Journal of Mathematical Research & Exposition
May, 2010, Vol. 30, No. 3, pp. 451–456
DOI:10.3770/j.issn:1000-341X.2010.03.009
Http://jmre.dlut.edu.cn

General Induced Matching Extendability of G^3

Long Shu WU[1], **Yun Sheng YAN**[2], **Qin WANG**[1,*]
1. *Department of Mathematics, China Jiliang University, Zhejiang* 310018*, P. R. China;*
2. *College of Science, Henan University of Technology, Henan* 450052*, P. R. China*

Abstract A graph G is induced matching extendable if every induced matching of G is included in a perfect matching of G. A graph G is generalized induced matching extendable if every induced matching of G is included in a maximum matching of G. A graph G is claw-free, if G dose not contain any induced subgraph isomorphic to $K_{1,3}$. The k-th power of G, denoted by G^k, is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if the distance between them is at most k in G. In this paper we show that, if the maximum matchings of G and G^3 have the same cardinality, then G^3 is generalized induced matching extendable. We also show that this result is best possible. As a result, we show that if G is a connected claw-free graph, then G^3 is generalized induced matching extendable.

Keywords near perfect matching; induced matching extendable; general induced matching extendability; power of graph.

Document code A
MR(2000) Subject Classification 05C05; 05C50
Chinese Library Classification O157.5

1. Introduction

The graphs considered in this paper are finite and simple. For a graph G, $V(G)$ and $E(G)$ denote, respectively, its vertex set and its edge set. For two vertex subsets X and Y in G, the distance between them, denoted by $d_G(X,Y)$, is the minimum length of a path connecting X and Y. $d_G(\{x\},\{y\})$ is written in shorter form as $d_G(x,y)$ for $x,y \in V(G)$. A component H of G is odd (even) if $|V(H)|$ is odd (even). The component number of G is denoted by $c(G)$ and the odd component number of G is denoted by $o(G)$. A graph G is claw-free, if G dose not contain any induced subgraph isomorphic to $K_{1,3}$. The k-th power of G, denoted by G^k, is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they have distance at most k in G. For two graphs G and H, $G \cup H$ is used to denote the union of them. The join $G \vee H$ of disjoint graphs G and H is the graph obtained from the disjoint union $G \cup H$ by joining each vertex of G to each vertex of H. For $X \subseteq V(G)$, the neighbor set $N_G(X)$ of X is defined by

$$N_G(X) = \{y \in V(G) \setminus X: \text{ there is } x \in X \text{ such that } xy \in E(G)\}.$$

Received December 1, 2008; Accepted September 18, 2009
Supported by the National Natural Science Foundation of China (Grant Nos. 10601051; 90818020) and the Natural Science Foundation of Zhejiang Province (Grant No. Y6090472).
* Corresponding author
E-mail address: wq@cjlu.edu.cn (Q. WANG)

$N_G(\{x\})$ is written in shorter form as $N_G(x)$ for $x \in V(G)$. For $S \subseteq V(G)$, set

$$E(S) = \{uv \in E(G) : u, v \in S\}.$$

If $I \subseteq V(G)$ such that $E(I) = \emptyset$, I is called an independent set of G. For $M \subseteq E(G)$, set

$$V(M) = \{v \in V(G) : \text{ there is } x \in V(G) \text{ such that } vx \in M\}.$$

$V(\{e\})$ is written in shorter form as $V(e)$ for $e \in E(G)$. $M \subseteq E(G)$ is a matching of G, if $V(e) \cap V(f) = \emptyset$ for every two distinct edges $e, f \in M$. A matching M of G is a maximum matching, if $|M| \geq |M'|$ for every matching M' of G. A matching M of G is a perfect matching, if $V(M) = V(G)$. A matching M of G is a near perfect matching, if $|V(M)| = |V(G)| - 1$. A matching M of G is induced [1, 2], if $E(V(M)) = M$. A graph G is induced matching extendable [3] (shortly, IM-extendable), if every induced matching of G is included in a perfect matching of G. A graph G is strongly IM-extendable, if every spanning supergraph of G is IM-extendable. A graph G is nearly induced matching extendable (shortly, nearly IM-extendable), if $G \vee K_1$ is induced matching extendable. A graph G is strongly nearly IM-extendable, if every spanning supergraph of G is nearly IM-extendable. A graph G is generalized induced matching extendable (shortly, generalized IM-extendable), if every induced matching of G is included in a maximum matching of G. A graph G is strongly generalized IM-extendable, if every spanning supergraph of G is generalized IM-extendable. The following is the famous Tutte's Theorem.

