2.7}$$

where $\rho>0$ is a constant, $J_\rho^\varphi := (I+\rho\partial_\eta\varphi)^{-1}$ is the so-called proximal map and I stands for identity operator on H.

Proof From the definition of J_ρ^φ,

$$g(x)=J_\rho^\varphi(g(x)-\rho(T(x)-A(x)))$$

if and only if

$$g(x)-\rho(T(x)-A(x))\in g(x)+\rho\partial_\eta\varphi(g(x))$$

if and only if

$$A(x)-T(x)\in\partial_\eta\varphi(g(x))$$

if and only if

$$\langle A(x)-T(x),\eta(y,g(x))\rangle\le\varphi(y)-\varphi(g(x)),\quad \forall y\in H,$$

by using the definition of the η-subdifferential. This implies that x is a solution of (2.1). □

Lemma 2.5 ([6]) *Let $\eta : H\times H\to H$ be strongly monotone and Lipschitz continuous with constants $\sigma>0$ and $\delta>0$, respectively, which satisfies $\eta(x,y)+\eta(y,x)=0$, $\forall x,y\in H$. Then*

$$\|J_\rho^\varphi(x)-J_\rho^\varphi(y)\|\le\tau\|x-y\|,\quad \forall x,y\in H,$$

where $\tau=\delta/\sigma$.

Remark 2.2 From Remrk 2.1, we have $\tau\ge 1$.

Let φ be a proper convex lower semicontinuous function on H. The subdifferential of φ at $x \in \mathrm{dom}\varphi$ is the set

$$\partial\varphi(x) = \{\xi \in H : \varphi(y) - \varphi(x) \geq \langle \xi, y - x\rangle, \text{ for all } y \in H\}.$$

It is known that $\partial\varphi(\cdot)$ is a maximal monotone mapping [8], and for each $\rho > 0$, the mapping $(I + \rho\partial\varphi)^{-1}$ is a single-valued mapping defined on the whole space H, where I denotes the identity mapping.

Lemma 2.6 ([8]) *For a given $z \in H$, $x \in H$ satisfies the inequality*

$$\langle x - z, y - x\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \quad \forall y \in H,$$

if and only if

$$x = J_\varphi(z),$$

where $J_\varphi := (I + \rho\partial\varphi)^{-1}$ is the proximal map and I stands for identity operator on H. Furthermore, J_φ is nonexpansive, that is,

$$\|J_\varphi(x) - J_\varphi(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Lemma 2.7 *For a given $z \in H$, $x \in K$ satisfies the inequality*

$$\langle x - z, y - x\rangle \geq 0, \quad \forall y \in K,$$

if and only if

$$x = P_K(z),$$

where P_K is the projection of H onto the convex set K. Furthermore, P_K is nonexpansive, that is,

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Lemma 2.8 ([9]) *Assume $\{\delta_n\}_{n=0}^{\infty}$ is a sequence of nonnegative real numbers such that*

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n + \sigma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in R such that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n/\lambda_n \leq 0$ or $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} \delta_n = 0$.

3. Algorithms

Lemma 2.4 enables us to reformulate (2.1) as the fixed point problem of solving $x = F(x)$, where

$$F(x) = x - g(x) + J_\rho^\varphi(g(x) - \rho(T(x) - A(x))). \tag{3.1}$$

Let S be a nonexpansive mapping. We denote the set of the fixed points of S by $F(S)$ and the set of the solutions of the variational inequalities (2.1) by NMVI(T, A, η, φ). If $x^* \in$

$F(S) \cap \mathrm{NMVI}(T, A, \eta, \varphi)$, then $x^* \in F(S)$ and $x^* \in \mathrm{NMVI}(T, A, \eta, \varphi)$. Thus it follows from Lemma 2.4 that

$$\begin{aligned} x^* = Sx^* &= x^* - g(x^*) + J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*))) \\ &= P_K[x^* - g(x^*) + J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*)))] \\ &= SP_K[x^* - g(x^*) + J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*)))], \end{aligned}$$

where $\rho > 0$ is a constant.

The fixed point formulation is used to suggest the following perturbed proximal-projection methods for finding a common element of two different sets of the fixed points of the nonexpansive mappings and of solutions of the variational inequalities.

