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Periodic Solutions to an Evolution p-Laplacian System

Ying Jie WEI*, **Wen Jie GAO**
Institute of Mathematics, Jilin University, Jilin 130012*, P. R. China*

Abstract In this paper, the authors study the existence of periodic solutions to an evolution p-Laplacian system. The authors prove a comparison principle of the system in general form. Then the existence of periodic solutions to the system of evolution p-Laplacian equations is obtained with the help of the comparison principle and the monotone iteration technique.

Keywords existence; periodic solutions; p-Laplacian system.

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1. Introduction

In this paper, we study the existence of periodic solutions of the evolution p-Laplacian system

$$\frac{\partial u_i}{\partial t} = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i) + f_i(t,u_1,u_2), \quad (x,t)\in\Omega\times(0,+\infty), \tag{1.1}$$

$$u_i(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,+\infty), \tag{1.2}$$

$$u_i(x,t+\omega)=u_i(x,t), \quad (x,t)\in\overline{\Omega}\times(0,+\infty), \tag{1.3}$$

where $p_i > 2$, $\omega > 0$, $f_i(t+\omega,u_1,u_2) = f_i(t,u_1,u_2)$, $f_i(t,u_1,u_2)$ is quasimonotonic for u_j $(j \neq i)$, $i,j = 1,2$, $\Omega\subset R^n$ is an open connected bounded domain with smooth boundary $\partial\Omega$.

System (1.1) models heat propagations in a two-component combustible mixture [1], chemical processes [2], interaction of two biological groups without self-limiting [3, 4], etc.

Many authors have studied the properties of the periodic solution to scalar semi-linear reaction diffusion equations and semi-linear reaction diffusion systems [5–10]. In [11], the authors studied the periodic solution of a scalar evolution p-Laplacian equation with nonlinear sources, and in [12], Wang studied the following degenerate nonlinear reaction diffusion system:

$$\frac{\partial u_i}{\partial t} = \Delta u_i^{m_i} + b_i(t)u_1^{p_i}u_2^{q_i}, \quad (x,t)\in\Omega\times R, \tag{1.4}$$

$$u_i(x,t)=0, \quad (x,t)\in\partial\Omega\times R, \tag{1.5}$$

$$u_i(x,t+\omega)=u_i(x,t), \quad (x,t)\in\overline{\Omega}\times R, \tag{1.6}$$

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* Corresponding author
E-mail address: weiyj@jlu.edu.cn (Y. J. WEI); wjgao@jlu.edu.cn (W. J. GAO)

where $m_i > 1$, $\omega > 0$, $p_i, q_i \geqslant 1$, $b_i(t) > 0$ and $b_i(t+\omega) = b_i(t)$, $i = 1, 2$, $\Omega \subset R^n$ is an open connected bounded domain with smooth boundary $\partial\Omega$.

Motivated by [11] and [12], we study the existence of periodic solutions to (1.1)–(1.3). Since the system is coupled with nonlinear terms, it is in general difficult to study the system. Our treatment is based on global existence [13], regularity to the solutions of a scalar equation [14] and a comparison principle which we will prove in this paper. We mainly use the monotone iteration technique to construct a monotone sequence of solutions and hence obtain the existence of periodic solutions to the system (1.1)–(1.3) by a standard limiting process.

System (1.1) degenerates when $\nabla u_i = 0$. In general, there would be no classical solutions and hence we have to study generalized solutions to Problem (1.1)–(1.3).

In this paper, $C_\omega(\bar{\Omega}_\omega)$ is used to denote the space of continuous functions of (x, t) and of ω-periodic with t. The following are the constrains to the nonlinear functions f_i, $i = 1, 2$ involved in this paper.

Definition 1 *A function $f_i = f_i(u_1, u_2)$ is said to be quasimonotone nondecreasing (resp., nonincreasing) if for fixed u_i, f_i is nondecreasing (resp., nonincreasing) in u_j for $j \neq i$.*

The definition of a periodic solution in this work is the following.

Definition 2 *A nonnegative vector valued function $u = (u_1, u_2)$ is called a generalized solution of the system (1.1)–(1.3), if $u_i \in L^\infty(\Omega_T) \cap L^{p_i}(0, T; W_0^{1,p_i}(\Omega))$, $u_{it} \in L^2(\Omega_T)$, $\forall T > 0$, $i = 1, 2$, and satisfy*

i) $u_i \in C_\omega(\bar{\Omega}_\omega)$, $u_i(x, t) = 0$, $(x, t) \in \partial\Omega \times (0, \omega)$, $i = 1, 2$, where $\Omega_\omega = \Omega \times (0, \omega)$;

ii) For any $\varphi_i \in C^1(\Omega_\omega)$, with $\varphi_i(x, t) = 0$, $(x, t) \in \partial\Omega \times (0, \omega)$, and $\varphi_i(x, 0) = \varphi_i(x, \omega)$,

$$-\iint_{\Omega_\omega} \Big(u_i \frac{\partial \varphi_i}{\partial t} - |\nabla u_i|^{p_i - 2} \nabla u_i \nabla \varphi_i + f_i(t, u_1, u_2)\varphi_i\Big) \mathrm{d}x\mathrm{d}t = 0. \tag{1.7}$$

