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A Weighted Weak Type Endpoint Estimate for the Multilinear Calderón-Zygmund Operators and Its Applications

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Abstract A weighted weak type endpoint estimate is established for the m-linear operator with Calderón-Zygmund kernel, which was introduced by Coifman and Meyer. As applications, the mapping properties on weighted $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ with weight M_Bw for certain maximal operator M_B and general weight w, and a two-weight weighted norm estimate for this operator, are obtained.

Keywords multilinear Calderón-Zygmund operator; maximal operator; weighted norm inequality; interpolation.

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1. Introduction

During the last several years, considerable attention has been paid to the study of boundedness of multilinear singular integral operators on function spaces [5–7]. Lu [11] studied L^p boundedness of multilinear oscillatory singular integrals with Calderón-Zygmund kernel; Meng [12] introduced multilinear Calderón-Zygmund operators on the product of Lebesgue spaces and Hardy-type spaces with non-doubling measures. Let $K(x;y_1,\ldots,y_m)$, $m\ge 1$, be a locally integrable function defined on $R^{(m+1)n}\backslash\{(x,y_1,\ldots,y_m): x=y_1=\cdots=y_m; x,y_1,\ldots,y_m\in R^n\}$ and $\gamma\in(0,1]$ be two constants. We say that K is a kernel in m-CZK(A,ϵ) if it satisfies the size condition that for all $(x,y_1,\ldots,y_m)$ with $x\ne y_j$ for some j with $1\le j\le m$,

$$|K(x;\,y_1,\,\ldots,\,y_m)|\le\frac{A}{(|x-y_1|+\cdots+|x-y_m|)^{mn}}\tag{1.1}$$

and satisfies the regularity condition that

$$|K(x;\,y_1,\,\ldots,\,y_m)-K(x';\,y_1,\,\ldots,\,y_m)|\le\frac{A|x-x'|^{\gamma}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\gamma}}\tag{1.2}$$

whenever $\max_{1\le j\le m}|x-y_j|\ge 2|x-x'|$, and also that for each fixed j with $1\le j\le m$,

$$|K(x;\,y_1,\,\ldots,y_j,\,\ldots,\,y_m)-K(x;\,y_1,\,\ldots,y_j',\,\ldots,\,y_m)|\le\frac{A|y_j-y_j'|^{\gamma}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\gamma}}\tag{1.3}$$

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whenever $\max_{1\le j\le m}|x-y_j|\ge 2|y_j-y_j'|$. Let T be an m-linear operator. We say that T is an operator with Calderón-Zygmund kernel K if for $f_1,\ldots,f_m\in L^2(\mathbb{R}^n)$ with compact supports, and for $x\notin\cap_{j=1}^m \operatorname{supp} f_j$

$$T(f_1,f_2,\ldots,f_m)(x)=\int_{(\mathbb{R}^n)^m}K(x;y_1,\ldots,y_m)f_1(y_1),\ldots,f_m(y_m)\mathrm{d}y_1,\ldots,\mathrm{d}y_m, \tag{1.4}$$

and K is in m-CZK(A,ϵ) for some constants A and ϵ. It is obvious that when $m=1$, this operator is just the classical Calderón-Zygmund operator. For the case of $m\ge 2$, this operator has intimate connections with operator theory and partial differential equations, and was considered first by Coifman and Meyer [1, 2], and then by many authors. In the remarkable work [6], Grafakos and Torrea considered the mapping properties of T on the space of type $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ with $1\le p_1,\ldots,p_m<\infty$, and established a $T1$ type theorem for the operator T. Grafakos and Kalton [5] established the $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times\cdots\times H^1(\mathbb{R}^n)\to L^{\frac{1}{m}}(\mathbb{R}^n)$ boundedness of T. For other works about m-linear operaor with Calderón-Zygmund, see [7] and the references therein.

The purpose of this paper is to give some weighted norm inequalities for the m-linear operator with Calderón-Zygmund kernel, in analogy with the weighted estimate for the classical Calderón-Zygmund operators which were established by Pérez [8] and Cruz-Uribe and Pérez [3]. To state our results, we first recall some notations.