Algorithm 3.1 For a given $x_0 \in H$, compute the approximate solution x_n of (2.1) by the iterative schemes

$$z_n = (1 - c_n)x_n + c_n SP_K[x_n - g(x_n) + J_\rho^\varphi(g(x_n) - \rho(T(x_n) - A(x_n)))] + f_n, \tag{3.2}$$

$$y_n = (1 - b_n)x_n + b_n SP_K[z_n - g(z_n) + J_\rho^\varphi(g(z_n) - \rho(T(z_n) - A(z_n)))] + e_n, \tag{3.3}$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - g(y_n) + J_\rho^\varphi(g(y_n) - \rho(T(y_n) - A(y_n)))] + d_n, \tag{3.4}$$

where $a_n, b_n, c_n \in [0, 1]$ for all $n \geq 0$, $d_n,\ e_n,\ f_n \in H$ $(n = 0, 1, 2, \dots)$ are errors, S is the nonexpansive operator, P_K is the projection of H onto the convex set K, and $J_\rho^\varphi := (I + \rho\partial_\eta\varphi)^{-1}$ is the proximal map.

If $\eta(x, y) = x - y$, for all $x, y \in H$ and φ is the proper convex lower semicontinuous function on H, Algorithm 3.1 reduces to the following method of solving (2.2).

Algorithm 3.2 For a given $x_0 \in H$, compute the approximate solution x_n of (2.2) by the iterative schemes

$$z_n = (1 - c_n)x_n + c_n SP_K[x_n - g(x_n) + J_\varphi(g(x_n) - \rho(T(x_n) - A(x_n)))] + f_n,$$

$$y_n = (1 - b_n)x_n + b_n SP_K[z_n - g(z_n) + J_\varphi(g(z_n) - \rho(T(z_n) - A(z_n)))] + e_n,$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - g(y_n) + J_\varphi(g(y_n) - \rho(T(y_n) - A(y_n)))] + d_n,$$

where $a_n, b_n, c_n \in [0, 1]$ for all $n \geq 0$, $d_n,\ e_n,\ f_n \in H$ $(n = 0, 1, 2, \dots)$ are errors, S is the nonexpansive operator, P_K is the projection of H onto the convex set K, and $J_\varphi := (I + \rho\partial\varphi)^{-1}$ is the proximal map.

If $\eta(x, y) = x - y$, $A(x) \equiv 0$, for all $x, y \in H$, and φ is the proper convex lower semicontinuous function on H, Algorithm 3.1 reduces to the following method of solving (2.3).

Algorithm 3.3 For a given $x_0 \in H$, compute the approximate solution x_n of (2.3) by the iterative schemes

$$z_n = (1 - c_n)x_n + c_n SP_K[x_n - g(x_n) + J_\varphi(g(x_n) - \rho T(x_n))] + f_n,$$

$$y_n = (1 - b_n)x_n + b_n SP_K[z_n - g(z_n) + J_\varphi(g(z_n) - \rho T(z_n))] + e_n,$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - g(y_n) + J_\varphi(g(y_n) - \rho T(y_n))] + d_n,$$

where $a_n,\ b_n,\ c_n \in [0,1]$ for all $n \geq 0$, $d_n,\ e_n,\ f_n \in H$ $(n = 0,1,2,\ldots)$ are errors, S is the nonexpansive operator, P_K is the projection of H onto the convex set K, $J_\varphi := (I + \rho\partial\varphi)^{-1}$ is the proximal map.

If $A(x) \equiv 0$, $\eta(y, g(x)) = g(y) - g(x)$, for all $x, y \in H$, and φ is the indicator of the closed convex set K, Algorithm 3.1 reduces to the following method of solving (2.4).

Algorithm 3.4 For a given $x_0 \in H$, compute the approximate solution x_n of (2.4) by the iterative schemes

$$z_n = (1 - c_n)x_n + c_n SP_K[x_n - g(x_n) + P_K(g(x_n) - \rho T(x_n))] + f_n, \tag{3.5}$$

$$y_n = (1 - b_n)x_n + b_n SP_K[z_n - g(z_n) + P_K(g(z_n) - \rho T(z_n))] + e_n, \tag{3.6}$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - g(y_n) + P_K(g(y_n) - \rho T(y_n))] + d_n, \tag{3.7}$$

where $a_n,\ b_n,\ c_n \in [0,1]$ for all $n \geq 0$, $d_n,\ e_n,\ f_n \in H$ $(n = 0,1,2,\ldots)$ are errors, S is the nonexpansive operator, and P_K is the projection of H onto the convex set K.

Remark 3.1 In Algorithm 2.1 of [4], for example, (2.10) of Algorithm 2.1 the domain of the map S is the nonempty closed and convex set K, but the value of $x_n - g(x_n) + P_K(g(x_n) - \rho T(x_n))$ $(n = 0,1,2,\ldots)$ does not always belong to the closed and convex set K. It follows that Algorithm 2.1 of [4] is not rigorous. Algorithm 3.4 modifies and generalizes Algorithm 2.1 of [4].

