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The Poincaré Series of Relative Invariants of Finite Pseudo-Reflection Groups

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Abstract Let F be a field with characteristic 0, $V=F^n$ the n-dimensional vector space over F and let G be a finite pseudo-reflection group which acts on V. Let $\chi: G\longrightarrow F^*$ be a 1-dimensional representation of G. In this article we show that $\chi(g)=(\det g)^{\alpha}(0\leq\alpha\leq r-1)$, where $g\in G$ and r is the order of g. In addition, we characterize the relation between the relative invariants and the invariants of the group G, and then we use Molien's Theorem of invariants to compute the Poincaré series of relative invariants.

Keywords Poincaré series; finite pseudo-reflection group; relative invariants.

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1. Introduction

Let F be a field with characteristic 0 and V be the n-dimensional vector space over F. The pseudo-reflection and the reflecting hyperplane are defined as follows:

$$\sigma\in GL(V), H=\{\xi\in V|\sigma\xi=\xi\}.$$

If $\dim H=n-1$, then σ is called a pseudo-reflection, and subspace H is called the reflecting hyperplane of σ. A vector $v\neq 0$ in $\mathrm{Im}(\sigma-1)$ is called a reflecting vector of σ (see [1, 2]).

Throughout this paper F denotes a fixed field with characteristic 0, unless the contrary is explicitly stated. σ has finite order, so the characteristic of the field F does not divide the order of σ (which we shall call the nonmodular case), thus σ must be diagonalizable.

For convenience, we always suppose G is a finite pseudo-reflection group that is generated by the fundamental pseudo-reflections $s_1,\ldots,s_n$. The definition of relative invariants is needed in the paper. Let $\chi: G\mapsto F^*$ be a 1-dimensional representation of G. For $f\in F[V^*]$, if $\sigma\cdot f=\chi(\sigma)f$, then f is called the χ-relative invariant of G.

$$\det: G\mapsto F^*$$
$$\sigma\mapsto \det\sigma$$

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is a 1-dimensional representation of G. The det-relative invariants of G have been discussed completely in [5]. In section 3, we will calculate the Poincaré series of det-relative invariants.

This brought about the questions: How to characterize the other 1-dimensional representation of the group G? What is the relation between relative invariants and invariants?

In Section 2, we shall discuss these questions, and obtain the conclusions:

Let $\chi : G \mapsto F^*$ be a 1-dimensional representation of G. If each $\sigma \in G$, $\sigma \cdot P = \chi(\sigma)P$, then $\chi(\sigma) = 1$ or $\chi(\sigma) = (\det\sigma)^\alpha$, $1 \le \alpha \le r-1$, $r = |\sigma|$.

The difference between the relative invariants and the invariants was described by Larry Smith in [3], which is only a divisor $L_\chi = c\prod_{U\in H(G)} l_{s_U}^{a_U}$, $c \in F^*$.

In Section 3, the Poincaré series of relative invariants of the group G can be computed.

2. The relative invariants of the finite pseudo-reflection group

Theorem 2.1 *If P is a χ-relative invariant of the group G, i.e., for each $\sigma \in G$, $\sigma \cdot P = \chi(\sigma)P$, $P \neq 0$, then $\chi(\sigma) = 1$ or $\chi(\sigma) = (\det\sigma)^\alpha$, $1 \le \alpha \le r-1$, where r is the order of σ.*

Proof Let U be a reflecting hyperplane of a pseudo-reflection σ, $G_U = \langle\sigma\rangle$, $|\sigma| = r$. Choose a basis $\varepsilon_1, \dots, \varepsilon_n$, such that

$$\sigma^i(\varepsilon_j) = \varepsilon_j \ (1 \le j \le n-1), \ \sigma^i(\varepsilon_n) = \xi_{\sigma^i}\varepsilon_n, \ \xi_{\sigma^i} = \xi_\sigma^i,$$

where ξ_σ is a primitive r-root of unity. Suppose $\{x_1, \dots, x_n\} \in V^*$ is the dual basis of $\varepsilon_1, \dots, \varepsilon_n$, thus the reflecting hyperplane U is determined by $x_n = 0$. Since

$$\sigma^i \cdot x_j = x_j, \ (1 \le j \le n-1), \ \sigma^i \cdot x_n = \xi_{\sigma^i}^{-1} x_n$$

and

$$\sigma \cdot P = \chi(\sigma)P,$$

we have

$$P(x_1, \dots, x_{n-1}, \xi_\sigma^{-1} x_n) = \chi(\sigma)P(x_1, \dots, x_n).$$