By a weight w we mean that w is a nonnegative and locally integrable function. For a measurable set E and a weight w, $w(E)$ denotes the integral of w over E, namely, $w(E)=\int_E w(x)\mathrm{d}x$. For $p\in(0,\infty)$ and a suitable function f, $\|f\|_{L^{p,\infty}(\mathbb{R}^n,w)}$ denotes the weighted weak L^p "norm" with respect to the weight w, that is,

$$\|f\|^p_{L^{p,\infty}(\mathbb{R}^n,w)}=\sup_{\lambda>0}\lambda^p w(\{x\in\mathbb{R}^n:|f(x)|>\lambda\}).$$

Let E be a measurable set with $\mu(E)<\infty$. For fixed $p\in(1,\infty)$ and $\delta\ge 0$ and suitable function f, set

$$\|f\|_{L^p(\log L)^\delta,E}=\inf\Big\{\lambda:\frac{1}{\mu(E)}\int_E\Big(\frac{|f(x)|}{\lambda}\Big)^p\log^\delta\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x\le 1\Big\}.$$

The maximal operator $M_{L^p(\log L)^\delta}$ is defined by

$$M_{L^p(\log L)^\delta}f(x)=\sup_{Q\ni x}\|f\|_{L^p(\log L)^\delta,Q},$$

where the sup is taken over all cubes containing x. In the following, we denote $M_{L^1(\log L)^\delta}$ by $M_{L(\log L)^\delta}$ for simplicity. It is easy to see that for $\delta=0$, $M_{L(\log L)^\delta}$ is just the operator M, the standard maximal operator.

Our main result can be stated as follows.

Theorem 1 *Let $m\ge 1$, T be an m-linear operator with Calderón-Zygmund kernel. Suppose that for some $q_1,q_2,\ldots,q_m\in[1,\infty]$ and some $q\in(0,\infty)$ with $1/q=\sum_{k=1}^m 1/q_k$, T is bounded from $L^{q_1}(\mathbb{R}^n)\times L^{q_2}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Then for any $\delta>0$, there exists a constant $C>0$ depending only on n and δ, such that for all weight w and all bounded functions $f_1,f_2,\ldots,f_m$*

with compact support,

$$\|T(f_1,\, f_2,\, \ldots,\, f_m)\|_{L^{1/m,\,\infty}(\mathbb{R}^n,\, w)} \le C \prod_{k=1}^{m} \|f_k\|_{L^1(\mathbb{R}^n,\, M_{L(\log L)^{\delta}} w)}.$$

As an application of Theorem 1, we have the following weighted estimate with general weight for the m-linear operator with Calderón-Zygmund kernel.

Corollary 1 *Let $m \ge 2$, T be an m-linear operator with Calderón-Zygmund kernel. Let $p_1,\, p_2,\, \ldots,\, p_m \in [1,\, \infty)$ with $\max_{1\le k\le m} p_k > 1$ and $p \in (0,\, \infty)$ with $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$. Suppose that for some $q_1,\, q_2,\, \ldots,\, q_m \in [1,\, \infty]$ and some $q \in (0,\, \infty)$ with $1/q = \sum_{k=1}^{m} 1/q_k$, T is bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Then there exists a constant $C > 0$ depending only on n, m, $p_1, \ldots p_m$ and δ, such that for any weight w and any bounded functions $f_1,\, f_2,\, \ldots,\, f_m$ with compact supports,*

$$\|T(f_1,\, f_2,\, \ldots,\, f_m)\|_{L^p(\mathbb{R}^n,\, w)} \le C \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\mathbb{R}^n,\, M_{L(\log L)^{p_0-1+\delta}} w)},$$

where $p_0 = \min_{1\le k\le m} p_k$.

(1) If $p_1, p_2, \ldots, p_m \in [1, \infty)$ with $1 = \min_{1\le j\le m} p_j < \max_{1\le j\le m} p_j$, and $p \in (0, \infty)$ with $1/p = \sum_{j=1}^{m} 1/p_j$, then for any $\delta > 0$, there exisits a constant $C > 0$ such that for any weight w,T is bounded from $L^{p_1}(\mathbb{R}^n, M_{L(\log L)^{\delta}w}) \times L^{p_2}(\mathbb{R}^n, M_{L(\log L)^{\delta}w}) \times \cdots L^{p_m}(\mathbb{R}^n, M_{L(\log L)^{\delta}w})$ to $L^p(\mathbb{R}^n, w)$;