Example 3.1 Let $H = R^2$, $K = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1,\ x_1,\ x_2 \in R\} \subset R^2$, $T(x) = -\frac{1}{\rho}x$, $g(x) = \frac{1}{2}x$, where $x = (x_1, x_2) \in K$, $\rho > 0$ is a constant. Then K is the nonempty closed and convex set on R^2, $T : K \to R^2$, $g : K \to K$. Let $x_0 = (\frac{2}{3}, \frac{2}{3})$. Then $x_0 \in K$, $T(x_0) = -\frac{1}{\rho}(\frac{2}{3}, \frac{2}{3})$, $g(x_0) = (\frac{1}{3}, \frac{1}{3})$, hence $x_0 - g(x_0) + P_K[g(x_0) - \rho T(x_0)] = (\frac{2}{3}, \frac{2}{3}) - (\frac{1}{3}, \frac{1}{3}) + P_K[(1,1)] = (\frac{1}{3}, \frac{1}{3}) + (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = (\frac{3\sqrt{2}+2}{6}, \frac{3\sqrt{2}+2}{6}) \notin K$.

If $A(x) \equiv 0$, $\eta(x,y) = x - y$, for all $x, y \in H$, $g = I$, the identity operator, $d_n = e_n = f_n \equiv 0$ $(n = 0,1,2,\ldots)$ and φ is the indicator of the closed convex set K, Algorithm 3.1 reduces to the following method of solving (2.5), which is basically as in [3].

Algorithm 3.5 For a given $x_0 \in K$, compute the approximate solution x_n of (2.5) by the iterative schemes

$$z_n = (1 - c_n)x_n + c_n SP_K[x_n - \rho T(x_n)],$$

$$y_n = (1 - b_n)x_n + b_n SP_K[z_n - \rho T(z_n)],$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - \rho T(y_n)],$$

where $a_n, b_n, c_n \in [0,1]$ for all $n \geq 0$, S is the nonexpansive operator, and P_K is the projection of H onto the convex set K.

4. Main results

In this section, we investigate the strong convergence of Algorithms 3.1, 3.4 and 3.5 in finding the common element of two sets of solutions of the variational inequalities and $F(S)$.

Theorem 4.1 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let the operator $T : H \to H$ be relaxed (γ, r)-cocoercive and μ-Lipschitz continuous. Let the operator $A : H \to H$ be Lipschitz continuous with constant $k > 0$. Let the operator $g : H \to H$ be relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitz continuous. Let $\varphi : H \to R \cup \{+\infty\}$ be a proper function and the operator $\eta(\cdot,\cdot)$ be strongly monotone and Lipschitz continuous with constants $\sigma > 0$ and $\delta > 0$, respectively, which satisfy Assumption 1. Let $S : K \to K$ be a nonexpansive mapping such that $F(S) \cap \mathrm{NMVI}(T, A, \eta, \varphi) \neq \varnothing$. If*

$$a_n, b_n, c_n \in [0,1]\ (n = 0, 1, 2, \ldots),\ \sum_{n=0}^{\infty} a_n = \infty,\ \sum_{n=0}^{\infty} \|d_n\| < \infty,\ \sum_{n=0}^{\infty} \|e_n\| < \infty,\ \sum_{n=0}^{\infty} \|f_n\| < \infty,$$

and the following conditions are satisfied:

$$2r_1 < (2\gamma_1 + 1)\mu_1^2 + 1,\ \xi = \sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2},\ \tau = \delta/\sigma,\ \xi < \frac{1}{1+\tau},\ \mu > k, \tag{4.1}$$

$$\frac{\gamma\mu^2\tau + k(1 - (1+\tau)\xi) + \sqrt{(\mu^2 - k^2)[\tau^2 - (1 - (1+\tau)\xi)^2]}}{\tau} < r < \gamma\mu^2 + \mu, \tag{4.2}$$

$$m_1 < \rho < m_2, \tag{4.3}$$

where

$$m_1 = \frac{a - \sqrt{a^2 - b}}{\tau(\mu^2 - k^2)}, \quad m_2 = \min\{w_1, w_2\},$$

$$w_1 = \frac{a + \sqrt{a^2 - b}}{\tau(\mu^2 - k^2)}, \quad w_2 = \frac{1 - (1+\tau)\xi}{k\tau},$$

$$a = r\tau - \gamma\mu^2\tau - k[1 - (1+\tau)\xi], \quad b = (\mu^2 - k^2)\{\tau^2 - [1 - (1+\tau)\xi]^2\},$$

then x_n obtained from Algorithm 3.1 converges strongly to $x^* \in F(S) \cap \mathrm{NMVI}(T, A, \eta, \varphi)$.

