Journal of Mathematical Research & Exposition
May, 2010, Vol. 30, No. 3, pp. 407–414
DOI:10.3770/j.issn:1000-341X.2010.03.004
Http://jmre.dlut.edu.cn

Extensions of Generalized Fitting Modules

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Abstract In this paper, we study the closeness of strongly (co)-hopfian properties under some constructions such as the ring of Morita context, direct products, triangular matrix, fraction ring etc. Also, we prove that if $M[X]$ is strongly hopfian (resp. strongly co-hopfian) in $R[X]$-Mod, then M is strongly hopfian (resp. strongly co-hopfian) in R-Mod.

Keywords Generalized Fitting modules; strongly hopfian modules; strongly co-hopfian modules.

Document code A
MR(2000) Subject Classification 16D50; 16E40; 16E50
Chinese Library Classification O153.3

1. Introduction

Throughout this paper, the rings considered will be associative with unit and all modules considered are left unitary modules. A module M is said to be strongly hopfian (resp. strongly co-hopfian) [1], if for every endomorphism f of M the ascending (resp. descending) chain $\mathrm{Ker} f \subseteq \mathrm{Ker} f^2 \subseteq \cdots \subseteq \mathrm{Ker} f^n \subseteq \cdots$ (resp. $\mathrm{Im} f \supseteq \mathrm{Im} f^2 \supseteq \cdots \supseteq \mathrm{Im} f^n \supseteq \cdots$) stabilizes. A strongly hopfian or strongly co-hopfian module will be called a Generalized Fitting module. A ring R is called left strongly hopfian (resp. strongly co-hopfian) if ${}_RR$ is strongly hopfian (resp. strongly co-hopfian). It was proved in [1] that if M is quasi-injective (resp. quasi-projective) strongly hopfian (resp. strongly co-hopfian), then M is strongly co-hopfian (resp. strongly hopfian), and every strongly co-hopfian ring R is strongly hopfian. From [1], we know that the class of strongly hopfian (resp. strongly co-hopfian) modules is situated properly between the class of Noetherian (resp. Artinian) and the class of hopfian (resp. co-hopfian) modules, so it seems a natural question to study the properties of strongly hopfian (resp. strongly co-hopfian) modules which are similar to the ones of Noetherian (resp. Artinian) modules and hopfian (resp. co-hopfian) modules, and this is the main goal of this paper. By [1], neither submodules nor quotients of strongly hopfian (resp. strongly co-hopfian) modules need to be strongly hopfian (resp. strongly co-hopfian), but if M is strongly hopfian (resp. strongly co-hopfian) and N is a direct summand of M, then N and M/N are both strongly hopfian (resp. strongly co-hopfian). Also if R is a commutative strongly hopfian ring, then the polynomial ring $R[x]$ is strongly hopfian [1]. In this paper we continue the study of Generalized Fitting modules and rings.

Supported by National Natural Science Foundation of China (Grant No. 10961021).
Received October 16, 2007; Accepted April 16, 2008
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For the ring of Morita context and its modules, the notions of hopficity and co-hopficity were studied in [2]. It was proved that if the ring of Morita context $T = (A, B, M, N, (-,-), [-,-])$ is hopfian, then A and B are hopfian. Furthermore $A \oplus M$ and $N \oplus B$ are hopfian objects in T-Mod. Also if A and B are hopfian and the pairings are zero maps, then T is hopfian [2, Proposition 2.3]. Motivated by these results, in Section 2, we show that (1) If ${}_TT$ is strongly hopfian (resp. strongly co-hopfian), then ${}_AA$ and ${}_BB$ are strongly hopfian (resp. strongly co-hopfian), moreover $A \oplus M$ and $N \oplus B$ are strongly hopfian (resp. strongly co-hopfian) objects in T-Mod. (2) Let T be the triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$, and let $X \in T$-Mod. If e_1X is strongly hopfian (resp. strongly co-hopfian) in A-Mod and e_2X is strongly hopfian (resp. strongly co-hopfian) in B-Mod, then X is strongly hopfian (resp. strongly co-hopfian) in T-Mod. Moreover a similar result holds for right T-Mod Y and corresponding right modules Ye_1 and Ye_2 over A and B, respectively.

