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Paired Domination of Cartesian Products of Graphs

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Abstract Let $\gamma_{pr}(G)$ denote the paired domination number and $G \,\Box\, H$ denote the Cartesian product of graphs G and H. In this paper we show that for all graphs G and H without isolated vertex, $\gamma_{pr}(G)\gamma_{pr}(H) \le 7\gamma_{pr}(G \,\Box\, H)$.

Keywords domination; paired domination; Cartesian product.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The open neighborhood of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$, the set of vertices adjacent to v. The closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V$, the open neighborhood of S is defined by $N_G(S) = \cup_{v\in S} N_G(v)$, and the closed neighborhood of S by $N_G[S] = N_G(S) \cup S$. The subgraph of G induced by the vertices in S is denoted by $G[S]$.

A set of vertices or a set of edges is independent if no two of its elements are adjacent. A matching in a graph G is a set of independent edges in G. A perfect matching M in G is a matching such that every vertex of G is incident with an edge of M. The ends of an edge in M are called paired vertices (with respect to M). Let $S \subseteq V(G)$. We say that S contains a perfect matching in G if $G[S]$ has a perfect matching.

For $S \subseteq V(G)$, the set S is a dominating set if $N[S] = V$, a total dominating set, denoted TDS, if $N(S) = V$, and a paired dominating set, denoted PDS, if $N(S) = V$ and S contains a perfect matching in G. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. The paired domination number $\gamma_{pr}(G)$ and the total domination number $\gamma_t(G)$ can be defined similarly. By the definitions, we can easily have

$$\gamma(G) \le \gamma_t(G) \le \gamma_{pr}(G) \le 2\gamma(G),$$

for each graph G without isolated vertex. For a detailed treatment of total domination and paired domination in graphs, the reader can refer to [2] and [7].

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A set $S \subseteq V(G)$ is a k-packing if the vertices in S are pairwise at distance at least $k+1$ apart in G, i.e., if $u, v \in S$, then $d_G(u,v) \geq k+1$. The k-packing number $\rho_k(G)$ is the maximum cardinality of a k-packing. In [1], the authors proved that $\gamma_{pr}(G)$ is at least twice its 3-packing number $\rho_3(G)$. And they defined a graph G to be a (γ_{pr}, ρ_3)-graph if $\gamma_{pr}(G) = 2\rho_3(G)$.

For graphs G and H, the Cartesian product $G \Box H$ is the graph with vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

In 1968, Vizing [9] conjectured that for any graphs G and H,

$$\gamma(G)\gamma(H) \leq \gamma(G \Box H).$$

The best general upper bound to date on $\gamma(G)\gamma(H)$ in terms of $\gamma(G \Box H)$ is the following theorem due to Clark and Suen [3].

Theorem 1 ([3]) *For any graphs G and H, $\gamma(G)\gamma(H) \leq 2\gamma(G \Box H)$.*

The inability to resolve Vizing's conjecture has lead authors to pose different variations of the original problem. Several such variations were studied by Nowakowski and Rall in [8]. The total domination version has been studied by Henning and Rall [4]. They proved that for any graphs G and H without isolated vertices, $\gamma_t(G)\gamma_t(H) \leq 6\gamma_t(G \Box H)$. The bound has been improved by Hou [6]. Recently, Pak Tung Ho in [5] proved that $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \Box H)$, which resolved the conjecture proposed by Henning and Rall in [4]. The paired domination version was studied by Brešar, Henning, and Rall [1]. They proved that for any graphs G and H without isolated vertices,

$$\gamma_{pr}(G \Box H) \geq \max\{\gamma_{pr}(G)\rho_3(H), \gamma_{pr}(H)\rho_3(G)\}.$$

As a corollary, they deduced that for any graphs G and H without isolated vertices, at least one of which is a (γ_{pr}, ρ_3)-graph,

$$\gamma_{pr}(G)\gamma_{pr}(H) \leq 2\gamma_{pr}(G \Box H),$$

and this bound is sharp. But they did not give a general bound of $\gamma_{pr}(G)\gamma_{pr}(H)$ in terms of $\gamma_{pr}(G \Box H)$ for any graphs G and H without isolated vertices as given in [4–6].

In this paper, we give a general bound as follows.

Theorem 2 *For any graphs G and H without isolated vertices,*

$$\gamma_{pr}(G)\gamma_{pr}(H) \leq 7\gamma_{pr}(G \Box H).$$

By Theorem 1 and $\gamma(G) \leq \gamma_{pr}(G) \leq 2\gamma(G)$, we have a trivial bound $\gamma_{pr}(G)\gamma_{pr}(H) \leq 8\gamma_{pr}(G \Box H)$. Then Theorem 2 improves the trivial bound. Some known results imply that for any graphs G and H without isolated vertices, $\gamma_{pr}(G)\gamma_{pr}(H) \leq 2\gamma_{pr}(G \Box H)$. We leave this as an open question.