Proof Let $x^* \in K$ be the solution of $F(S) \cap \mathrm{NMVI}(T, A, \eta, \varphi)$. Then

$$\begin{aligned} x^* &= (1 - c_n)x^* + c_n SP_K[x^* - g(x^*) + J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*))] && (4.4)\\ &= (1 - b_n)x^* + b_n SP_K[x^* - g(x^*) + J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*))] && (4.5)\\ &= (1 - a_n)x^* + a_n SP_K[x^* - g(x^*) + J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*))] && (4.6) \end{aligned}$$

where a_n, b_n, $c_n \in [0,1]$ $(n = 0, 1, 2, \ldots)$ are some constants. To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$. From (3.4) and (4.6), the nonexpansive property of the projection P_K, the nonexpansive mapping S and Lemma 2.5, we have

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|(1 - a_n)(x_n - x^*) + a_n(SP_K[y_n - g(y_n) + J_\rho^\varphi(g(y_n) - \rho(T(y_n) - A(y_n))] - \\ &\quad SP_K[x^* - g(x^*) + J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*))])\| + \|d_n\| \\ &\leq (1 - a_n)\|x_n - x^*\| + a_n\|y_n - g(y_n) + J_\rho^\varphi(g(y_n) - \rho(T(y_n) - A(y_n))) - x^* + g(x^*) - \end{aligned}$$

$$
\begin{aligned}
&J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*)))\| + \|d_n\| \\
&\le (1-a_n)\|x_n - x^*\| + a_n\|y_n - x^* - g(y_n) + g(x^*)\| + a_n\|J_\rho^\varphi(g(y_n) - \\
&\quad \rho(T(y_n) - A(y_n))) - J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*)))\| + \|d_n\| \\
&\le (1-a_n)\|x_n - x^*\| + a_n\|y_n - x^* - g(y_n) + g(x^*)\| + a_n\tau\|g(y_n) - \\
&\quad \rho(T(y_n) - A(y_n)) - g(x^*) + \rho(T(x^*) - A(x^*))\| + \|d_n\| \\
&\le (1-a_n)\|x_n - x^*\| + a_n(1+\tau)\|(y_n - x^*) - (g(y_n) - g(x^*))\| + \\
&\quad a_n\tau\|y_n - x^* - \rho(T(y_n) - T(x^*))\| + a_n\rho\tau\|A(y_n) - A(x^*)\| + \|d_n\|.
\end{aligned} \tag{4.7}
$$

From the relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian definition on g,

$$
\begin{aligned}
&\|y_n - x^* - (g(y_n) - g(x^*))\|^2 \\
&\quad = \|y_n - x^*\|^2 - 2\langle g(y_n) - g(x^*), y_n - x^*\rangle + \|g(y_n) - g(x^*)\|^2 \\
&\quad \le \|y_n - x^*\|^2 - 2[-\gamma_1\|g(y_n) - g(x^*)\|^2 + r_1\|y_n - x^*\|^2] + \|g(y_n) - g(x^*)\|^2 \\
&\quad \le \|y_n - x^*\|^2 + 2\gamma_1\mu_1^2\|y_n - x^*\|^2 - 2r_1\|y_n - x^*\|^2 + \mu_1^2\|y_n - x^*\|^2 \\
&\quad = [1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2]\|y_n - x^*\|^2.
\end{aligned} \tag{4.8}
$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T,

$$
\begin{aligned}
&\|y_n - x^* - \rho(T(y_n) - T(x^*))\|^2 \\
&\quad = \|y_n - x^*\|^2 - 2\rho\langle T(y_n) - T(x^*), y_n - x^*\rangle + \rho^2\|T(y_n) - T(x^*)\|^2 \\
&\quad \le \|y_n - x^*\|^2 - 2\rho[-\gamma\|T(y_n) - T(x^*)\|^2 + r\|y_n - x^*\|^2] + \rho^2\|T(y_n) - T(x^*)\|^2 \\
&\quad \le \|y_n - x^*\|^2 + 2\gamma\mu^2\rho\|y_n - x^*\|^2 - 2r\rho\|y_n - x^*\|^2 + \mu^2\rho^2\|y_n - x^*\|^2 \\
&\quad = [1 + 2\gamma\mu^2\rho - 2r\rho + \mu^2\rho^2]\|y_n - x^*\|^2.
\end{aligned} \tag{4.9}
$$

By the Lipschitz continuity of A and (4.7)–(4.9), we obtain

$$
\|x_{n+1} - x^*\| \le (1-a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\| + \|d_n\| \tag{4.10}
$$

where

$$
\theta = (1+\tau)\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2} + \tau\sqrt{1 + 2\gamma\mu^2\rho - 2r\rho + \mu^2\rho^2} + k\tau\rho. \tag{4.11}
$$

It follows from (4.1)–(4.3) that $\theta < 1$.