In the following, we will give the definition of the generalized solution of system (1.1), (1.2) with

$$u_i(x, 0) = u_{i0}(x). \tag{1.8}$$

Definition 3 *A continuous vector valued function $u = (u_1, u_2)$ is called a generalized solution of the system (1.1), (1.2) and (1.8), if*

i) u satisfies boundary condition (1.2), and for any $\tau > 0$, $u_i \in L^\infty(\Omega_\tau) \cap L^{p_i}(0, \tau; W_0^{1,p_i}(\Omega))$, $u_{it} \in L^2(\Omega_\tau)$, $i = 1, 2$, where $\Omega_\tau = \Omega \times (0, \tau)$;

ii) For any $\tau > 0$, and for any nonnegative $\varphi_i \in W^{1,\infty}(\bar{\Omega}_\tau)$, with $\varphi_i(x, t) = 0$, $\partial\Omega \times (0, \tau)$,

$$\begin{aligned} &-\iint_{Q_\tau} \Big(u_i \frac{\partial \varphi_i}{\partial t} - |\nabla u_i|^{p_i - 2} \nabla u_i \nabla \varphi_i + f_i(t, u_1, u_2)\varphi_i\Big) \mathrm{d}x\mathrm{d}t \\ &= \int_\Omega u_{i0}(x)\varphi_i(x, 0)\mathrm{d}x - \int_\Omega u_i(x, \tau)\varphi_i(x, \tau)\mathrm{d}x. \end{aligned} \tag{1.9}$$

If we replace $=$ with $\geqslant$ ($\leqslant$) in above equality, and $u_i(x, t) \geq 0 (u_i(x, t) \leq 0)$, $(x, t) \in \partial\Omega \times (0, \tau)$, $i = 1, 2$, then u is called a supersolution (subsolution) of the system (1.1), (1.2) and (1.8).

Similarly, we define the periodic supersolution and subsolution of (1.1)–(1.3) as follows:

Definition 4 *A continuous vector valued function $u=(u_1,u_2)$ is called a periodic supersolution (subsolution) of the system (1.1)–(1.3), if*

i) $u_i(x,t)\geqslant 0$ $(u_i(x,t)\leqslant 0)$, $(x,t)\in\partial\Omega\times(0,\tau)$, *and* $u_i(x,0)\geqslant(\leqslant)0$ *for* $x\in\Omega$;

ii) $u_i(x,t)\geqslant u_i(x,t+\omega)$ $(u_i(x,t)\leqslant u_i(x,t+\omega))$, $(x,t)\in\Omega_\tau$,

u_i *satisfies (1.9) replacing* $=$ *with* $\geqslant(\leqslant)$, $i=1,2$, *i.e.,*

$$-\iint_{Q_\tau}\Big(u_i\frac{\partial\varphi_i}{\partial t}-|\nabla u_i|^{p_i-2}\nabla u_i\nabla\varphi_i+f_i(t,u_1,u_2)\varphi_i\Big)\mathrm{d}x\mathrm{d}t$$
$$\geqslant(\leqslant)\int_\Omega u_{i0}(x)\varphi_i(x,0)\mathrm{d}x-\int_\Omega u_i(x,\tau)\varphi_i(x,\tau)\mathrm{d}x.$$

2. Main results

Our main existence result is the following:

Theorem 1 *Let* $p_i>2$, m_1, $n_2\geqslant 0$, m_2, $n_1>0$, $(p_1-1-m_1)(p_2-1-n_2)-m_2n_1>0$, f_i *is quasimonotonic and satisfies Lipschitz condition, and there exist nonnegative functions* $c_{i1}(t)$ *and* $c_{i2}(t)$, *s.t.,* $c_{i2}(t)u_1^{m_i}u_2^{n_i}\leqslant f_i(t,u_1,u_2)\leqslant c_{i1}(t)u_1^{m_i}u_2^{n_i}$, $c_{ij}(t)=c_{ij}(t+\omega)$, $i=1,2$, $j=1,2$. *Then there exists a nontrivial nonnegative periodic solution to the problem (1.1)–(1.3).*

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 *Let* $f_i(u_1,u_2)$ *be quasimonotone nondecreasing and satisfy the Lipschitz condition. Let* $\underline{u}=(\underline{u}_1,\underline{u}_2)$ *and* $\overline{u}=(\overline{u}_1,\overline{u}_2)$ *be the subsolution and supersolution of the system (1.1), (1.2) and (1.8) satisfying* $u_0=(\underline{u}_{10},\underline{u}_{20})$ *and* $u_0=(\overline{u}_{10},\overline{u}_{20})$, *respectively, and* $\underline{u}_{i0}\leqslant\overline{u}_{i0}$. *Then* $\underline{u}_i(x,t)\leqslant\overline{u}_i(x,t)$, $i=1,2$.