Let R and S be hopfian rings. How to determine whether $R \times S$ is hopfian as a ring is a basic question in [3], and it was proved that if R and S are hopfian rings and the only central idempotents in S are 0 and 1_S and that S is not a homomorphic image of R, then $R \times S$ is a hopfian ring. In Section 3, we show that if R and S are strongly hopfian rings, then $R \times S$ is a strongly hopfian ring. Also in [4, Theorem 2.1] Varadarajan showed that M is hopfian in R-mod if and only if $M[X]$ is hopfian in $R[X]$-mod if and only if $M[[X]]$ is hopfian in $R[[X]]$-mod. At the end of this paper, we study whether this is true for strongly hopfian (resp. strongly co-hopfian) modules.

2. Strong hopficity and strong co-hopficity for the ring of morita contexts

A Morita context denoted by $T = (A, B, M, N, (-,-), [-,-])$ consists of two rings A, B, two bimodules ${}_AN_B$, ${}_BM_A$ and a pair of bimodule homomorphisms

$$(-,-) : N \otimes_B M \longrightarrow A; \quad [-,-] : M \otimes_A N \longrightarrow B$$

which satisfy the following associativity conditions:

$$(v, w)v' = v[w, v'], \quad [w, v]w' = w(v, w').$$

These conditions will insure that the set T of generalized matrices

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}, \quad a \in A, b \in B, n \in N, m \in M$$

will form a ring, called the ring of the context.

We start with a known result needed in the following.

Lemma 2.1 ([1, Proposition 2.9]) *For a ring R, we have:*

(1) R is left strongly hopfian if and only if for every $a \in R$ there exists $n \in N$ such that $l(a^n) = l(a^{n+1})$.

(2) R is left strongly co-hopfian if and only if for every $a \in R$ there exist $n \in N$ and $b \in R$

such that $a^n = ba^{n+1}$ (if and only if R is right strongly co-hopfian, if and only if R is strongly π-regular).

It is true that if ${}_TT$ is hopfian, then ${}_AA$ and ${}_BB$ are hopfian, furthermore $A \oplus M$ and $N \oplus B$ are hopfian objects in T-Mod [2, Proposition 2.1]. Replacing "hopfian" with "strongly hopfian" or "strongly co-hopfian", we have

Proposition 2.2 *If ${}_TT$ is strongly hopfian (resp. strongly co-hopfian), then ${}_AA$ and ${}_BB$ are strongly hopfian (resp. strongly co-hopfian). Moreover $A \oplus M$ and $N \oplus B$ are strongly hopfian (resp. strongly co-hopfian) objects in T-Mod.*

Proof Let $a \in A$. Then $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in T$. Since T is left strongly hopfian, there exists an integer $m \geq 1$ such that $l_T\left[\begin{pmatrix} a^m & 0 \\ 0 & 0 \end{pmatrix}\right] = l_T\left[\begin{pmatrix} a^{m+1} & 0 \\ 0 & 0 \end{pmatrix}\right]$. Take $t \in l(a^{m+1})$, then $\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \in l_T\begin{pmatrix} a^{m+1} & 0 \\ 0 & 0 \end{pmatrix} = l_T\begin{pmatrix} a^m & 0 \\ 0 & 0 \end{pmatrix}$. So $t \in l(a^m)$ which implies that $l(a^{m+1}) = l(a^m)$. Thus ${}_AA$ is strongly hopfian.