If $P(x_1, \dots, x_n)$ is described as follows:

$$P(x_1, \dots, x_n) = \sum_{m\ge 0} P_m(x_1, \dots, x_{n-1})x_n^m,$$

then

$$\sum_{m\ge 0} P_m(x_1, \dots, x_{n-1})\xi_\sigma^{-m} x_n^m = \chi(\sigma)\sum_{m\ge 0} P_m(x_1, \dots, x_{n-1})x_n^m,$$

i.e.,

$$\begin{aligned} &P_0 + \xi_\sigma^{-1}P_1x_n + \cdots + \xi_\sigma^{-r+1}P_{r-1}x_n^{r-1} + P_rx_n^r + \cdots \\ &\quad = \chi(\sigma)(P_0 + P_1x_n + \cdots + P_{r-1}x_n^{r-1} + P_rx_n^r + \cdots). \end{aligned}$$

Equating coefficients of the x_n, we obtain:

$$\chi(\sigma) = 1 \ \text{ or } \ P_0 = 0;$$

$$\chi(\sigma)\xi_\sigma = 1 \ \text{ or } \ P_1 = 0;$$

$$\cdots$$

$$\chi(\sigma)\xi_\sigma^{r-1} = 1 \ \text{ or } \ P_{r-1} = 0.$$

We prove that only one of

$$\chi(\sigma) = 1,\ \chi(\sigma)\xi_\sigma = 1, \ldots, \chi(\sigma)\xi_\sigma^{r-1} = 1$$

occurs. Otherwise, suppose there exist two equalities

$$\chi(\sigma)\xi_\sigma^m = 1,\ \chi(\sigma)\xi_\sigma^n = 1,\ 0 \le m, n \le r-1,$$

which implies $\xi_\sigma^{m-n} = 1$. Clearly, $m - n < r$, which contradicts the fact that ξ_σ is a primitive r-th root of unity. Hence it is impossible for no less than two cases to exist at the same time. In fact, if none of

$$\chi(\sigma) = 1,\ \chi(\sigma)\xi_\sigma = 1, \ldots, \chi(\sigma)\xi_\sigma^{r-1} = 1$$

exists, then

$$P_0 = 0,\ P_1 = 0, \ldots,$$

i.e.,

$$P = 0$$

which contradicts $P \neq 0$. Therefore, there must exist only one of

$$\chi(\sigma) = 1,\ \chi(\sigma)\xi_\sigma = 1, \ldots, \chi(\sigma)\xi_\sigma^{r-1} = 1.$$

Suppose $\chi(\sigma)\xi_\sigma^u = 1$, $0 \le u \le r-1$, then

$$\chi(\sigma) = \xi^{r-m} = \xi^\alpha = (\det\sigma)^\alpha, \quad 0 \le \alpha \le r-1.$$

This completes the proof. □

For the remainder of this section we shall characterize the relation between the χ-relative invariants and invariants of G. Let $H(G) = \{H_s | s \in G\}$ denote the set of reflecting hyperplanes of all pseudo-reflections in G.

$$H_s = \{\lambda \in V | l_s(x_1, \ldots, x_n)(\lambda) = 0\}$$

is defined by $l_s(x_1, \ldots, x_n) = 0$, where $l_s(x_1, \ldots, x_n) = 0$ is a homogeneous linear polynomial. If $U \in H(G)$ is a reflecting hyperplane of G, we denote by G_U the pointwise stabilizer of U in G. This is the group generated by all the pseudo-reflections in G with U as a reflecting hyperplane together with 1. For every $U \in H(G)$, choose $a_U \in N$ minimal such that $\chi(s_U) = \det(s_U)^{a_U}$ and introduce the form

$$L_\chi = c \prod_{U \in H(G)} l_{s_U}^{a_U},\ c \in F^*.$$

In the following, we shall show that $L_\chi = c\prod_{U\in H(G)} l_{s_U}^{a_U},\ c \in F^*$ divides every χ-relative invariant of G. To this end, we require two lemmas.