Proof Since $\underline{u}$ and $\overline{u}$ are the subsolution and supersolution of system (1.1), (1.2) and (1.8), for any $\varphi_i\in W^{1,\infty}(\overline{\Omega}_\tau)$,$\forall\tau\in(0,T)$, with $\varphi_i=0$, for $(x,t)\in\partial\Omega\times(0,\tau)$, we have

$$\int_\Omega\underline{u}_i(x,\tau)\varphi_i(x,\tau)\mathrm{d}x+\iint_{\Omega_\tau}|\nabla\underline{u}_i|^{p_i-2}\nabla\underline{u}_i\nabla\varphi_i\mathrm{d}x\mathrm{d}t$$
$$\leqslant\iint_{\Omega_\tau}(f_i(\underline{u})\varphi_i+\varphi_{it}\underline{u}_i)\mathrm{d}x\mathrm{d}t+\int_\Omega\underline{u}_{i0}(x)\varphi_i(x,0)\mathrm{d}x,$$

and

$$\int_\Omega\overline{u}_i(x,\tau)\varphi_i(x,\tau)\mathrm{d}x+\iint_{\Omega_\tau}|\nabla\overline{u}_i|^{p_i-2}\nabla\overline{u}_i\nabla\varphi_i\mathrm{d}x\mathrm{d}t$$
$$\geqslant\iint_{\Omega_\tau}(f_i(\overline{u})\varphi_i+\varphi_{it}\overline{u}_i)\mathrm{d}x\mathrm{d}t+\int_\Omega\overline{u}_{i0}(x)\varphi_i(x,0)\mathrm{d}x.$$

Taking $\varphi_i=(\underline{u}_i-\overline{u}_i)^+$ as a test function, where $a^+=\max(0,a)\geqslant 0$, subtracting the two inequalities, we get

$$\frac{1}{2}\int_\Omega((\underline{u}_i(x,\tau)-\overline{u}_i(x,\tau))^+)^2\mathrm{d}x$$
$$=-\iint_{\Omega_\tau}(|\nabla\underline{u}_i|^{p_i-2}\nabla\underline{u}_i-|\nabla\overline{u}_i|^{p_i-2}\nabla\overline{u}_i)\nabla((\underline{u}_i-\overline{u}_i)^+)\mathrm{d}x\mathrm{d}t+$$

$$
\begin{aligned}
&\iint_{\Omega_\tau}(f_i(\underline{u})-f_i(\overline{u}))(\underline{u}_i-\overline{u}_i)^+\mathrm{d}x\mathrm{d}t\\
=&-\iint_{\Omega_\tau\bigcap\{\underline{u}_i>\overline{u}_i\}}(|\nabla\underline{u}_i|^{p_i-2}\nabla\underline{u}_i-|\nabla\overline{u}_i|^{p_i-2}\nabla\overline{u}_i)\nabla(\underline{u}_i-\overline{u}_i)\mathrm{d}x\mathrm{d}t+\\
&\iint_{\Omega_\tau}(f_i(\underline{u})-f_i(\overline{u}))(\underline{u}_i-\overline{u}_i)^+\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Notice that

$$(|\nabla\underline{u}_i|^{p_i-2}\nabla\underline{u}_i-|\nabla\overline{u}_i|^{p_i-2}\nabla\overline{u}_i)\nabla(\underline{u}_i-\overline{u}_i)\geqslant 0.$$

In view of the above inequality and the Lipschitz condition, by simple calculation, we obtain that

$$
\begin{aligned}
&\int_\Omega(|(\underline{u}_1-\overline{u}_1)^+|^2+|(\underline{u}_2-\overline{u}_2)^+|^2)\mathrm{d}x\\
&\leqslant 2K\iint_{\Omega_\tau}(|(\underline{u}_1-\overline{u}_1)^+|+|(\underline{u}_2-\overline{u}_2)^+|)^2\mathrm{d}x\mathrm{d}t\\
&\leqslant 4K\int_0^\tau\int_\Omega(|(\underline{u}_1-\overline{u}_1)^+|^2+|(\underline{u}_2-\overline{u}_2)^+|^2)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Setting $F(\tau)=\int_0^\tau\int_\Omega(|(\underline{u}_1-\overline{u}_1)^+|^2+|(\underline{u}_2-\overline{u}_2)^+|^2)\mathrm{d}x\mathrm{d}t$, then the above inequality can be written as

$$F'(\tau)\leqslant 4KF(\tau).$$

A standard argument shows that $F(\tau)\equiv 0$ since $F(0)\equiv 0$, hence $(\underline{u}_i-\overline{u}_i)^+=0$, i.e., $\underline{u}_i\leqslant\overline{u}_i$.