Suppose T is left strongly co-hopfian. By Lemma 2.1(2), there exists an integer $m \geq 1$ such that $\begin{pmatrix} a^m & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\begin{pmatrix} a^{m+1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1a^{m+1} & 0 \\ b_3a^{m+1} & 0 \end{pmatrix}$ for some $\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in T$. So $a^m = b_1a^{m+1}$ which implies that A is left strongly co-hopfian. By using a similar argument, we have that ${}_BB$ is strongly hopfian (resp. strongly co-hopfian). Since $A \oplus M$ and $N \oplus B$ are direct summands of T, the final statement follows from [1, Proposition 2.17]. $\square$

It follows from [2, Proposition 2.3] that if ${}_AA$, ${}_BB$ are hopfian and the pairings are zero maps, then ${}_TT$ is hopfian. We do not know whether this is true for strongly hopfian rings, but we have the following proposition.

Proposition 2.3 *Let A be a strongly hopfian ring. If $l(b) = 0$ for any $0 \neq b \in B$ and the pairings are zero maps, then $T = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ is strongly hopfian.*

Proof If $b = 0$, put $X = \begin{pmatrix} a & n \\ 0 & 0 \end{pmatrix} \in T$. We easily see that $X^k = \begin{pmatrix} a^k & a^{k-1}n \\ 0 & 0 \end{pmatrix}$ for any $k \geq 2$. By assumption, there exists an integer $m \geq 1$ such that $l(a^m) = l(a^{m+1})$. Take any $\begin{pmatrix} x_1 & t \\ 0 & x_2 \end{pmatrix} \in l_T(X^{m+2})$, then $x_1a^{m+2} = 0$ and it follows that $x_1a^m = x_1a^{m+1} = 0$. Hence $\begin{pmatrix} x_1 & t \\ 0 & x_2 \end{pmatrix} X^{m+1} = \begin{pmatrix} x_1a^{m+1} & x_1a^mn \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which implies $\begin{pmatrix} x_1 & t \\ 0 & x_2 \end{pmatrix} \in l_T(X^{m+1})$.

If $b \neq 0$, put $\begin{pmatrix} a & n \\ 0 & b \end{pmatrix} \in T$. Then $a \in A, b \in B$. By assumption, there exists an integer

$m \geq 1$ such that $l(a^m) = l(a^{m+1})$ and $l(b) = 0$. Take any $\begin{pmatrix} x_1 & t \\ 0 & x_2 \end{pmatrix} \in l_T \left[\begin{pmatrix} a & n \\ 0 & b \end{pmatrix}^{m+1} \right]$ $= l_T \begin{pmatrix} a^{m+1} & a^m n + a^{m-1}nb + \cdots + anb^{m-1} + nb^m \\ 0 & b^{m+1} \end{pmatrix}$, then $\begin{pmatrix} x_1 & t \\ 0 & x_2 \end{pmatrix} \left[\begin{pmatrix} a & n \\ 0 & b \end{pmatrix}^{m+1} \right] =$ $\begin{pmatrix} x_1 a^{m+1} & x_1 a^m n + x_1 a^{m-1} nb + \cdots + x_1 nb^m + tb^{m+1} \\ 0 & x_2 b^{m+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $x_1 \in l(a^{m+1}) =$ $l(a^m)$, $x_2 \in l(b^{m+1}) = 0$ and $x_1 a^{m-1} nb + \cdots + x_1 nb^m + tb^{m+1} = (x_1 a^{m-1} n + \cdots + x_1 nb^{m-1} +$ $tb^m)b = 0$. So $x_1 a^{m-1} n + \cdots + x_1 nb^{m-1} + tb^m = 0$. Hence $\begin{pmatrix} x_1 & t \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & n \\ 0 & b \end{pmatrix}^m =$ $\begin{pmatrix} x_1 a^m & x_1 a^{m-1} n + x_1 a^{m-2} nb + \cdots + tb^m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} x_1 & t \\ 0 & 0 \end{pmatrix} \in l_T \left[\begin{pmatrix} a & n \\ 0 & b \end{pmatrix}^m \right]$. It follows that $l_T \left[\begin{pmatrix} a & n \\ 0 & b \end{pmatrix}^m \right] = l_T \left[\begin{pmatrix} a & n \\ 0 & b \end{pmatrix}^{m+1} \right]$ and we have that ${}_T T$ is strongly hopfian.□