(2) If $p_1, p_2, \ldots, p_m \in [1, \infty)$ with $1 < \min_{1\le j\le m} p_j$, and $p \in (0, \infty)$ with $1/p = \sum_{j=1}^{m} 1/p_j$. Then for any j_0 with $p_{j_0} \in (1, \infty)$, there exists a constant $C > 0$ depending on δ, such that for any weight w and all bounded functions $f_1, f_2, \ldots, f_m$ with compact support,

$$\begin{aligned}\|T(f_1, f_2, \ldots, f_m)\|_{L^p(\mathbb{R}^n, w)} \le & C\|f\|_{L^{p_{j_0}}(\mathbb{R}^n, M_{L(\log L)^{p_{j_0}-1+\delta}w})} \times \\ & \prod_{1\le j\le m, j\ne j_0} \|f_k\|_{L^{p_j}(\mathbb{R}^n, M_{L(\log L)^{\delta}w})}.\end{aligned}$$

To give another application of Theorem 1, we consider the two-weight, weighted norm estimate for the m-linear operator with Calderón-Zygmund kernel. Let u, v be two weights on $\mathbb{R}^n$. We say that $(u,\, v) \in A_{p,\,(\log L)^{\sigma}}(\mathbb{R}^n)$ with some $\sigma > 0$, if there exists a constant $C > 0$ such that for any cube Q,

$$\|u\|_{L(\log L)^{\sigma},\, Q} \Big(\frac{1}{|Q|} \int_Q \big(v(x)\big)^{p'/p}\, \mathrm{d}x\Big)^{p-1} \le C.$$

Corollary 2 *Let $m \ge 1$, T be an m-linear operator with Calderón-Zygmund kernel. Suppose that for some $q_1,\, q_2,\, \ldots,\, q_m \in [1,\, \infty]$ and some $q \in (0,\, \infty)$ with $1/q = \sum_{k=1}^{m} 1/q_k$, T is bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$.*

(i) If $p \in (1/m,\, \infty)$ and $(u,\, v) \in A_{mp,\,(\log L)^{mp-1+\sigma}}(\mathbb{R}^n)$ for some $\sigma > 0$, then for any bounded functions $f_1,\, f_2,\, \ldots,\, f_m$ with compact supports,

$$\|T(f_1,\, f_2,\, \ldots,\, f_m)(x)\|_{L^{p,\,\infty}(\mathbb{R}^n,\, u)} \le C \prod_{k=1}^{m} \|f_k\|_{L^{mp}(\mathbb{R}^n,\, v)};$$

(ii) If $p_1, p_2, \dots, p_m \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$ and $1 < p_0 = \min_{1\le k\le m} p_k < \max_{1\le k\le n} p_k$, and if $(u, v) \in A_{p_0,\,(\log L)^{p_0-1+\sigma}}(\mathbb{R}^n)$ for $\sigma > 0$, then for any bounded functions $f_1, f_2, \dots, f_m$ with compact supports,

$$\|T(f_1, f_2, \dots, f_m)\|_{L^p(\mathbb{R}^n,\,u)} \le C\prod_{k=1}^{m}\|f_k\|_{L^{p_k}(\mathbb{R}^n,\,v)}.$$

We now make some conventions. Throughout this paper, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constant with subscript, say, C_1, does not change in different occurrences. For a measurable set E, χ_E denotes the characteristic function of E. Given $\lambda > 0$ and a cube Q, λQ denotes the cube with the same center as Q and whose side length is λ times that of Q. For a fixed p with $p \in [1, \infty)$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$. For a locally integrable function f on $\mathbb{R}^n$ and bounded measurable set E, $m_E(f)$ denotes the mean value of f over E, that is, $m_E(f) = \frac{1}{|E|}\int_E f(x)\mathrm{d}x$.

2. Proof of Theorem 1

We begin with some preliminary lemmas.

Lemma 1 *Let $m \ge 2$, T be an m-linear operator with Calderón-Zygmund kernel K in m-CZK(A, ϵ). Then for all positive integer l with $1 \le l < m$ and all bounded functions $f_1, f_2, \dots, f_l$ with compact support, the operator $T_{f_1, f_2, \dots, f_l}$ defined by*

$$T_{f_1, f_2, \dots, f_l}(f_{l+1}, \dots, f_m)(x) = T(f_1, f_2, \dots, f_m)(x)$$

is an $(m-l)$-linear operator with kernel in $(m-l)$-CZK$(A\prod_{j=1}^{l}\|f_j\|_{L^\infty(\mathbb{R}^n)}, \epsilon)$. Moreover, if T is bounded from $L^{q_1}(\mathbb{R}^n)\times L^{q_2}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $q_1, q_2, \dots, q_m \in [1, \infty]$ and $q \in (0, \infty)$ with $1/q = \sum_{k=1}^{m} 1/q_k$. Then