Lemma 2 ([14]) *Let u be a solution of the homogeneous Dirichlet problem to the equation*

$$\frac{\partial u}{\partial t}=\mathrm{div}(|\nabla u|^{p-2}\nabla u)+f(x,t),$$

where $f\in L^\infty(\Omega\times(0,\tau))$. Then there exist an $\alpha>0$ and a constant K depending only on $\tau'\in(0,\tau)$ and the upper-bound of $\|f\|_{L^\infty(\Omega\times(0,\tau))}$, s.t.,

$$|u(x_1,t_1)-u(x_2,t_2)|\leqslant K(|x_1-x_2|^\alpha+|t_1-t_2|^{\frac{\alpha}{2}}),\quad (x_i,t_i)\in\overline{\Omega}\times[\tau',\tau].$$

Lemma 3 *Under the assumptions of Theorem 1, there exist a nontrivial subsolution and a supersolution to the problem (1.1), (1.2) and (1.8).*

Proof Motivated by [15], we use the eigenfunction to construct the subsolution and the supersolution of system (1.1), (1.2) and (1.8).

Let μ_1 and λ_1 be the first eigenvalue of the following eigenvalue problem.

$$-\mathrm{div}(|\nabla\psi_1|^{p_1-2}\nabla\psi_1)=\mu_1\psi_1^{p_1-1},\ x\in\Omega,\psi_1=0,x\in\partial\Omega, \tag{2.1}$$

$$-\mathrm{div}(|\nabla\phi_1|^{p_2-2}\nabla\phi_1)=\lambda_1\phi_1^{p_2-1},\ x\in\Omega,\phi_1=0,x\in\partial\Omega, \tag{2.2}$$

where ψ_1 and ϕ_1 are the corresponding eigenfunctions, satisfying $\psi_1(x)>0,\phi_1(x)>0,x\in\Omega$, $|\nabla\psi_1|>0,|\nabla\phi_1|>0,x\in\partial\Omega,i=1,2$. Without loss of generality, let $\|\psi_1\|_{p_1}=\|\phi_1\|_{p_2}=1$. Since $(p_1-1-m_1)(p_2-1-n_2)-m_2n_1>0$, we can choose k, s.t.,

$$\frac{m_2}{p_2-1-n_2}<k<\frac{p_1-1-m_1}{n_1}. \tag{2.3}$$

We now prove that $(\underline{u}_1, \underline{u}_2) = (a\psi_1^m(x), a^k\phi_1^n(x))$ is a subsolution to Problem (1.1), (1.2) and (1.8), where $m = \frac{p_1}{p_1-1}, n = \dfrac{p_2}{p_2-1}$, and $a > 0$ is small number to be specified later.

Let $\varphi_1(x,t) \in C^1(\overline{\Omega}_\tau), \varphi_1(x,t) \geqslant 0$, be a test function. Then it follows from (2.1) that

$$\begin{aligned}
&\iint_{\Omega_\tau} (\underline{u}_1 \frac{\partial \varphi_1}{\partial t} + \mathrm{div}(|\nabla \underline{u}_1|^{p_1-2}\nabla \underline{u}_1)\varphi_1 + f_1(t, \underline{u}_1, \underline{u}_2)\varphi_1)\mathrm{d}x\mathrm{d}t + \\
&\quad \int_\Omega \underline{u}_1(x,0)\varphi_1(x,0)\mathrm{d}x - \int_\Omega \underline{u}_1(x,\tau)\varphi_1(x,\tau)\mathrm{d}x \\
&= \iint_{\Omega_\tau} (f_1(t,\underline{u}_1,\underline{u}_2) + \mathrm{div}(|\nabla \underline{u}_1|^{p_1-2}\nabla \underline{u}_1))\varphi_1 \mathrm{d}x\mathrm{d}t \\
&\geqslant \int_0^\tau \int_\Omega \min_{(0,\omega)} c_{12}(t)\underline{u}_1^{m_1}\underline{u}_2^{n_1}\varphi_1\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega (am)^{p_1-1}(\mu_1\psi_1^{p_1} - |\nabla\psi_1|^{p_1})\varphi_1\mathrm{d}x\mathrm{d}t. \quad (2.4)
\end{aligned}$$