Given a ring R and a bimodule ${}_R M_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

Corollary 2.4 *If for any $0 \neq s \in S$, $l(s) = 0$, then the trivial extension $R = T(S, M) \cong \{\begin{pmatrix} s & m \\ 0 & s \end{pmatrix} | s \in S, m \in M\}$ is left strongly hopfian.*

Corollary 2.5 *If for any $0 \neq a \in A$, $0 \neq b \in B$, $l(a) = 0 = l(b)$ and the pairings are zero maps, then $T = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ are left strongly hopfian.*

Recall that from Lam [5] a ring R is called M_n-unique if a ring isomorphism $Mat_{n\times n}(R) \cong Mat_{n\times n}(S)$ implies that $R \cong S$. It follows from [2, Theorem 3.1] that if R is M_n-unique and R is hopfian (resp. co-hopfian) as a ring, then so is the matrix ring $Mat_{n\times n}(R)$. We do not know whether this is true for strongly hopfian (resp. strongly co-hopfian) rings, but we have

Proposition 2.6 *Let R be a ring and $n \geq 1$ an integer. If $M_n(R)$ is left strongly hopfian (resp. strongly co-hopfian) as a ring, then R is left strongly hopfian (resp. strongly co-hopfian).*

Proof Let $a \in R$. Then $\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$. Since $M_n(R)$ is left strongly hopfian, there exists an integer $m \geq 1$ such that $l \left[\begin{pmatrix} a^m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \right] = l \left[\begin{pmatrix} a^{m+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \right]$.

Take $b \in l(a^{m+1})$, then

$$\begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \left[\begin{pmatrix} a^{m+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right] = l\left[\begin{pmatrix} a^{m} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right].$$

So $b \in l(a^m)$ which implies that $l(a^{m+1}) = l(a^m)$. Thus R is left strongly hopfian.

Suppose $M_n(R)$ is left strongly co-hopfian. By Lemma 2.1 (2), there exists an integer $m \geq 1$ such that $\begin{pmatrix} a^{m} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} a^{m+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ $= \begin{pmatrix} b_{11}a^{m+1} & 0 & \cdots & 0 \\ b_{21}a^{m+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}a^{m+1} & 0 & \cdots & 0 \end{pmatrix}$ for some $\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ n_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \in M_n(R)$. So $a^m = b_{11}a^{m+1}$, this shows that R is left strongly co-hopfian. □

For the rest of this section we consider triangular matrix rings, so assume $N = 0$. Let X be a left T-module and Y be a right T-module. Put $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $e_1Te_1 = e_1T$, $e_2Te_2 = Te_2$ are rings, $A \cong e_1T$ and $B \cong Te_2$. Hence we can view e_1X as a left A-module and Ye_2 as a right B-module. From [2, Theorem 4.3], we know that if e_1X is hopfian (resp. co-hopfian) in A-Mod and e_2X is hopfian (resp. co-hopfian) in B-Mod , then X is hopfian (resp. co-hopfian) in T-Mod. In the following, we show that this is true for strongly hopfian (resp. strongly co-hopfian) modules.