$$\|T_{f_1, f_2, \dots, f_l}(f_{l+1}, \dots f_m)\|_{L^p(\mathbb{R}^n)} \le C\prod_{k=1}^{l}\|f_k\|_{L^\infty(\mathbb{R}^n)}\prod_{k=l+1}^{m}\|f_k\|_{L^{p_k}(\mathbb{R}^n)}$$

with $p_k \in (1, \infty)$ $(l+1 \le k \le m)$ and $1/p = \sum_{k=l+1}^{m} 1/p_k$.

This lemma is a combination of Lemma 3 and Theorem 2 in [6].

Lemma 2 *Let $q \in (1, \infty)$, (u, v) be a pair of weights and $(u, v) \in A_{q,\,(\log L)^{q-1+\sigma}}(\mathbb{R}^n)$ for some $\sigma > 0$. Then for any $\delta \in (0, \sigma/q)$, there exists a constant $C > 0$ such that*

$$\|M_{L(\log L)^\delta} f\|_{L^{q'}(\mathbb{R}^n,\,v^{-q'/q})} \le C\|f\|_{L^{q'}(\mathbb{R}^n,\,u^{-q'/q})}.$$

For a proof, see [3, p.424].

Lemma 3 *Let $q_0, \delta > 0$. S be an operator from $\mathscr{S}(\mathbb{R}^n)\times\cdots\times\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{M}$ (the set of measurable functions on $\mathbb{R}^n$). Suppose that there exists a constant $C > 0$ such that for any* *weight* w,

$$w(\{x\in\mathbb{R}^n:\,|S(f_1,\,f_2,\,\dots,f_m)(x)|>\lambda\})\le C\lambda^{-q_0}\prod_{k=1}^{m}\|f_k\|^{q_0}_{L^1(\mathbb{R}^n,\,M_{L(\log L)^\delta}w)}.$$

Then for any $q>q_0$ *and* $\sigma>\delta q/q_0$, *there exists a constant* $C>0$ *such that for any weight* w,

$$\|S(f_1,\,f_2,\,\dots,f_m)\|_{L^{q,\,\infty}(\mathbb{R}^n,\,w)}\le C\prod_{k=1}^{m}\|f_k\|_{L^{q/q_0}(\mathbb{R}^n,\,M_{L(\log L)^{q/q_0-1+\sigma}}w)}. \tag{2.1}$$

Proof We will employ the idea of Cruz-Uribe and Pérez [4]. Let $q>q_0$ and set $r=q/q_0$. For any fixed $\lambda>0$, set

$$\mathscr{F}_\lambda=\{x\in\mathbb{R}^n:\,|S(f_1,\,f_2,\,\dots,\,f_m)(x)|>\lambda\}.$$

By duality and our hypothesis,

$$\begin{aligned}\big\{w(\mathscr{F}_\lambda)\big\}^{/}=\|\chi_{\mathscr{F}_\lambda}\|_{\mathscr{L}(\mathbb{R},\,)}&=\sup_{\|h\|_{L^{r'}(\mathbb{R}^n,\,w)}\le 1}\Big|\int_{\mathscr{F}_\lambda}h(x)w(x)\mathrm{d}x\Big|\\&\le C\lambda^{-q_0}\prod_{k=1}^{m}\|f_k\|^{q_0}_{L^1\left(\mathbb{R}^n,\,M_{L(\log L)^\delta}(hw)\right)}.\end{aligned} \tag{2.2}$$

For any fixed $\sigma>\delta r$, set $\eta=\sigma-\delta r$. As it was pointed out in [3, p.424], we have

$$t\log^{-\delta}(2+t)\le\frac{t^{1/r}}{\log^{\delta+(r-1+\eta)/r}(2+t)}\times t^{1/r'}\log^{(r-1+\eta)/r}(2+t).$$

This via the generalization of Hölder inequality [10, p.64] in turn implies that

$$\begin{aligned}M_{L(\log L)^\delta}(hw)(x)&\le CM_{L^r(\log L)^{(1+\delta)r-1+\eta}}(w^{1/r})(x)M_{L^{r'}(\log L)^{-1-(r'-1)\eta}}(hw^{1/r'})(x)\\&\le C\Big\{M_{L(\log L)^{r-1+\sigma}}w(x)\Big\}^{1/r}M_{L^{r'}(\log L)^{-1-(r'-1)\eta}}(hw^{1/r'})(x).\end{aligned}$$