Lemma 2.2 *Let $l \in F[V^*]_1$ be a linear polynomial function and $s \in G$ a pseudo-reflection. Suppose $s(l) = \alpha \cdot l$, for some $\alpha \in F^*$. Then either $\alpha = 1$ or $\alpha = \det(s)$, where l and l_s are nonzero multiples of l and l_s.*

Proof The case $\alpha = 1$ is trivial. On the other hand, for each $f \in F[V^*]$, l_s is a divisor of $sf - f$. In fact, by appropriate choice of basis, without loss of generality, assume that $s(\varepsilon_n) = \lambda_s \varepsilon_n$. Then $l_s = \frac{(\lambda-1)}{\|\varepsilon_n\|} x_n$ is a divisor of $sf - f$ if and only if x_n is. We know

$$(sf - f)(v) = f(s^{-1}v) - f(v) = f(v - \frac{1-\lambda^{-1}}{\|\varepsilon_n\|} x_n(v)\varepsilon_n) - f(v),$$

which becomes $f(v) - f(v) = 0$ for all $v \in V$ if we substitute 0 for x_n. Then x_n must appear in each monomial summand of $sf - f$, unless $sf - f$ is itself 0. Therefore, $sf - f$ is divisible by l_s. We denote

$$\triangle_s(f) = \frac{sf - f}{l_s}.$$

By [4, Lemma 7.1.5], $\triangle_s(l_s) = \lambda_s - 1$, so

$$(\lambda_s - 1)l_s = \triangle_s(l)l_s = s(l) - l = (\alpha - 1)l.$$

If $\alpha \neq 1$, then $\lambda_s \neq 1, \alpha = \lambda_s = \det(s)$, l and l_s are proportional.

Lemma 2.3 *Let $l_1, \ldots, l_m \in F[V^*]_1$ be linear polynomial functions and $s \in G$ a pseudo-reflection. Suppose there are constants $\alpha_1, \ldots, \alpha_m \in F^*$ such that*

$$s(l_i) = \begin{cases} \alpha_i l_{i+1}, & 1 \leq i \leq m-1; \\ \alpha_m l_1, & i = m. \end{cases}$$

If none of $l_1, \ldots, l_m$ is nonzero multiples of l_s, then $\alpha_1 \cdots \alpha_m = 1$ and $L = \alpha_1 \cdots \alpha_m$ is an invariant of s.

Proof Clearly, $s(L) = \alpha_1 \cdots \alpha_m L$ and $s^m(l_1) = \alpha_1 \cdots \alpha_m l_1$. If $s^m \neq 1$, then it is a pseudo-reflection with the reflecting hyperplane $\ker(l_s)$. Since l_1 is not a nonzero multiple of l_s, by Lemma 2.2, $\alpha_1 \cdots \alpha_m = 1$. On the other hand, if $s^m = 1$, then $l_1 = s^m(l_1) = \alpha_1 \cdots \alpha_m l_1$, so again $\alpha_1 \cdots \alpha_m = 1$.

Theorem 2.4 *Let $\chi : G \mapsto F^*$ be a 1-dimensional representation of G. If $U \in H(G)$, f is a χ-relative invariant of G, then $l_{s_U}^{a_U}$ divides f.*

Proof Choose a basis $u_1, \ldots, u_{n-1}$ for the reflecting hyperplane U and extend it to a basis $u_1, \ldots, u_{n-1}, u_n$ for V, where u_n is an eigenvector corresponding to the eigenvalue $\det(s_U)$. Let $z_1, \ldots, z_n \in V^*$ be the dual basis of $u_1, \ldots, u_n$. Then $l_{s_U}(z_1, \ldots, z_n)$ can be regarded as z_n equally. Since f is a χ-relative invariant of G, we only consider the case that f is a monomial polynomial function. Suppose that $f = z_1^{e_1} \cdots z_n^{e_n}$. By Lemma 2.2, we have

$$\chi(s_U) \cdot f = s_U \cdot f = (\det s_U)^{e_n} \cdot f.$$

Since a_U is the smallest natural number such that $\chi(s_U) = (\det s_U)^{a_U}$, we must have $a_U \leq e_n$, therefore, the result follows.