Similarly, for all $\varphi_2(x,t) \in C^1(\overline{\Omega}_\tau), \varphi_2(x,t) \geqslant 0$, following (2.2), we have

$$\begin{aligned}
&\iint_{\Omega_\tau} (\underline{u}_2 \frac{\partial \varphi_2}{\partial t} + \mathrm{div}(|\nabla \underline{u}_2|^{p_2-2}\nabla \underline{u}_2)\varphi_2 + f_2(t, \underline{u}_1, \underline{u}_2)\varphi_2)\mathrm{d}x\mathrm{d}t + \\
&\quad \int_\Omega \underline{u}_2(x,0)\varphi_2(x,0)\mathrm{d}x - \int_\Omega \underline{u}_2(x,\tau)\varphi_2(x,\tau)\mathrm{d}x \\
&= \iint_{\Omega_\tau} (f_2(t,\underline{u}_1,\underline{u}_2) + \mathrm{div}(|\nabla \underline{u}_2|^{p_2-2}\nabla \underline{u}_2))\varphi_2 \mathrm{d}x\mathrm{d}t \\
&\geqslant \int_0^\tau \int_\Omega \min_{(0,\omega)} c_{22}(t)\underline{u}_1^{m_2}\underline{u}_2^{n_2}\varphi_2\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega (a^k n)^{p_2-1}(\lambda_1\phi_1^{p_2} - |\nabla\phi_1|^{p_2})\varphi_2\mathrm{d}x\mathrm{d}t. \quad (2.5)
\end{aligned}$$

We need to prove that the right hand side of (2.4) and (2.5) are nonnegative.

Since $\psi_1 = 0$, $\phi_1 = 0$, $|\nabla\psi_1| > 0$, $|\nabla\phi_1| > 0$, $x \in \partial\Omega$, there exists an $\eta > 0$, s.t.,

$$\mu_1\psi_1^{p_1} - |\nabla\psi_1|^{p_1} \leqslant 0, \quad \lambda_1\phi_1^{p_2} - |\nabla\phi_1|^{p_2} \leqslant 0, \quad x \in \overline{\Omega}_\eta, \quad (2.6)$$

where $\overline{\Omega}_\eta = \{x \in \Omega | \mathrm{dist}(x, \partial\Omega) \leqslant \eta\}$. This shows that

$$\int_0^\tau \int_{\overline{\Omega}_\eta} (am)^{p_1-1}(\mu_1\psi_1^{p_1} - |\nabla\psi_1|^{p_1})\varphi_1\mathrm{d}x\mathrm{d}t \leqslant 0 \leqslant \int_0^\tau \int_{\overline{\Omega}_\eta} \min_{(0,\omega)} c_{12}(t)\underline{u}_1^{m_1}\underline{u}_2^{n_1}\varphi_1\mathrm{d}x\mathrm{d}t, \quad (2.7)$$

and

$$\int_0^\tau \int_{\overline{\Omega}_\eta} (a^k n)^{p_2-1}(\lambda_1\phi_1^{p_2} - |\nabla\phi_1|^{p_2})\varphi_2\mathrm{d}x\mathrm{d}t \leqslant 0 \leqslant \int_0^\tau \int_{\overline{\Omega}_\eta} \min_{(0,\omega)} c_{22}(t)\underline{u}_1^{m_2}\underline{u}_2^{n_2}\varphi_2\mathrm{d}x\mathrm{d}t. \quad (2.8)$$

(2.7) and (2.8) show that $(\underline{u}_1, \underline{u}_2)$ is a subsolution on $\overline{\Omega}_\eta \times (0, +\infty)$. Furthermore, we note that $\psi_1(x), \phi_1(x) \geqslant \mu > 0$ for some $\mu > 0$ in $\Omega_0 = \Omega \setminus \overline{\Omega}_\eta$. Then from (2.3) there exists an $a_0 > 0$, s.t., if $a \in (0, a_0)$, the following inequalities hold:

$$a^{k(p_2-1-n_2)-m_2}\lambda_1 n^{p_2-1}\phi_1^{p_2-nn_2} \leqslant \min_{(0,\omega)} c_{12}(t)\mu^{mm_2} \leqslant \min_{(0,\omega)} c_{12}(t)\psi_1^{mm_2}, \quad x \in \Omega_0, \quad (2.9)$$

$$a^{p_1-1-m_1-kn_1}\mu_1 m^{p_1-1}\psi_1^{p_1-mm_1} \leqslant \min_{(0,\omega)} c_{22}(t)\mu^{nn_1} \leqslant \min_{(0,\omega)} c_{22}(t)\phi_1^{nn_1}, \quad x \in \Omega_0. \quad (2.10)$$