Theorem 2.7 *Let T be the triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$, and let $X \in T$-Mod. If e_1X is strongly hopfian (resp. strongly co-hopfian) in A-Mod and e_2X is strongly hopfian (resp. strongly co-hopfian) in B-Mod , then X is strongly hopfian (resp. strongly co-hopfian) in T-Mod. Moreover a similar result holds for right T-Mod Y and corresponding right modules Ye_1 and Ye_2 over A and B, respectively.*

Proof Let $\alpha \in \mathrm{End}_T(X)$. Then $\alpha_1 \in \mathrm{End}_A(e_1X)$, $\alpha_2 \in \mathrm{End}_B(e_2X)$ where α_1, α_2 are the restrictions of α to e_1X and e_2X, respectively. By assumption, there exist integers $m \geq 1$ and $n \geq 1$ such that $\mathrm{Ker}\alpha_1^m = \mathrm{Ker}\alpha_1^{m+1}$ and $\mathrm{Ker}\alpha_2^n = \mathrm{Ker}\alpha_2^{n+1}$. Let $K = \max\{m, n\}$. Then $\mathrm{Ker}\alpha_1^k = \mathrm{Ker}\alpha_1^{k+1}$ and $\mathrm{Ker}\alpha_2^k = Ker\alpha_2^{k+1}$. But $e_1 + e_2 = 1_T$, so for any $h \in \mathrm{Ker}\alpha^{k+1}$ we have $0 = \alpha^{k+1}(h) = \alpha^{k+1}(e_1h + e_2h) = \alpha_1^{k+1}(e_1h) + \alpha_2^{k+1}(e_2h) = e_1\alpha_1^{k+1}(h) + e_2\alpha_2^{k+1}(h)$. Since $e_2e_1 = 0$, we get $0 = e_2^2\alpha_2^{k+1}(h) = \alpha_2^{k+1}(e_2h)$. So $e_2h \in \mathrm{Ker}\alpha_2^{k+1} = \mathrm{Ker}\alpha_2^k$. Similarly, $e_1h \in \mathrm{Ker}\alpha_1^{k+1} = \mathrm{Ker}\alpha_1^k$. Thus $\alpha^k(h) = \alpha^k(e_1h + e_2h) = \alpha_1^k(e_1h) + \alpha_2^k(e_2h) = 0$, $h \in \mathrm{Ker}\alpha^k$ which implies that $\mathrm{Ker}\alpha^k = \mathrm{Ker}\alpha^{k+1}$. So X is strongly hopfian in T-Mod.

Suppose e_1X is strongly co-hopfian in A-Mod and e_2X is strongly co-hopfian in B-Mod. By assumption, there exists an integer $k \geq 1$ such that $\mathrm{Im}\alpha_1^k = \mathrm{Im}\alpha_1^{k+1}$ and $\mathrm{Im}\alpha_2^k = \mathrm{Im}\alpha_2^{k+1}$. For any $h \in \mathrm{Im}\alpha^k$, there exists $t \in X$ such that $\alpha^k(t) = h$. But $e_1 + e_2 = 1_T$, so $h = \alpha^k(t) = \alpha^k(e_1t + e_2t) = \alpha_1^k(e_1t) + \alpha_2^k(e_2t) = e_1\alpha_1^k(t) + e_2\alpha_2^k(t)$. Since $e_1e_2 = 0$, $e_1h = e_1^2\alpha_1^k(t) = \alpha_1^k(e_1t) \in \mathrm{Im}\alpha_1^k = \mathrm{Im}\alpha_1^{k+1}$. Similarly, $e_2h = \alpha_2^k(e_2t) \in \mathrm{Im}\alpha_2^k = \mathrm{Im}\alpha_2^{k+1}$. Thus there exist $e_1b \in e_1X$, $e_2c \in e_2X$ such that $\alpha_1^{k+1}(e_1b) = e_1h$ and $\alpha_2^{k+1}(e_1c) = e_2h$. So $h = (e_1 + e_2)h = \alpha_1^{k+1}(e_1b) + \alpha_2^{k+1}(e_2c) = \alpha^{k+1}(e_1b + e_2c) \in \mathrm{Im}\alpha^{k+1}$. Hence $\mathrm{Im}\alpha^k = \mathrm{Im}\alpha^{k+1}$, which shows that X is strongly co-hopfian in T-Mod. □