Corollary 2.5 *Let $\chi : G \mapsto F^*$ be a 1-dimensional representation of G. If f is a χ-relative invariant of G, then $L_\chi = c\prod_{U\in H(G)} l_{s_U}^{a_U}$, $c \in F^*$ divides f.*

To prove that $L_\chi = c\prod_{U\in H(G)} l_{s_U}^{a_U}$, $c \in F^*$ is a χ- relative invariant of G, we shall write L_χ in the form

$$L_\chi = c l_{s_{U'}}^{a_{U'}} \prod_{U'\neq U''} l_{s_{U''}}^{a_{U''}}.$$

Since $l_{s_{U'}} \neq l_{S_{U'}}$, if $U' \neq U''$, by Lemma 2.3, the product $\prod_{U'\neq U''} l_{s_{U'}}^{a_{U''}}$ is an invariant of $s_{U'}$. It follows from Lemma 2.2

$$s_{U'}(l_{s_{U'}}^{a_{U'}}) = s_{U'}(l_{s_{U'}})^{a_{U'}} = (\det s_{U'} \cdot l_{s_{U'}})^{a_{U'}} = (\det s_{U'})^{a_{U'}} \cdot l_{s_{U'}}^{a_{U'}} = \chi(s_{U'}) \cdot l_{s_{U'}}^{a_{U'}},$$

so

$$\begin{aligned} s_{U'}(L_\chi) &= s_{U'}(l_{s_{U'}}^{a_{U'}} \prod_{U'\neq U''} l_{s_{U''}}^{a_{U''}}) = s_{U'}(l_{s_{U'}}^{a_{U'}}) s_{U'}(\prod_{U'\neq U''} l_{s_{U''}}^{a_{U''}}) \\ &= \chi(s_{U'}) l_{s_{U'}}^{a_{U'}} \prod_{U'\neq U''} l_{s_{U''}}^{a_{U''}} = \chi(s_{U'}) \prod_{U'\neq U''} l_{s_{U''}}^{a_{U''}}. \end{aligned}$$

Namely,

$$L_\chi = c \prod_{U\in H(G)} l_{s_U}^{a_U}, \quad c \in F^*$$

is a χ-relative invariant.

Theorem 2.6 *Let $\chi : G \mapsto F^*$ be a 1-dimensional representation of G. $L_\chi = c\prod_{U\in H(G)} l_{s_U}^{a_U}$, $c \in F^*$, then $L_\chi = c\prod_{U\in H(G)} l_{s_U}^{a_U}$, $c \in F^*$ is a χ-relative invariant.*

Proof If s is a fundamental pseudo-reflection, then $s(L_\chi) = \chi(s)L_\chi$. For each $g \in G$, we may write $g = s_1 \cdots s_k$, where s_i $(i = 1, 2, \ldots, k)$ are fundamental pseudo-reflections. Therefore,

$$g(L_\chi) = (s_1 \cdots s_k)(L_\chi) = (s_1 \cdots (s_k L_\chi)) = \chi(s_1) \cdots \chi(s_k) L_\chi = \chi(g) L_\chi$$

and $L_\chi = c\prod_{U\in H(G)} l_{s_U}^{a_U}$, $c \in F^*$ is a χ-relative invariant. □

Theorem 2.7 *Let $\chi : G \mapsto F^*$ be a 1-dimensional representation of G. If f is a χ-relative invariant, then $f = h \cdot L_\chi$, h is an invariant.*

Proof Since G is generated by fundamental pseudo-reflections, for every $g \in G$, g may be denoted as the product of some suitable fundamental pseudo-reflections. Hence

$$g(L_\chi h) = g(L_\chi) g(h) = g(L_\chi) h = \chi(g) L_\chi h = \chi(g)(L_\chi h)$$

and the result follows.

From Theorem 2.7, we obtain the conclusion that the difference between relative invariants and invariants is only one divisor $L_\chi = c\prod_{U\in H(G)} l_{s_U}^{a_U}$, $c \in F^*$.

3. The Poincaré series of relative invariants of finite pseudo-reflection groups

Suppose $F[V^*]$ is graded F-algebra. The Poincaré series of $F[V^*]$ is defined as follows:

$$P(F[V^*],t)=\sum_d \dim F[V^*]_d t^d,$$

where $F[V^*]_d$ is an F-subspace consisting of all homogeneous polynomial functions of degree d in $F[V^*]$. For the finite subgroup of the general linear group, its Poincaré series of invariants can be characterized by Molien's Theorem [4, 5].