It then follows from the Hölder inequality that for each k with $1\le k\le m$,

$$\begin{aligned}\|f_k\|^{q_0}_{L^1\left(\mathbb{R}^n,\,M_{L(\log L)^\delta}(hw)\right)}\le&C\Big(\int_{\mathbb{R}^n}|f_k(x)|^rM_{L(\log L)^{r-1+\epsilon}}w(x)\mathrm{d}x\Big)^{q_0/r}\times\\&\Big(\int_{\mathbb{R}^n}\Big(M_{L^{r'}(\log L)^{-1-(r'-1)\eta}}(hw^{1/r'})(x)\Big)^{r'}\mathrm{d}x\Big)^{q_0/r'}\\\le&C\Big(\int_{\mathbb{R}^n}|f_k(x)|^rM_{L(\log L)^{r-1+\sigma}}w(x)\mathrm{d}x\Big)^{q_0/r},\end{aligned} \tag{2.3}$$

where in the last inequality, we have invoked the fact that the operator $M_{L^{r'}(\log L)^{-1-(r'-1)\eta}}$ is bounded on $L^{r'}(\mathbb{R}^n)$ ([9]). Combining the estimates (2.2) and (2.3) yields (2.1) and then completes the proof of Lemma 3. □

Proof of Theorem 1 We will proceed by an induction argument on m. If $m=1$, Theorem 1 is just Theorem 1 in [8]. Now let $m\ge1$ be a positive integer. We assume that Theorem 1 holds for any l-linear operator with Calderón-Zygmund kernel for any l with $1\le l\le m$. Let $f_1,\,f_2,\,\dots,\,f_m,\,f_{m+1}\in L^1(\mathbb{R}^n,\,M_{L(\log L)^\delta}w)$ such that

$$\|f_1\|_{L^1(\mathbb{R}^n,\,M_{L(\log L)^\delta}w)}=\|f_2\|_{L^1(\mathbb{R}^n,\,M_{L(\log L)^\delta}w)}=\cdots=\|f_{m+1}\|_{L^1(\mathbb{R}^n,\,M_{L(\log L)^\delta}w)}=1.$$

For each fixed $\lambda > 0$ and each fixed k with $1 \le k \le m+1$, applying the Calderón-Zygmund decomposition to each f_j at level $\lambda^{1/(m+1)}$, we then obtain a sequence of cubes $\{Q^k_{k_j}\}_j$ with disjoint interiors, such that

(i) For any fixed j,

$$\lambda^{1/(m+1)} \le \frac{1}{|Q^k_{k_j}|} \int_{Q^k_{k_j}} |f_k(y)| \mathrm{d}y \le 2^n \lambda^{1/(m+1)}.$$

(ii) $|f_k(x)| \le C\lambda^{1/(m+1)}$ a.e., $x \in \mathbb{R}^n \setminus \cup_j Q^k_{k_j}$.

Set

$$g_k(x) = f_k(x)\chi_{\mathbb{R}^n \setminus \cup_j Q^k_{k_j}}(x) + \sum_j m_{Q^k_{k_j}}(f_k)\chi_{Q^k_{k_j}}(x)$$

and

$$b_k(x) = \sum_j \Big(f_k(x) - m_{Q^k_{k_j}}(f_k)\Big)\chi_{Q^k_{k_j}}(x).$$

Lemma 1, together with our inductive hypothesis, tells us that

$$\begin{aligned} &w(\{x \in \mathbb{R}^n : |T(f_1, f_2, \dots, f_m, g_{m+1})(x) > \lambda/2\}) \\ &\le C\lambda^{-1/m} \|g_{m+1}\|^{1/m}_{L^\infty(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|^{1/m}_{L^1(\mathbb{R}^n,\, M_{L(\log L)^\delta} w)} \\ &\le C\lambda^{-1/(m+1)}. \end{aligned}$$