3. Strong hopficity and strong co-hopficity of $S^{-1}R$, $R \times S$ and $M[X]$

In this section, a multiplicative set in a ring R means a subset $S \subseteq R$ such that S is closed under multiplication, $0 \overline{\in} S$ and $1 \in S$. Let S be a multiplicative subset of a commutative ring R. The relation defined in the set $R \times S$ by

$$(r, s) \sim (r', s') \Longleftrightarrow urs' = ur's \quad \text{for some} \quad u \in S$$

is an equivalence relation. The equivalence class of $(r, s) \in R \times S$ will be denoted by r/s, the set of all equivalence classes of $R \times S$ under $\sim$ is denoted by $S^{-1}R = (R \times S)/\sim = \{r/s | r \in R, s \in S\}$. Then $S^{-1}R$ is a commutative ring with identity $1/1$, where addition and multiplication are defined by

$$r/s + r'/s' = (rs' + r's)/ss' \quad (r/s)(r'/s') = rr'/ss'.$$

We call $S^{-1}R$ the fraction ring of R by S.

Proposition 3.1 *Let S be a multiplicative set in a commutative ring R, M be a torsionfree module. If $S^{-1}M$ is strongly hopfian (resp. strongly co-hopfian), then M is strongly hopfian (resp. strongly co-hopfian).*

Proof Let $f : M \longrightarrow M$ be any endomorphism in R-Mod. Then $S^{-1}f : S^{-1}M \longrightarrow S^{-1}M$ defined by $S^{-1}f(m/s) = f(m)/s$ is an endomorphism in $S^{-1}R$-Mod, also $(S^{-1}f)^n : S^{-1}M \longrightarrow S^{-1}M$ defined by $(S^{-1}f)^n(m/s) = f^n(m)/s$ is an endomorphism in $S^{-1}R$-Mod. Since $S^{-1}M$ is strongly hopfian, there exists an integer $k \geq 1$ such that $\mathrm{Ker}(S^{-1}f)^k = Ker(S^{-1}f)^{k+1}$. For any $x \in \mathrm{Ker}f^{k+1}$, $x/1 \in S^{-1}M$ and $f^{k+1}(x) = 0$. So $(S^{-1}f)^{k+1}(x/1) = f^{k+1}(x)/1 = 0/1$, $x/1 \in \mathrm{Ker}(S^{-1}f)^{k+1} = Ker(S^{-1}f)^k$. Thus $(S^{-1}f)^k(x/1) = f^k(x)/1 = 0/1$, and there exists $u \in S$ such that $uf^k(x) = 0$. Since M is torsionfree, $f^k(x) = 0$ and $x \in \mathrm{Ker}f^k$ which implies that $\mathrm{Ker}f^k = \mathrm{Ker}f^{k+1}$. Hence M is strongly hopfian. Suppose $S^{-1}M$ is strongly co-hopfian. By Lemma 2.1 (2), there exists an integer $k \geq 1$ such that $\mathrm{Im}(S^{-1}f)^k = \mathrm{Im}(S^{-1}f)^{k+1}$. For any $x \in \mathrm{Im}f^k$, there exists $y \in M$ such that $f^k(y) = x$. So $(S^{-1}f)^k(y/1) = x/1$, $x/1 \in \mathrm{Im}(S^{-1}f)^k = \mathrm{Im}(S^{-1}f)^{k+1}$. We have $t/1 \in S^{-1}M$ such that $(S^{-1}f)^{k+1}(t/1) = f^{k+1}(t)/1 = x/1$. Thus there exists $u \in S$ such that $uf^{k+1}(t) = ux$, $u(f^{k+1}(t) - x) = 0$. Since M is torsionfree, $f^{k+1}(t) - x = 0$ and $x = f^{k+1}(t) \in \mathrm{Im}f^{k+1}$ which implies that $\mathrm{Im}f^k = \mathrm{Im}f^{k+1}$. Hence M is strongly co-hopfian. □