From (3.3) and (4.5), the nonexpansive property of the projection P_K, the nonexpansive mapping S and Lemma 2.5, we obtain

$$
\begin{aligned}
&\|y_n - x^*\| \\
&\quad = \|(1-b_n)(x_n - x^*) + b_n(SP_K[z_n - g(z_n) + J_\rho^\varphi(g(z_n) - \rho(T(z_n) - A(z_n))] - \\
&\qquad SP_K[x^* - g(x^*) + J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*))])\| + \|e_n\| \\
&\quad \le (1-b_n)\|x_n - x^*\| + b_n\|z_n - g(z_n) + J_\rho^\varphi(g(z_n) - \rho(T(z_n) - A(z_n))) - x^* + g(x^*) - \\
&\qquad J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*)))\| + \|e_n\| \\
&\quad \le (1-b_n)\|x_n - x^*\| + b_n\|z_n - x^* - g(z_n) + g(x^*)\| + b_n\|J_\rho^\varphi(g(z_n) - \\
&\qquad \rho(T(z_n) - A(z_n))) - J_\rho^\varphi(g(x^*) - \rho(T(x^*) - A(x^*)))\| + \|e_n\|
\end{aligned}
$$

$$\le (1-b_n)\|x_n - x^*\| + b_n\|z_n - x^* - g(z_n) + g(x^*)\| + b_n\tau\|g(z_n)-$$
$$\rho(T(z_n) - A(z_n)) - g(x^*) + \rho(T(x^*) - A(x^*))\| + \|e_n\|$$
$$\le (1-b_n)\|x_n - x^*\| + b_n(1+\tau)\|(z_n - x^*) - (g(z_n) - g(x^*))\|+$$
$$b_n\tau\|z_n - x^* - \rho(T(z_n) - T(x^*))\| + a_n\rho\tau\|A(z_n) - A(x^*)\| + \|e_n\|. \tag{4.12}$$

From the relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian definition on g,

$$\|z_n - x^* - (g(z_n) - g(x^*))\|^2 = \|z_n - x^*\|^2 - 2\langle g(z_n) - g(x^*), z_n - x^*\rangle + \|g(z_n) - g(x^*)\|^2$$
$$\le \|z_n - x^*\|^2 - 2[-\gamma_1\|g(z_n) - g(x^*)\|^2 + r_1\|z_n - x^*\|^2] + \|g(z_n) - g(x^*)\|^2$$
$$\le \|z_n - x^*\|^2 + 2\gamma_1\mu_1^2\|z_n - x^*\|^2 - 2r_1\|z_n - x^*\|^2 + \mu_1^2\|z_n - x^*\|^2$$
$$= [1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2]\|z_n - x^*\|^2. \tag{4.13}$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T,

$$\|z_n - x^* - \rho(T(z_n) - T(x^*))\|^2$$
$$= \|z_n - x^*\|^2 - 2\rho\langle T(z_n) - T(x^*), z_n - x^*\rangle + \rho^2\|T(z_n) - T(x^*)\|^2$$
$$\le \|z_n - x^*\|^2 - 2\rho[-\gamma\|T(z_n) - T(x^*)\|^2 + r\|z_n - x^*\|^2] + \rho^2\|T(z_n) - T(x^*)\|^2$$
$$\le \|z_n - x^*\|^2 + 2\gamma\mu^2\rho\|z_n - x^*\|^2 - 2r\rho\|z_n - x^*\|^2 + \mu^2\rho^2\|z_n - x^*\|^2$$
$$= [1 + 2\gamma\mu^2\rho - 2r\rho + \mu^2\rho^2]\|z_n - x^*\|^2. \tag{4.14}$$

By the Lipschitz continuity of A and (4.11)–(4.14), we obtain

$$\|y_n - x^*\| \le (1-b_n)\|x_n - x^*\| + b_n\theta\|z_n - x^*\| + \|e_n\|. \tag{4.15}$$

In a similar way, from (3.2) and (4.4), it follows that

$$\|z_n - x^*\| \le (1-c_n)\|x_n - x^*\| + c_n\theta\|x_n - x^*\| + \|f_n\|$$
$$= [1 - c_n(1-\theta)]\|x_n - x^*\| + \|f_n\| \le \|x_n - x^*\| + \|f_n\|. \tag{4.16}$$