Let $\Omega = \bigcup_{k=1}^{m+1} \cup_j 4\sqrt{n} Q^k_{k_j}$. A trivial computation leads to that

$$\begin{aligned} w(\Omega) &\le \sum_{k=1}^{m+1} \sum_j \frac{w(Q^k_{k_j})}{|Q^k_{k_j}|} |Q^k_{k_j}| \\ &\le \lambda^{-1/(m+1)} \sum_{k=1}^{m+1} \sum_j \int_{Q^k_{k_j}} |f_k(x)| \mathrm{d}x \inf_{y \in Q^k_{k_j}} Mw(y) \\ &\le C(m+1)\lambda^{-1/(m+1)}. \end{aligned}$$

Set $w^*(x) = w(x)\chi_{\mathbb{R}^n \setminus \Omega}(x)$. The proof of Theorem 1 is now reduced to proving that

$$w^*(\{x \in \mathbb{R}^n \setminus \Omega : |T(f_1, f_2, \dots, f_m, b_{m+1})(x)| > \lambda/2\}) \le C\lambda^{-1/(m+1)}. \tag{2.4}$$

We now prove (2.4). Let Λ_j $(1 \le j \le 2^m)$ be a nonempty subset of $\{1, 2, \dots, m\}$ and set

$$E_j = \{x \in \mathbb{R}^n \setminus \Omega : |T(h_1, h_2, \dots, h_m, b_{m+1})(x)| > \lambda/2^{m+2} : \text{ where } h_l = g_l \text{ when } l \in \Lambda_j,$$
$$h_l = b_l \text{ when } l \notin \Lambda_j,\ 1 \le l \le m\}.$$

Denote by N_j the cardinal number of Λ_j. Lemma 1, together with our inductive hypothesis, tells us that for any j with $1 \le j \le 2^m$,

$$\begin{aligned} w^*(E_j) &\le C\lambda^{-1/(m+1-N_j)} \prod_{k \in \Lambda_j} \|g_k\|^{1/(m+1-N_j)}_{L^\infty(\mathbb{R}^n)} \prod_{1 \le k \le m+1,\, k \notin \Lambda_j} \|b_k\|^{1/(m+1-N_j)}_{L^1(\mathbb{R}^n,\, M_{L(\log L)^\delta} w^*)} \\ &\le C\lambda^{-1/(m+1)} \prod_{1 \le k \le m+1,\, k \notin \Lambda_j} \|b_k\|^{1/(m+1-N_j)}_{L^1(\mathbb{R}^n,\, M_{L(\log L)^\delta} w^*)}. \end{aligned}$$

Recall that $\operatorname{supp} w^* \subset \mathbb{R}^n \backslash \Omega$. There exists a constant $C > 0$ such that for any k and j,

$$\sup_{x \in Q^k_{k_j}} M_{L(\log L)^\delta} w(x) \le C \inf_{y \in Q^k_{k_j}} M_{L(\log L)^\delta} w(y)$$

(see [8]). Thus, for any k with $1 \le k \le m+1$,

$$\begin{aligned} \|b_k\|_{L^1(\mathbb{R}^n,\, M_{L(\log L)^\delta} w^*)} &\le \sum_j \int_{Q^k_{k_j}} |f_k(y)| \mathrm{d}y \inf_{y \in Q^k_{k_j}} M_{L(\log L)^\delta} w(y) \\ &\le C \int_{\mathbb{R}^n} |f_k(y)| M_{L(\log L)^\delta} w(y) \mathrm{d}y. \end{aligned}$$

This, in turn, implies that

$$w\Big(\bigcup_{j=1}^{2^m} E_j\Big) \le C \lambda^{-1/(m+1)}. \tag{2.5}$$

Now we claim that

$$w^*(\{x \in \mathbb{R}^n \backslash \Omega :\, |T(b_1, b_2, \ldots, b_m, b_{m+1})(x)| > \lambda/4\}) \le C \lambda^{-1/(m+1)}. \tag{2.6}$$

In fact, for each fixed k and k_j, denote by $c^k_{k_j}$ and $l(Q^k_{k_j})$ the center and side length of $Q^k_{k_j}$, respectively. By an estimate of Grafakos and Torres [6, pp.137–138], we know that for any $x \in \mathbb{R}^n \backslash \Omega$,

$$|T(b_1, b_2, \ldots, b_m, b_{m+1})(x)| \le C \lambda \prod_{k=1}^{m+1} \mathcal{M}_k(x),$$

where $\mathcal{M}_k$ is the Marcinkiewicz function defined by

$$\mathcal{M}_k(x) = \sum_j \frac{\{l(Q_j)\}^{n+\epsilon/(m+1)}}{\big(l(Q_j) + |x - c^k_{k_j}|\big)^{n+\epsilon/(m+1)}}.$$