Corollary 3.2 *Let S be a multiplicative set in a commutative ring R such that S does not* *contain divisor of R. If $S^{-1}R$ is strongly hopfian (resp. strongly co-hopfian), then R is strongly hopfian (resp. strongly co-hopfian).*

Theorem 3.3 *Let S be a multiplicative set in a commutative ring R. If R is strongly hopfian, then $S^{-1}R$ is strongly hopfian.*

Proof Let $a/s \in S^{-1}R$. Then $a \in R$. By assumption, there exists an integer $k \geq 1$ such that $l(a^k) = l(a^{k+1})$. Clearly, $l_{S^{-1}R}(a^k/s^k) \subseteq l_{S^{-1}R}(a^{k+1}/s^{k+1})$. For any $b/t \in l_{S^{-1}R}(a^{k+1}/s^{k+1})$, $(b/t)(a^{k+1}/s^{k+1}) = ba^{k+1}/ts^{k+1} = 0/1$. Thus there exists $u \in S$ such that $uba^{k+1} = 0$. So $ub \in l(a^{k+1}) = l(a^k)$, $uba^k = 0$. Hence $ba^k/ts^k = 0/1$, $b/t \in l_{S^{-1}R}(a^k/s^k)$ which implies that $l_{S^{-1}R}(a^k/s^k) = l_{S^{-1}R}(a^{k+1}/s^{k+1})$. This shows that $S^{-1}R$ is strongly hopfian. $\Box$

Let R and S be any two rings. It was proved in [3] that if $R \times S$ is hopfian as a ring, then R and S are hopfian rings. Conversely, if R and S are hopfian rings, and the only central idempotents in S are 0 and 1_S and that S is not a homomorphic image of R, then $R \times S$ is a hopfian ring. It is easily seen that if $R \times S$ is strongly hopfian as a ring, then R and S are strongly hopfian rings. Conversely, we have

Theorem 3.4 *Let R and S be left strongly hopfian rings. Then $R \times S$ is a left strongly hopfain ring.*

Proof Let $(r, s) \in R \times S$. Then $r \in R$, $s \in S$. Since R and S are left strongly hopfian, there exist integer $m \geq 1$ and $n \geq 1$ such that $l(r^m) = l(r^{m+1})$ and $l(s^n) = l(s^{n+1})$. Let $k = \max\{m, n\}$. Then $l((r, s)^k) = l((r, s)^{k+1})$. By Lemma 2.1, $R \times S$ is left strongly hopfian. $\Box$

We conclude this paper with the following results.

Proposition 3.5 *Let R be a commutative strongly hopfian ring. Then $R[x]/(x^{n+1})$ is strongly hopfian.*

Proof Let $u = \overline{x} \in R[x]/(x^{n+1})$. Then $R[x]/(x^{n+1}) = R + Ru + \cdots + Ru^n$ with $u^{n+1} = 0$. For any $P(u) \in R[x]/(x^{n+1})$, put $P(u) = a_0 + a_1u + \cdots + a_nu$ where $a_i \in R$. Since R is strongly hopfian, there exists an integer $m \geq 1$ such that $l(a_i^{m+1}) = l(a_i^m)$ for all $i \in \{0, 1, \ldots, n\}$. By [1, Lemma 5.2] we get $l(P(u)^{(n+1)m+1}) = l(P(u)^{(n+1)m})$, so $R[x]/(x^{n+1})$ is strongly hopfian. $\Box$

Proposition 3.6 *Let $n_1, n_2, \ldots, n_k$ be non-negative integer and $x_1, x_2, \ldots, x_k$ be k commuting indeterminates over R. If R is also a commutative strongly hopfian ring, then $R[x_1, x_2, \ldots, x_k]/(x_1^{n_1+1}, x_2^{n_2+1}, \ldots, x_k^{n_k+1})$ is strongly hopfian.*