2. Proof of Theorem 2

We first give some notation which will be used in our proofs. Let G be a graph without isolated vertices and T a subgraph of G. We say that $S \subseteq V(G)$ dominates T in G if $N_G[S] \supseteq V(T)$, and S is called a dominating set of T in G. And S is called a paired dominating set (denoted PDS) of T in G if $N_G[S] \supseteq V(T)$ and S contains a perfect matching in G. In the product $G \Box H$, we define H_x to be the subgraph induced by $\{x\} \times V(H)$, for any $x \in V(G)$, G_y can be defined similarly for any $y \in V(H)$.

For any vertex (x,u) of $G \Box H$, the vertex u of H is the H-projection of (x,u), denoted $u = \phi_H(x,u)$. For any subset $A = \{(x_1,u_1),\dots,(x_k,u_k)\}$ of $V(G \Box H)$, the H-projection of A, denoted $\phi_H(A)$, is defined by $\phi_H(A) = \bigcup_{i=1}^{k}\{\phi_H(x_i,u_i)\} = \{u_1,u_2,\dots,u_k\}$, which is a subset of $V(H)$. For a vertex $(x,u) \in V(G \Box H)$, an edge joining (x,u) and $(y,u)(y \in N_G(x))$ is called a G-edge of $G \Box H$. Similarly, an edge joining (x,u) and $(x,v)(v \in N_H(u))$ is called an H-edge of $G \Box H$. The following is a useful lemma to prove the main theorem.

Lemma 1 *Let H be a graph without isolated vertex. Suppose G is a graph and D is a set of vertices in $G \Box H$ such that $\phi_H(D)$ dominates H, and $D = D_1 \cup D_2$ where D_1 has a perfect matching in $G \Box H$. Then $\gamma_{pr}(H) \le |D_1| + 2|D_2|$.*

Proof Let M_1 be a perfect matching of D_1 in $G \Box H$. If M_1 contains no H-edge, then $\phi_H(D_1) \le \frac{1}{2}|D_1|$. Hence $\gamma_{pr}(H) \le 2\gamma(H) \le 2|\phi_H(D)| \le 2(|\phi_H(D_1)| + |\phi_H(D_2)|) \le |D_1| + 2|D_2|$.

Now, assume that M_1 contains H-edges. Let M_{11} be a maximum subset of M_1 such that $\phi_H(V(M_{11}))$ has a perfect matching M'_{11} in H and $|M_{11}| = |M'_{11}|$. Let $D_{11} = V(M_{11})$ and $D_{12} = D_1 - D_{11}$. Then, by the maximal of D_{11}, for any vertex $\alpha \in D_{12}$, there exists either a vertex $\beta \in D_{12}$ such that $\phi_H(\beta) = \phi_H(\alpha)$ or a vertex $\beta \in D_{11}$ such that $\phi_H(\beta) = \phi_H(\alpha)$ or $\phi_H(\beta) = \phi_H(p(\alpha))$, where $p(\alpha)$ denotes the paired vertex of α (with respect to M_1). Hence $|\phi_H(D_1)| = |\phi_H(D_{11})| + |\phi_H(D_{12})| - |\phi_H(D_{11}) \cap \phi_H(D_{12})| \le |D_{11}| + \frac{1}{2}|D_{12}|$.

Let M be a maximum matching of the subgraph of H induced by $\phi_H(D)$ and S be the set of vertices saturated by M. Then $|S| \ge |\phi_H(D_{11})| = |D_{11}|$. Let $\bar{S} = \phi_H(D) - S$. Let M' be a maximum matching of the bipartite subgraph of H with partite sets $\bar{S}$ and $N_H(\bar{S}) - S$ and with edge set all the edges of H connecting vertices in $\bar{S}$ and vertices in $N_H(\bar{S}) - S$. Let S' be the set of all vertices saturated by M'. If the bipartite subgraph defined above has isolated vertices, let S_1 denote the isolated vertex set (then $S_1 \subseteq \bar{S}$ and, for each vertex $u \in S_1$, $N_H(u) \subseteq S$ by the above definition), and $S_2 = \bar{S} - S_1$. Then S' is a PDS of $S_2 \cup (N_H(\bar{S}) - S)$ in H and $|S'| \le 2|\bar{S}|$. Note that S_1 does not contribute to the domination of H and $\phi_H(D)$ dominates H, $S \cup S'$ is a PDS of H. Hence

$$\begin{aligned}\gamma_{pr}(H) &\le |S| + 2|\bar{S}| \le 2|\phi_H(D)| - |S| \le 2(|\phi_H(D_1)| + |\phi_H(D_2)|) - |D_{11}| \\ &\le 2|D_{11}| + |D_{12}| + 2|D_2| - |D_{11}| = |D_1| + 2|D_2|. \quad \Box\end{aligned}$$

In the following proof, we will use $N(S)$ instead of $N_{G\Box H}(S)$ if the index is clear.