On the other hand, a straightforward computation leads to that for any k with $1 \le k \le m+1$,

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{M}_k(x) w(x) \mathrm{d}x &\le \sum_j \int_{\mathbb{R}^n} \frac{\{l(Q_j)\}^{n+\epsilon/(m+1)}}{\big(l(Q_j) + |x - c^k_{k_j}|\big)^{n+\epsilon/(m+1)}} w(x)\, \mathrm{d}x \\ &\le C \sum_j |Q^k_{k_j}| \inf_{y \in Q^k_{k_j}} Mw(y) \\ &\le C \lambda^{-1/(m+1)} \int_{\mathbb{R}^n} f(y) Mw(y) \mathrm{d}y. \end{aligned}$$

Therefore,

$$\begin{aligned} & w^*(\{x \in \mathbb{R}^n \backslash \Omega :\, |T(b_1, b_2, \ldots, b_m, b_{m+1})(x)| > \lambda\}) \\ & \quad \le \lambda^{-1/(m+1)} \int_{\mathbb{R}^n \backslash \Omega} \Big| T(b_1, b_2, \ldots, b_m, b_{m+1})(x) \Big|^{1/(m+1)} w^*(x) \mathrm{d}x \\ & \quad \le C \prod_{k=1}^{m+1} \Big(\int_{\mathbb{R}^n \backslash \Omega} \mathcal{M}_k(x) w^*(x) \mathrm{d}x \Big)^{1/(m+1)} \\ & \quad \le C \lambda^{-1/(m+1)}, \end{aligned}$$

and so (2.6) holds. Combining the estimates (2.5) and (2.6) then leads to our desired inequality (2.4).

Proof of Corollary 1 We only consider the case $m=2$. The conclusion for the case $m\geq 3$ can be proved in the same way, along with an inductive argument involving Lemma 1. By Theorem 1, we see that for any $\delta>0$ and weight w, T is bounded from $L^1(R^n,\, M_{L(\log L)^\delta}w)\times L^1(R^n,\, M_{L(\log L)^\delta}w)$ to $L^{1/2,\infty}(R^n,\, w)$. On the other hand, the weighted weak type endpoint estimated for the Calderon-Zygmund operater [7, Theorem 1.6], together with Lemma 1, states that T is bounded from $L^1(R^n,\, M_{L(\log L)^\delta}w)\times L^\infty(R^n)$ to $L^{1,\infty}(R^n,\, w)$. Therefore, by the classical interpolation theorem of Marcinkiewicz, it follows that for any $p\in(1,\infty)$

$$\|T(f_1,\, f_2\|_{L^{p/(p+1)}(\mathbb{R}^n,\, w)}\leq C\|f_1\|_{L^1(\mathbb{R}^n,\, M_{L(\log L)^\delta}w)}\|f_2\|_{L^p(\mathbb{R}^n,\, M_{L(\log L)^\delta}w)}. \tag{2.4}$$

Now let $p_1,p_2\in(1,\infty)$. Again by Lemma 1 and the weighted L^{p_2} estimate for the Calderon-Zygmund operator [7, Theorem 1.1], we know that for any δ and weight w,

$$\|T(f_1,\, f_2)\|_{L^{p_2}(\mathbb{R}^n,\, w)}\leq C\|f_1\|_{L^\infty(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,\, M_{L(\log L)^{p_2-1+\delta}}w)}. \tag{2.5}$$

Interpolation between the equalities (2.5) and the trivial estimate give that

$$\|T(f_1,\, f_2)\|_{L^{p_2/(p_2+1)}(\mathbb{R}^n,\, w)}\leq \|f_1\|_{L^1(\mathbb{R}^n,\, M_{L(\log L)^\delta}w)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,\, M_{L(\log L)^{p_2-1+\delta}}w)}. \tag{2.6}$$

It then follows that $p\in(0,\infty)$ with $1/p=1/p_1+1/p_2$,

$$\|T(f_1,\, f_2)\|_{L^p(\mathbb{R}^n,\, w)}\leq C\|f_1\|_{L^{p_1}(\mathbb{R}^n,\, M_{L(\log L)^\delta}w)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,\, M_{L(\log L)^{p_2-1+\delta}}w)}.$$