Proof Since

$$\begin{aligned}&(R[x_1, x_2, \ldots, x_{k-1}]/(x_1^{n_1+1}, x_2^{n_2+1}, \ldots, x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1})\\ &\quad\cong R[x_1, x_2, \ldots, x_k]/(x_1^{n_1+1}, x_2^{n_2+1}, \ldots, x_k^{n_k+1}),\end{aligned}$$

the result follows from Proposition 3.1 and the induction. $\Box$

It is well known that M is hopfian in R-mod if and only if $M[X]$ is hopfian in $R[X]$-mod (see [4, Theorem 2.1]). For strongly hopfian (resp. strongly co-hopfian) modules, we have

Proposition 3.7 *If $M[X]$ is strongly hopfian (resp. strongly co-hopfian) in $R[X]$-Mod, then M is strongly hopfian (resp. strongly co-hopfian) in R-Mod.*

Proof Let $f : M \longrightarrow M$ be any endomorphism in R-Mod. Then $f[X] : M[X] \longrightarrow M[X]$ defined by $f[X](\sum_{j=0}^{k} a_j X^j) = \sum_{j=0}^{k} f(a_j)X^j$ is a endomorphism in $R[X]$-Mod, also $f[X]^m : M[X] \longrightarrow M[X]$ defined by $f[X]^m(\sum_{j=0}^{k} a_j X^j) = \sum_{j=0}^{k} f^m(a_j)X^j$ is a endomorphism in $R[X]$-Mod. Since $M[X]$ is strongly hopfian, there exists an integer $m \geq 1$ such that $\mathrm{Ker} f[X]^m = \mathrm{Ker} f[X]^{m+1}$. Take $b \in \mathrm{Ker} f^{m+1}$, then $\sum_{j=0}^{k} f^{m+1}(b)X^j = 0$. So $\sum_{j=0}^{k} bX^j \in \mathrm{Ker} f[X]^{m+1} = \mathrm{Ker} f[X]^m$, $f[X]^m(\sum_{j=0}^{k} bX^j) = \sum_{j=0}^{k} f^m(b)X^j = 0$. Thus $f^m(b) = 0$, $b \in \mathrm{Ker} f^m$ which implies that $\mathrm{Ker} f^m = \mathrm{Ker} f^{m+1}$. Hence M is strongly hopfian.

Suppose now that $M[X]$ is strongly co-hopfian in $R[X]$-Mod. By Lemma 2.1 (2), there exists an integer $m \geq 1$ such that $\mathrm{Im} f[X]^m = \mathrm{Im} f[X]^{m+1}$. Take $b \in \mathrm{Im} f^m$, then there exists $t \in M$ such that $f^m(t) = h$, so $\sum_{j=0}^{k} hX^j = \sum_{j=0}^{k} f^m(t)X^j = f[X]^m(\sum_{j=0}^{k} tX^j) \in \mathrm{Im} f[X]^m = \mathrm{Im} f[X]^{m+1}$. Thus there exists $\sum_{j=0}^{k} p_j X^j \in M[X]$ such that $f[X]^{m+1}(\sum_{j=0}^{k} p_j X^j) = \sum_{j=0}^{k} f^{m+1}(p_j)X^j = \sum_{j=0}^{k} hX^j$, so $h = f^{m+1}(p_j) \in \mathrm{Im} f^{m+1}$ which implies that $\mathrm{Im} f^m = \mathrm{Im} f^{m+1}$. Hence M is strongly co-hopfian. $\square$

Corollary 3.8 *If $R[X]$ is strongly hopfian, then so is R.*

Remark Kerr (see [7]) constructed a commutative Goldie ring R for which $R[X]$ does not have the ACC on annihilator ideals.

Question (1) If M is strongly hopfian (resp. strongly co-hopfian), is $M[X]$ strongly hopfian (resp.strongly co-hopfian)?

(2) If R is strongly hopfian, is it true that $R[[x]]$ is strongly hopfian?

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