(2.9) and (2.10) show that

$$\int_0^\tau \int_{\Omega_0} |\nabla \underline{u}_1|^{p_1-2}\nabla \underline{u}_1 \nabla\varphi_1\mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\Omega_0} (am)^{p_1-1}(\mu_1\psi_1^{p_1} - |\nabla\psi_1|^{p_1})\varphi_1\mathrm{d}x\mathrm{d}t$$

$$\leqslant \int_0^\tau \int_{\Omega_0} \min_{(0,\omega)} c_{12}(t) \underline{u}_1^{m_1} \underline{u}_2^{n_1} \varphi_1 \mathrm{d}x\mathrm{d}t, \tag{2.11}$$

and

$$\int_0^\tau \int_{\Omega_0} |\nabla \underline{u}_2|^{p_2-2} \nabla \underline{u}_2 \nabla \varphi_2 \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\Omega_0} (a^k n)^{p_2-1} (\lambda_1 \phi_1^{p_2} - |\nabla \phi_1|^{p_2}) \varphi_2 \mathrm{d}x\mathrm{d}t$$

$$\leqslant \int_0^\tau \int_{\Omega_0} \min_{(0,\omega)} c_{22}(t) \underline{u}_1^{m_2} \underline{u}_2^{n_2} \varphi_2 \mathrm{d}x\mathrm{d}t. \tag{2.12}$$

Therefore $(\underline{u}_1, \underline{u}_2) = (a\psi_1^m(x), a^k \phi_1^n(x))$ is a subsolution of (1.1), (1.2) and (1.8).

We now construct a supersolution $(\overline{u}_1, \overline{u}_2)$ of (1.1), (1.2) and (1.8). Let $w_1(x), w_2(x)$ be the positive solutions of the following problems, respectively.

$$-\mathrm{div}(|\nabla w_1|^{p_1-2} \nabla w_1) = 1, \ x \in \Omega, w_1 = 0, x \in \partial\Omega, \tag{2.13}$$

$$-\mathrm{div}(|\nabla w_2|^{p_2-2} \nabla w_2) = 1, \ x \in \Omega, w_2 = 0, x \in \partial\Omega. \tag{2.14}$$

Let

$$\overline{u}_1 = A w_1(x), \quad \overline{u}_2 = B w_2(x), \tag{2.15}$$

where the constants A, $B > 0$ are large and to be chosen later. We shall verify that $(\overline{u}_1, \overline{u}_2)$ is a supersolution of (1.1), (1.2) and (1.8). Let $\varphi_i \in C^1(\overline{\Omega}_\tau), \varphi_i \geqslant 0$, be test functions, $i = 1, 2$. Then from (2.13), (2.14), we obtain that

$$\iint_{\Omega_\tau} (\overline{u}_1 \frac{\partial \varphi_1}{\partial t} + \mathrm{div}(|\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1) \varphi_1 + f_1(t, \overline{u}_1, \overline{u}_2) \varphi_1) \mathrm{d}x\mathrm{d}t +$$

$$\int_\Omega \overline{u}_1(x,0) \varphi_1(x,0) \mathrm{d}x - \int_\Omega \overline{u}_1(x,\tau) \varphi_1(x,\tau) \mathrm{d}x$$

$$= \iint_{\Omega_\tau} (f_1(t, \overline{u}_1, \overline{u}_2) + \mathrm{div}(|\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1)) \varphi_1 \mathrm{d}x\mathrm{d}t$$

$$\leqslant \int_0^\tau \int_\Omega \max_{(0,\omega)} c_{11}(t) \overline{u}_1^{m_1} \overline{u}_2^{n_1} \varphi_1 \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega A^{p_1-1} \varphi_1 \mathrm{d}x\mathrm{d}t, \tag{2.16}$$

and

$$\iint_{\Omega_\tau} (\overline{u}_2 \frac{\partial \varphi_2}{\partial t} + \mathrm{div}(|\nabla \overline{u}_2|^{p_2-2} \nabla \overline{u}_2) \varphi_2 + f_2(t, \overline{u}_1, \overline{u}_2) \varphi_2) \mathrm{d}x\mathrm{d}t +$$

$$\int_\Omega \overline{u}_2(x,0) \varphi_2(x,0) \mathrm{d}x - \int_\Omega \overline{u}_2(x,\tau) \varphi_2(x,\tau) \mathrm{d}x$$

$$= \iint_{\Omega_\tau} (f_2(t, \overline{u}_1, \overline{u}_2) + \mathrm{div}(|\nabla \overline{u}_2|^{p_2-2} \nabla \overline{u}_2)) \varphi_2 \mathrm{d}x\mathrm{d}t$$

$$\leqslant \int_0^\tau \int_\Omega \max_{(0,\omega)} c_{21}(t) \overline{u}_1^{m_2} \overline{u}_2^{n_2} \varphi_2 \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega B^{p_2-1} \varphi_2 \mathrm{d}x\mathrm{d}t. \tag{2.17}$$