From (4.15) and (4.16),

$$\|y_n - x^*\| \le (1-b_n)\|x_n - x^*\| + b_n\theta\|x_n - x^*\| + b_n\theta\|f_n\| + \|e_n\|$$
$$= [1 - b_n(1-\theta)]\|x_n - x^*\| + b_n\theta\|f_n\| + \|e_n\| \le \|x_n - x^*\| + b_n\theta\|f_n\| + \|e_n\|. \tag{4.17}$$

Then from (4.10) and (4.17), we obtain that

$$\|x_{n+1} - x^*\| \le (1-a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\| + \|d_n\|$$
$$\le (1-a_n)\|x_n - x^*\| + a_n\theta\|x_n - x^*\| + a_nb_n\theta^2\|f_n\| + a_n\theta\|e_n\| + \|d_n\|$$
$$= [1 - a_n(1-\theta)]\|x_n - x^*\| + a_nb_n\theta^2\|f_n\| + a_n\theta\|e_n\| + \|d_n\|$$
$$\le [1 - a_n(1-\theta)]\|x_n - x^*\| + \|f_n\| + \|e_n\| + \|d_n\|. \tag{4.18}$$

By (4.18) and Lemma 2.8, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Example 4.1 Let $\eta(x, y) = x - y$, $T(x) = \frac{35}{9}x$, $A(x) = \frac{1}{2}x$, $g(x) = x$, for all $x, y \in H$. Then we have

(i) T is relaxed (γ, r)-cocoercive with $\gamma = \frac{1}{144}$, $r = \frac{35}{9}$, and μ-Lipschitz continuous with $\mu = 4$.

(ii) A is Lipschitz continuous with $k = 1$.

(iii) $\eta(\cdot,\cdot)$ is Lipschitz continuous with $\tau = 1$.

(iv) g is relaxed (γ_1, r_1)-cocoercive with $\gamma_1 = \frac{1}{72}$, $r_1 = 1$, and μ_1-Lipschitz continuous with $\mu_1 = 1$. Hence $\xi = \frac{1}{6}$.

After simple calculations, we have that $a = \frac{28}{9}$, $b = \frac{25}{3}$, $m_1 = \frac{28-\sqrt{109}}{135}$, $w_1 = \frac{28+\sqrt{109}}{135}$, $w_2 = \frac{2}{3}$, $m_2 = \frac{28+\sqrt{109}}{135}$ and hence condition (4.3) reduces to

$$\frac{28-\sqrt{109}}{135} < \rho < \frac{28+\sqrt{109}}{135},$$

which implies that $\theta \in (0,1)$.

If $A(x) \equiv 0$, $\eta(y, g(x)) = g(y) - g(x)$, for all $x, y \in H$, and φ is the indicator of the closed convex set K, we obtain the following convergence proof of Algorithm 3.4.

Theorem 4.2 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let the operator $T : K \to H$ be relaxed (γ, r)-cocoercive and μ-Lipschitz continuous. Let the operator $g : H \to K$ be relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitz continuous. Let $S : K \to K$ be a nonexpansive mapping such that $F(S) \cap \mathrm{GVI}(K,T,g) \neq \varnothing$, where $\mathrm{GVI}(K,T,g)$ denotes the set of solutions of (2.4). If*

$$a_n, b_n, c_n \in [0,1]\ (n = 0,1,2,\cdots),\ \sum_{n=0}^{\infty} a_n = \infty,\ \sum_{n=0}^{\infty} \|d_n\| < \infty,\ \sum_{n=0}^{\infty} \|e_n\| < \infty,\ \sum_{n=0}^{\infty} \|f_n\| < \infty,$$

and the following conditions are satisfied:

$$2r_1 < (2\gamma_1 + 1)\mu_1^2 + 1,\ \xi < 1,\ \gamma\mu^2 + \mu\sqrt{\xi(2-\xi)} < r < \gamma\mu^2 + \mu, \tag{4.19}$$

$$|\rho - \frac{r - \gamma\mu^2}{\mu^2}| < \frac{\sqrt{(r-\gamma\mu^2)^2 - \mu^2\xi(2-\xi)}}{\mu^2}, \tag{4.20}$$

where

$$\xi = 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2}.$$

Then x_n obtained from Algorithm 3.4 converges strongly to $x^* \in F(S) \cap \mathrm{GVI}(K,T,g)$.