Theorem 3 *For any graphs G and H without isolated vertices,*

$$\gamma_{pr}(G)\gamma_{pr}(H) \le 7\gamma_{pr}(G \Box H).$$

Proof Let D be a minimum PDS of $G \Box H$. Then the subgraph induced by D in $G \Box H$ contains a perfect matching M. Let $M = M_G \cup M_H$, where M_G is the set of all G-edges in M and M_H is the set of all H-edges in M. By the symmetry of the graphs G and H in $G \,\Box\, H$, we may assume that $|M_G| \le |M_H|$. Let $D_G = V(M_G)$ and $D_H = V(M_H)$. Then $D = D_G \cup D_H$ and $|D_G| \le |D_H|$. So $|D_G| \le \frac{1}{2}|D|$.

Let $A = \{x_1, y_1, \ldots, x_k, y_k\}$ be a minimum PDS of G where for each i, x_i is adjacent to y_i in G, and so $\gamma_{pr}(G) = 2k$. Let $\{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ be a partition of $V(G)$ such that $\{x_i, y_i\} \subseteq \Pi_i \subseteq N(\{x_i, y_i\})$ for each i, $1 \le i \le k$. For each $i = 1, 2, \ldots, k$, we introduce the following notations: $D_i = D \cap (\Pi_i \times V(H))$, $D_{G_i} = D_G \cap D_i$. Let $M_{H_i} = M_H \cap E(G \,\Box\, H[D_i])$, where $E(G \,\Box\, H[D_i])$ is the edge set of the subgraph of $G \,\Box\, H$ induced by D_i, and $D_{H_i} = V(M_{H_i})$ (note that $D_{H_i} = D_i - D_{G_i}$).

Let $F_i = \{(x_i, w) \mid w \in V(H) \text{ and } (\Pi_i \times \{w\}) \cap N(D_i) = \emptyset\}$, and denote $l_i = |F_i|$, $F_i' = \phi_H(F_i) = \{w \in V(H) \mid (x_i, w) \in F_i\}$. Then $\phi_H(D_i) \cup F_i'$ dominates H. Note that $D_i = D_{H_i} \cup D_{G_i}$ and D_{H_i} has a perfect matching in $G \,\Box\, H$. By Lemma 1,

$$\gamma_{pr}(H) \le |D_{H_i}| + 2|D_{G_i}| + 2|F_i| = |D_i| + |D_{G_i}| + 2l_i.$$

So,

$$\begin{aligned} \frac{1}{2}\gamma_{pr}(G)\gamma_{pr}(H) &= \sum_{i=1}^{k} \gamma_{pr}(H) \le \sum_{i=1}^{k} |D_i| + \sum_{i=1}^{k} |D_{G_i}| + 2\sum_{i=1}^{k} l_i \\ &= |D| + |D_G| + 2\sum_{i=1}^{k} l_i \le \frac{3}{2}|D| + 2\sum_{i=1}^{k} l_i. \end{aligned} \tag{1}$$

The set $\Pi_i \times \{w\}$ is called a cell and we say the cell $\Pi_i \times \{w\}$ is vertically undominated if $(\Pi_i \times \{w\}) \cap N(D_i) = \emptyset$, and vertically dominated otherwise. Let $D_w = D \cap G_w$ for any $w \in V(H)$. If a cell $\Pi_i \times \{w\}$ is vertically undominated, then, since D is a PDS of $G \,\Box\, H$, $\Pi_i \times \{w\} \subseteq N(D_w)$. Hence each vertex in a vertically undominated cell $\Pi_i \times \{w\}$ is dominated by D_w. Each vertex in a cell (in particular, in a vertically dominated cell) $\Pi_j \times \{w\}$ is paired dominated by $\{x_j, y_j\} \times \{w\}$.

Let $C_w = \bigcup_j (\{x_j, y_j\} \times \{w\})$, where j is taken over all vertically dominated cells $\Pi_j \times \{w\}$. Then $C_w \cup D_w$ dominates G_w and C_w contains a perfect matching. Let m_w denote the number of vertically undominated cells in G_w. Note that G_w is isomorphic to G, by Lemma 1,

$$\gamma_{pr}(G) \le 2(k - m_w) + 2|D_w|.$$

Hence $m_w \le |D_w|$. Therefore,

$$\sum_{i=1}^{k} l_i = \sum_{w \in V(H)} m_w \le \sum_{w \in V(H)} |D_w| = |D|.$$

Thus, by inequation (1), we have

$$\gamma_{pr}(G)\gamma_{pr}(H) \le 7|D| = 7\gamma_{pr}(G \,\Box\, H). \quad \Box$$

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