We need to prove that the right hand side of (2.16) and (2.17) are nonpositive. Let $l = \|w_1\|_\infty$, $L = \|w_2\|_\infty$, $C = \max\{\max_{(0,\omega)} c_{11}(t), \max_{(0,\omega)} c_{21}(t)\}$. Since $\theta > 0$, it is easy to prove that there exist positive large constants A, B, s.t.,

$$A^{p_1-1-m_1} = C B^{n_1} l^{m_1} L^{n_1}, \quad B^{p_2-1-n_2} = C A^{m_2} l^{m_2} L^{n_2}. \tag{2.18}$$

Therefore

$$A^{p_1-1} \geqslant C\overline{u}_1^{m_1}\overline{u}_2^{n_1} \geqslant \max_{(0,\omega)} c_{11}(t)\overline{u}_1^{m_1}\overline{u}_2^{n_1}, \tag{2.19}$$

$$B^{p_2-1} \geqslant C\overline{u}_1^{m_2}\overline{u}_2^{n_2} \geqslant \max_{(0,\omega)} c_{21}(t)\overline{u}_1^{m_2}\overline{u}_2^{n_2}. \tag{2.20}$$

These imply that the right hand side of (2.16) and (2.17) are nonpositive. Therefore, $(\overline{u}_1, \overline{u}_2)$ is a supersolution of (1.1), (1.2) and (1.8). We can choose large A, B such that $\underline{u}_i \leqslant \overline{u}_i$, $i = 1, 2$.

3. The proof of main results

Definition 3 (Poincaré Mapping) *Set* $T = (T_1, T_2)$: $C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$, $T(u_{10}(x), u_{20}(x)) = (u_1(x,\omega), u_2(x,\omega))$, *where* $u(x,t) = (u_1(x,t), u_2(x,t))$ *is the solution of the initial-boundary value problem*

$$\frac{\partial u_i}{\partial t} = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i) + f_i(t, u_1, u_2), \quad (x,t) \in \Omega \times (0, +\infty), \tag{3.1}$$

$$u_i(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0, +\infty), \tag{3.2}$$

$$u_i(x, 0) = u_{i0}, \quad x \in \overline{\Omega}. \tag{3.3}$$

The definition is reasonable due to the existence and uniqueness of the system (1.1), (1.2) and (1.8) in [13].

In the following, we will prove Theorem 1.

Proof Set $u_0 = \underline{u}$. By Lemma 1 and the fact that $\underline{u}$ is the subsolution of system (1.1), we get that $u_i(x,\omega) = T_i\underline{u}(x) \geqslant \underline{u}_i(x)$, $i = 1, 2$. Repeating the process, we can obtain a sequence $\{T^n\underline{u}\}_{n=1}^{\infty}$, where $T^1 = T$, $T^{n+1}\underline{u} = T(T^n\underline{u})$. By Lemma 1 and $T_i\underline{u} \geqslant \underline{u}_i$, we know that $\{T^n\underline{u}\}_{n=1}^{\infty}$ is nondecreasing. Similarly, we can obtain a nonincreasing sequence $\{T^n\overline{u}\}_{n=1}^{\infty}$.

Following Lemma 1, we know that $T_i\underline{u}(x) \leqslant T_i\overline{u}(x)$. Therefore

$$\underline{u}_i(x) \leqslant T_i\underline{u}(x) \leqslant \cdots \leqslant T_i^n\underline{u}(x) \leqslant T_i^n\overline{u}(x) \leqslant \cdots \leqslant T_i\overline{u}(x) \leqslant \overline{u}_i(x), \quad i = 1, 2. \tag{3.4}$$

Let $u_n(x,t)$ be the solution of system (1.1), (1.2) and (1.8) with $u_{i0} = T^{n-1}\underline{u}$. We get $T^n\underline{u}(x) = u_n(x,\omega)$. By Lemma 1, $u_{in}(x,t) \leqslant \overline{u}_i(x)$, $i = 1, 2$. So there exists a constant C_0 independent of n, s.t.,

$$f_i(t, u_{1n}, u_{2n}) \leqslant C_0, \quad i = 1, 2. \tag{3.5}$$

Following above inequality and Lemma 2, there exist an $\alpha > 0$ and a constant K depending only on $\omega > 0$, such that

$$|u_{in}(x_1, t_1) - u_{in}(x_2, t_2)| \leqslant K(|x_1 - x_2|^{\alpha} + |t_1 - t_2|^{\frac{\alpha}{2}}), (x_i, t_i) \in \overline{\Omega} \times [\frac{\omega}{2}, \omega]. \tag{3.6}$$

Particularly,

$$|T_i^n\underline{u}(x_1) - T_i^n\underline{u}(x_2)| \leqslant K|x_1 - x_2|^{\alpha}, \quad x_i \in \overline{\Omega}.$$

Due to Ascoli-Arzelá Theorem, there exist a function $v_0 \in C(\overline{\Omega}) \times C(\overline{\Omega})$ and a subsequence of $\{T^n\underline{u}\}_{n=1}^{\infty}$, without loss of generality, denoted again by $\{T^n\underline{u}\}_{n=1}^{\infty}$, s.t.,

$$T^n\underline{u} \to v_0, \quad \text{uniformly in } \ C(\overline{\Omega}) \times C(\overline{\Omega}), \quad n \to \infty. \tag{3.7}$$

We will prove that the solution to the initial boundary problem (1.1), (1.2) and (1.8) with $u(x,0)=v_0$ is a solution of Problem (1.1)–(1.3).