Proof Let $x^* \in K$ be the solution of $F(S) \cap \mathrm{GVI}(K,T,g)$. Then

$$x^* = (1-c_n)x^* + c_n SP_K[x^* - g(x^*) + P_K(g(x^*) - \rho T(x^*))] \tag{4.21}$$

$$= (1-b_n)x^* + b_n SP_K[x^* - g(x^*) + P_K(g(x^*) - \rho T(x^*))] \tag{4.22}$$

$$= (1-a_n)x^* + a_n SP_K[x^* - g(x^*) + P_K(g(x^*) - \rho T(x^*))] \tag{4.23}$$

where a_n, b_n, $c_n \in [0,1]$ $(n = 0,1,2,\ldots)$ are some constants. From (3.7) and (4.23), the nonexpansive property of the projection P_K and the nonexpansive mapping S, we have

$$\begin{aligned}&\|x_{n+1} - x^*\| \\ &\quad = \|(1-a_n)(x_n - x^*) + a_n(SP_K[y_n - g(y_n) + P_K(g(y_n) - \rho T(y_n))] -\end{aligned}$$

$$
\begin{aligned}
&SP_K[x^* - g(x^*) + P_K(g(x^*) - \rho T(x^*))])\| + \|d_n\| \\
&\le (1-a_n)\|x_n - x^*\| + a_n\|y_n - x^* - g(y_n) + g(x^*) + (P_K(g(y_n) - \rho T(y_n))]- \\
&\quad P_K(g(x^*) - \rho T(x^*))])\| + \|d_n\| \\
&\le (1-a_n)\|x_n - x^*\| + a_n\|y_n - x^* - g(y_n) + g(x^*)\| + \\
&\quad a_n\|g(y_n) - g(x^*) - \rho(T(y_n) - T(x^*))\| + \|d_n\| \\
&\le (1-a_n)\|x_n - x^*\| + 2a_n\|y_n - x^* - (g(y_n) - g(x^*))\| + \\
&\quad a_n\|y_n - x^* - \rho(T(y_n) - T(x^*))\| + \|d_n\|. \qquad (4.24)
\end{aligned}
$$

From the relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian definition on g,

$$
\begin{aligned}
&\|y_n - x^* - (g(y_n) - g(x^*))\|^2 = \|y_n - x^*\|^2 - 2\langle g(y_n) - g(x^*), y_n - x^*\rangle + \|g(y_n) - g(x^*)\|^2 \\
&\le \|y_n - x^*\|^2 - 2[-\gamma_1\|g(y_n) - g(x^*)\|^2 + r_1\|y_n - x^*\|^2] + \|g(y_n) - g(x^*)\|^2 \\
&\le \|y_n - x^*\|^2 + 2\gamma_1\mu_1^2\|y_n - x^*\|^2 - 2r_1\|y_n - x^*\|^2 + \mu_1^2\|y_n - x^*\|^2 \\
&= [1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2]\|y_n - x^*\|^2. \qquad (4.25)
\end{aligned}
$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T, we have

$$
\begin{aligned}
&\|y_n - x^* - \rho(T(y_n) - T(x^*))\|^2 \\
&= \|y_n - x^*\|^2 - 2\rho\langle T(y_n) - T(x^*), y_n - x^*\rangle + \rho^2\|T(y_n) - T(x^*)\|^2 \\
&\le \|y_n - x^*\|^2 - 2\rho[-\gamma\|T(y_n) - T(x^*)\|^2 + r\|y_n - x^*\|^2] + \rho^2\|T(y_n) - T(x^*)\|^2 \\
&\le \|y_n - x^*\|^2 + 2\rho\gamma\mu^2\|y_n - x^*\|^2 - 2\rho r\|y_n - x^*\|^2 + \rho^2\mu^2\|y_n - x^*\|^2 \\
&= [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|y_n - x^*\|^2. \qquad (4.26)
\end{aligned}
$$

By (4.24)–(4.26), we obtain

$$\|x_{n+1} - x^*\| \le (1-a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\| + \|d_n\|, \qquad (4.27)$$

where

$$\theta = \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2} + 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2}. \qquad (4.28)$$

It follows from (4.19) and (4.20) that $\theta < 1$.