Similarly, interpolating the equality

$$\|T(f_1,\, f_2)\|_{L^{p_1}(\mathbb{R}^n,\, w)}\leq C\|f_1\|_{L^{p_1}(\mathbb{R}^n,\, M_{L(\log L)^{p_1-1+\delta}}w)}\|f_2\|_{L^\infty(\mathbb{R}^n)}$$

and the inequality

$$\|T(f_1,\, f_2)\|_{L^{p_1/(p_1+1)}(\mathbb{R}^n,\, w)}\leq C\|f_1\|_{L^{p_1}(\mathbb{R}^n,\, M_{L(\log L)^{p_1-1+\delta}}w)}\|f_2\|_{L^1(\mathbb{R}^n,\, M_{L(\log L)^\delta}w)}$$

yields

$$\|T(f_1,\, f_2)\|_{L^p(\mathbb{R}^n,\, w)}\leq C\|f_1\|_{L^{p_1}(\mathbb{R}^n,\, M_{L(\log L)^{p_1-1+\delta}}w)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,\, M_{L(\log L)^\delta}w)}.$$

This completes the proof of Corollary 1. □

Proof of Corollary 2 The proof of (i) follows from the same argument used in the proof of Theorem 1.2 in [3]. Let

$$\Omega_\lambda=\{x\in\mathbb{R}^n:\,|T(f_1,\,f_2,\,\ldots,\,f_m)(x)|>\lambda\}.$$

Not that $|\Omega_\lambda|<\infty$ for any $\lambda>0$. By duality, we know that

$$\{u(\Omega_\lambda)\}^{1/(mp)}=\|u^{1/(mp)}\chi_{\Omega_\lambda}\|_{L^{mp}(\mathbb{R}^n)}=\sup_{\|h\|_{L^{(mp)'}(\mathbb{R}^n)}\leq 1}\int_{\Omega_\lambda}\big(u(x)\big)^{1/(mp)}h(x)\mathrm{d}x.$$

An application of Theorem 1 yields that when $h\in L^{(mp)'}(\mathbb{R}^n)$ with $\|h\|_{L^{(mp)'}(\mathbb{R}^n)}\leq 1$,

$$\begin{aligned}\int_{\Omega_\lambda}\big(u(x)\big)^{1/(mp)}h(x)\mathrm{d}x&\leq C\lambda^{-1/m}\prod_{k=1}^m\|f_k\|^{1/m}_{L^1(\mathbb{R}^n,\, M_{L(\log L)^\delta})}\\ &\leq C\lambda^{-1/m}\prod_{k=1}^m\|f_k\|^{1/m}_{L^{mp}(\mathbb{R}^n,\, v)}\Big\|M_{L(\log L)^\delta}(u^{1/(mp)}h)\Big\|^{1/m}_{L^{(mp)'}(\mathbb{R}^n,\, v^{-(mp)'/(mp)})}\end{aligned}$$

$$\leq C\Big(\lambda^{-p}\prod_{k=1}^{m}\|f_k\|^p_{L^{mp}(\mathbb{R}^n,\,v)}\Big)^{1/(mp)}.$$

Our desired conclusion (i) then follows directly.

We turn our attention to (ii). We only consider the case that $m = 2$. For the case that $m \geq 2$, (ii) can be proved by the same argument, along with the induction argument on m. Let $p_1, p_2 \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = 1/p_1 + 1/p_2$. Without loss of generality, we may assume that $p_1 < p_2$. By conclusion (i), we have

$$\|T(f_1,\, f_2)\|_{L^{p_1/2,\,\infty}(\mathbb{R}^n,\,u)} \leq C\prod_{k=1}^{2}\|f_k\|_{L^{p_1}(\mathbb{R}^n,\,v)}. \tag{2.11}$$

On the other hand, Lemma 1 states that for $f_2 \in L^\infty(\mathbb{R}^n)$ with compact support,

$$\|T(f_1,\, f_2)\|_{L^{p_1,\,\infty}(\mathbb{R}^n,\,u)} \leq C\|f_1\|_{L^{p_1}(\mathbb{R}^n,\,v)}\|f_2\|_{L^\infty(\mathbb{R}^n)}. \tag{2.12}$$

Interpolating the inequalities (2.11) and (2.12) then gives

$$\|T(f_1,\, f_2)\|_{L^{p}(\mathbb{R}^n,\,u)} \leq C\prod_{k=1}^{2}\|f_k\|_{L^{p_k}(\mathbb{R}^n,\,v)}.$$

This completes the proof of Corollary 2. □

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