Considering initial and boundary problem (1.1), (1.2) with

$$u_i(x,0)=T_i^n\underline{u}(x). \tag{3.8}$$

Since $\overline{u}(x)$ is a supersolution of (1.1) and $T_i^n\underline{u}(x)\leqslant\overline{u}_i(x)$, we have

$$u_{in}(x,t)\leqslant\overline{u}_i(x),\quad (x,t)\in\overline{\Omega}\times(0,+\infty). \tag{3.9}$$

Following above inequality and Lemma 2, we obtain that there exists a positive constant K depending only on ω and a $\beta>0$, s.t., $(x_i,t_i)\in\overline{\Omega}\times[\omega,2\omega]$,

$$|u_{in}(x_1,t_1)-u_{in}(x_2,t_2)|\leqslant K(|x_1-x_2|^{\beta}+|t_1-t_2|^{\frac{\beta}{2}}). \tag{3.10}$$

Following the proof of the global existence in [13], we know that there exists a positive constant C_0 independent of n, s.t.,

$$|\nabla u_{in}|_{L^{p_i}(\Omega\times(\omega,2\omega))}\leqslant C_0, \tag{3.11}$$

$$|u_{int}|_{L^2(\Omega\times(\omega,2\omega))}\leqslant C_0. \tag{3.12}$$

Due to (3.7)–(3.9), there exist functions $w_i(x,t)\in C(\overline{\Omega}\times(\omega,2\omega))$ and a subsequence of $\{u_{in}\}_{n=1}^{\infty}$, without loss of generality, denoted again by $\{u_{in}\}_{n=1}^{\infty}$, s.t.,

$$u_{in}\to w_i,\qquad \text{in}\quad C(\overline{\Omega}\times[\omega,2\omega]), \tag{3.13}$$

$$\nabla u_{in}\rightharpoonup\nabla w_i,\qquad \text{in}\quad L^{p_i}(\Omega\times(\omega,2\omega)), \tag{3.14}$$

$$u_{int}\rightharpoonup w_{it},\qquad \text{in}\quad L^2(\Omega\times(\omega,2\omega)), \tag{3.15}$$

$$|\nabla u_{in}|^{p_i-2}u_{inx_l}\rightharpoonup w_{ix_l},\quad \text{in}\quad L^{\frac{p_i}{p_i-1}}(\Omega\times[\omega,2\omega]), \tag{3.16}$$

where $\rightharpoonup$ stands for weak convergence, $i=1,2$. Following (3.4), (3.10)–(3.13), we get that $v_{i0}(x)=w_i(x,\omega)$.

By the definition of generalized solutions and (3.10)–(3.13), we obtain

$$\begin{aligned}&\iint_{\Omega'_\omega}(w_i\frac{\partial\varphi_i}{\partial t}-|\nabla w_i|^{p_i-2}\nabla w_i\nabla\varphi_i+f_i(t,w_1,w_2)\varphi_i)\mathrm{d}x\mathrm{d}t\\&=\int_\Omega w_i(x,2\omega)\varphi_i(x,2\omega)\mathrm{d}x-\int_\Omega w_i(x,\omega)\varphi_i(x,\omega)\mathrm{d}x,\quad i=1,2,\end{aligned}$$

where $\Omega'_\omega=\Omega\times(\omega,2\omega)$. It shows that function $w_i(x,t)$ is a solution of (1.1), on Ω'_ω. On the other hand, following (3.10) and the definition of the map T, we get

$$\begin{aligned}w(x,2\omega)&=\lim_{n\to\infty}u_n(x,2\omega)=\lim_{n\to\infty}T(u_n(x,\omega))(x)\\&=\lim_{n\to\infty}T(T(T^n\underline{u}))(x)=\lim_{n\to\infty}T^{n+2}\underline{u}(x)\\&=\lim_{n\to\infty}T^{n+1}\underline{u}(x)=\lim_{n\to\infty}T(T^n\underline{u})(x)\\&=\lim_{n\to\infty}u_n(x,\omega)=w(x,\omega).\end{aligned}$$

By the uniqueness of the solution to the initial and boundary problem, we know that $u(x,t)=w(x,t+\omega)$, $t\in[0,\omega]$. Therefore, $u(x,0)=w(x,\omega)=w(x,2\omega)=u(x,\omega)$. The proof is completed. □

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