From (3.6) and (4.22), the nonexpansive property of the projection P_K and the nonexpansive mapping S, we obtain

$$
\begin{aligned}
&\|y_n - x^*\| \\
&= \|(1-b_n)(x_n - x^*) + b_n(SP_K[z_n - g(z_n) + P_K(g(z_n) - \rho T(z_n))]- \\
&\quad SP_K[x^* - g(x^*) + P_K(g(x^*) - \rho T(x^*))])\| + \|e_n\| \\
&\le (1-b_n)\|x_n - x^*\| + 2b_n\|z_n - x^* - (g(z_n) - g(x^*))\| + \\
&\quad b_n\|z_n - x^* - \rho(T(z_n) - T(x^*))\| + \|e_n\|. \qquad (4.29)
\end{aligned}
$$

From the relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian definition on g,

$$\|z_n - x^* - (g(z_n) - g(x^*))\|^2$$

$$
\begin{aligned}
&= \|z_n - x^*\|^2 - 2\langle g(z_n) - g(x^*), z_n - x^*\rangle + \|g(z_n) - g(x^*)\|^2 \\
&\le \|z_n - x^*\|^2 - 2[-\gamma_1\|g(z_n) - g(x^*)\|^2 + r_1\|z_n - x^*\|^2] + \|g(z_n) - g(x^*)\|^2 \\
&= [1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2]\|z_n - x^*\|^2. \qquad (4.30)
\end{aligned}
$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T, we have

$$
\begin{aligned}
&\|z_n - x^* - \rho(T(z_n) - Tx^*)\|^2 \\
&\quad= \|z_n - x^*\|^2 - 2\rho\langle T(z_n) - Tx^*, z_n - x^*\rangle + \rho^2\|T(z_n) - Tx^*\|^2 \\
&\quad\le \|z_n - x^*\|^2 - 2\rho[-\gamma\|T(z_n) - Tx^*\|^2 + r\|z_n - x^*\|^2] + \rho^2\|T(z_n) - Tx^*\|^2 \\
&\quad= [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|z_n - x^*\|^2. \qquad (4.31)
\end{aligned}
$$

By (4.28)–(4.31), we obtain

$$\|y_n - x^*\| \le (1 - b_n)\|x_n - x^*\| + b_n\theta\|z_n - x^*\| + \|e_n\|. \qquad (4.32)$$

In a similar way, from (3.5) and (4.21), it follows that

$$
\begin{aligned}
\|z_n - x^*\| &\le (1 - c_n)\|x_n - x^*\| + c_n\theta\|x_n - x^*\| + \|f_n\| \\
&= [1 - c_n(1 - \theta)]\|x_n - x^*\| + \|f_n\| \le \|x_n - x^*\| + \|f_n\|. \qquad (4.33)
\end{aligned}
$$

From (4.32) and (4.33),

$$
\begin{aligned}
&\|y_n - x^*\| \le (1 - b_n)\|x_n - x^*\| + b_n\theta\|x_n - x^*\| + b_n\theta\|f_n\| + \|e_n\| \\
&\quad= [1 - b_n(1 - \theta)]\|x_n - x^*\| + b_n\theta\|f_n\| + \|e_n\| \le \|x_n - x^*\| + b_n\theta\|f_n\| + \|e_n\|. \quad (4.34)
\end{aligned}
$$

Then from (4.27) and (4.34), we obtain that

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \le (1 - a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\| + \|d_n\| \\
&\quad\le (1 - a_n)\|x_n - x^*\| + a_n\theta\|x_n - x^*\| + a_n b_n\theta^2\|f_n\| + a_n\theta\|e_n\| + \|d_n\| \\
&\quad= [1 - a_n(1 - \theta)]\|x_n - x^*\| + a_n b_n\theta^2\|f_n\| + a_n\theta\|e_n\| + \|d_n\| \\
&\quad\le [1 - a_n(1 - \theta)]\|x_n - x^*\| + \|f_n\| + \|e_n\| + \|d_n\|. \qquad (4.35)
\end{aligned}
$$

By (4.35) and Lemma 2.8, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. $\square$

Remark 4.1 Theorem 4.2 generalizes and complements Theorem 3.1 of [4].

If $A(x) \equiv 0$, $\eta(x, y) = x - y$, for all $x, y \in H$, $g = I$, the identity operator, $d_n = e_n = f_n \equiv 0$ $(n = 0, 1, 2, \ldots)$ and φ is the indicator of the closed convex set K, we obtain the following convergence proof of Algorithm 3.5 which is Theorem 3.1 of [3].

Theorem 4.3 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let the operator $T : K \to H$ be relaxed (γ, r)-cocoercive and μ-Lipschitz continuous. Let $S : K \to K$ be a nonexpansive mapping such that $F(S) \cap VI(T, K) \neq \varnothing$, where $VI(T, K)$ denotes the set of solutions of (2.5). If*

$$a_n, b_n, c_n \in [0, 1] \ (n = 0, 1, 2, \ldots), \ \sum_{n=0}^{\infty} a_n = \infty$$

and the following conditions are satisfied:

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2, \quad \gamma\mu^2 < r.$$

Then x_n obtained from Algorithm 3.5 converges strongly to $x^* \in F(S) \cap \mathrm{VI}(T, K)$.

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