Lemma 3.1 (Molien) *Let V be a finite dimension F vector space. Let $G\in \mathrm{GL}(V)$ be a finite nonmodular subgroup. Then*

$$P(F[V^*]^G,t)=\frac{1}{|G|}\sum_{\sigma\in G}\frac{1}{\det(1-\sigma t)}.$$

To compute the Poincaré series of relative invariants, we denote by A_k $(k=0,1,2\ldots)$ the subspace consisting of relative invariants of degree k. Suppose that $\deg(L_x)=M$. It follows from Theorem 2.7 that $\dim A_k=\dim F[V^*]^G_{k-M}$. So we can make use of the Molien's Theorem of invariants to compute the Poincaré series of relative invariants as follows

$$\begin{aligned}P(F[V^*]^G_\chi,t)=P(A_k,t))&=\sum_{k=M}^{\infty}(\dim A_k)t^k=\sum_{k=M}^{\infty}\dim F[V^*]^G_{k-M}\cdot t^k\\&=\sum_{d=0}^{\infty}\dim F[V^*]^G_d\cdot t^dt^M=(\sum_{d=0}^{\infty}\dim F[V^*]^G_d\cdot t^d)t^M\\&=\frac{t^M}{|G|}\sum_{\sigma\in G}\frac{1}{\det(1-\sigma t)}.\end{aligned}$$

The following example illustrates Theorem 2.7.

Example If a_U equals 1, where $\chi(\sigma)=(\det\sigma)^{a_U}$ and $U\in H(G)$, for every $\sigma\in G$, then a χ-relative invariant becomes a det-relative invariant. We have conclusions analogous to the preceding results.

Lemma 3.2 *Let $\sigma_1,\ldots,\sigma_N$ be all the pseudo-reflections in the group G, the hyperplanes of which are $U_1,\ldots,U_N$ respectively, where $U_i=\{\lambda\in V|l_i(x_1,\ldots,x_n)(\lambda)=0\}$, and $x_1,\ldots,x_n\in V^*$ is a dual basis relative to $\{\varepsilon_1,\ldots,\varepsilon_n\}$. Suppose $f_1,\ldots,f_n$ is a group fundamental invariants of G. If we regard $f_1,\ldots,f_n$ as polynomials in n indeterminates $x_1,\ldots,x_n$, then*

$$\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}=c\prod_{i=1}^{N}l_i(x_1,\ldots,x_n),\quad c\in F,\ c\neq 0.$$

Lemma 3.3 *Suppose that G is a finite pseudo-reflections group, $f_1,\ldots,f_n$ are homogeneous invariants which are independent algebraically, and they generate a algebra $F[V^*]^G$. Let*

$$J=\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}.$$

Then

(i) $\sigma \cdot J = (\det\sigma)J$;

(ii) *Suppose that* $P \in F[V^*]$, $\sigma \cdot P = (\det\sigma)P$, *for each* $\sigma \in G$, *then* $P = Jg,\ g \in F[V^*]^G$;

(iii) *For* $k = 0, 1, 2, \ldots$, *let* A_k *be a subspace consisting of* det*-relative invariants of degree* k. *Then*

$$\dim A_k = \dim F[V^*]^G_{k-N}.$$

Hence, we calculate the Poincaré series of det-relative invariants as follows:

$$\begin{aligned} P(F[V^*]^G_{\det}, t) = P(A_k, t)) &= \sum_{k=N}^{\infty} (\dim A_k) t^k \\ &= \sum_{k=N}^{\infty} \dim F[V^*]^G_{k-N} t^k = (\sum_{d=0}^{\infty} \dim F[V^*]^G_d t^d) t^N \\ &= \frac{t^N}{|G|} \sum_{\sigma \in G} \frac{1}{\det(1 - \sigma t)}. \end{aligned}$$

In view of the preceding methods, we have

Theorem 3.4 *Let* F_p *be a finite field with* p^n *elements and* G *be a finite pseudo-reflection group. If* $r|p^n - 1$, $|\sigma| = r$ *where* $\sigma \in G$, *then for any* $\sigma \in G$,

$$\chi(\sigma) = (\det\sigma)^{\alpha}, \quad 0 \le \alpha \le 1$$

and

$$f = h \cdot L_{\chi},\ L_{\chi} = c \prod_{U \in H(G)} l_{s_U}^{a_U},\ c \in F^*,$$

where f *is a* χ*-relative invariant and* h *is an invariant of* G.

Theorem 3.5 *Let* V *be a finite dimension* F_p *vector space, and* $G \in GL(V)$ *be a finite nonmodular subgroup. If* p *does not divide* G, *then*

$$P(F[V^*]^G, t) = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{\det(1 - \sigma t)}.$$

Hence, when F_p is a finite field with p^n elements, in the case $r|p^n - 1$, the Poincaré series of relative invariants of finite pseudo-reflection group G is the same as the preceding.

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