Tutte's Theorem ([4, 5]) *A graph G has a perfect matching if and only if for every $S \subset V(G)$, $o(G - S) \leq |S|$.*

Yuan proved in [6] that, for a connected graph G with $|V(G)|$ even, G^4 is strongly IM-extendable. Qian proved in [7] that, for a 2-connected graph G with $|V(G)|$ even, G^3 is strongly IM-extendable, and for a locally connected graph G with $|V(G)|$ even, G^2 is strongly IM-extendable. It was shown in [8] that, for a connected graph G with a perfect matching, G^3 is IM-extendable. In [9], it was shown that, if G is a graph with $|V(G)|$ even and without independent vertex cut, then G^2 is strongly IM-extendable. These results solved three conjectures posed in [10].

We further study the IM-extendability of the 3-power of graphs. We show in this paper that, if the maximum matchings of G and G^3 have the same cardinality, then G^3 is generalized induced matching extendable. We also show that this result is best possible. As a result, we show that if G is a connected graph and has a near perfect matching, then G^3 is nearly IM-extendable. We also show that if G is connected and claw-free, then G^3 is generalized induced matching extendable.

2. Main results and proof

The following Lemma was shown in [8].

Lemma 1 ([8]) *Suppose that G is a connected graph with $|V(G)| \geq 3$. If $I \subset V(G)$ such that $|I| \leq 2$ and $|I|$ has same parity as $|V(G)|$, then $G^3 - I$ has a perfect matching.*

Corollary 1 *Suppose that G is a connected graph with $|V(G)| \geq 3$. If $I \subset V(G)$ such that $1 \leq |I| \leq 2$ and $|I|$ has opposing parity as $|V(G)|$, then $G^3 - I$ has a near perfect matching.*

Proof Arbitrarily select a vertex $u \in I$. Then $|I \setminus \{u\}|$ has the same parity as $|V(G)|$. Write $I' = I \setminus \{u\}$. From Lemma 1, we know that $G^3 - I'$ has a perfect matching M. Suppose that $uv \in M$. Then $M \setminus \{uv\}$ is a near perfect matching of $G^3 - I$. The result follows. □

Corollary 2 *Suppose that G is a connected graph with $|V(G)| \geq 3$ and $|V(G)|$ odd. Then G^3 has a near perfect matching.*

Proof For an edge $uv \in E(G)$, from Corollary 1, we know that $G^3 - \{u, v\}$ has a near perfect matching M. Then $M \cup \{uv\}$ is a near perfect matching of G^3. The result follows. □

Theorem 1 *If the maximum matchings of G and G^3 have the same cardinality, then G^3 is generalized induced matching extendable.*

Proof Let M be a maximum matching of G. It is clear that $G - V(M)$ is an independent set of G. Let N be an induced matching of G^3. For $e = xy \in N$, let P_e be a shortest (x, y)-path in G. Set

$$E_e = E(P_e) \cup \{f \in M : \text{there is } u \in V(P_e) \text{ such that } f \text{ is incident to } u \text{ in } M\}.$$

Let H_e be the edge induced subgraph of G induced by E_e. Then H_e is connected. From the fact that $d_G(x, y) \leq 3$, it is easy to see that for every edge $zw \in M \cap E(H_e)$,

$$d_G(\{x, y\}, \{z, w\}) \leq 1.$$

Suppose that $e = xy$ and $f = uv$ are two distinct edges in N such that $V(H_e) \cap V(H_f) \neq \emptyset$. Let z be a vertex in $V(H_e) \cap V(H_f)$. Let zw be the edge such that $zw \in M$. By the definition of H_e and H_f, we know that $zw \in E(H_e) \cap E(H_f)$. Because $d_G(\{x, y\}, \{z, w\}) \leq 1$ and $d_G(\{u, v\}, \{z, w\}) \leq 1$, we must have $d_G(\{x, y\}, \{u, v\}) \leq 3$. This contradicts the fact that N is an induced matching in G^3. So we must have for every two distinct edges $e = xy$ and $f = uv$ in N, $V(H_e) \cap V(H_f) = \emptyset$.

Now we distinguish the following two cases.

Case 1 $V(H_e) \subseteq V(M)$. In this case, $E_e \cap M$ is a perfect matching of H_e. Then $|V(H_e)|$ must be even. By Lemma 1, $(H_e)^3 - \{x, y\}$ has a perfect matching. Now for each edge $e = xy \in N$ with $|V(H_e)|$ even, $V(H_e) \subseteq V(M)$ and $|V(H_e)| \geq 4$, let M_e be a perfect matching in $(H_e)^3 - \{x, y\}$. For $e = xy \in N$ with $|V(H_e)| = 2$, we know that $V(H_e) = \{x, y\} \subseteq V(M)$ and we define $M_e = \emptyset$.

Case 2 H_e contains a vertex $u \in G - V(M)$. Then $|V(H_e)|$ must be odd. Note that H_e can only contain one such vertex u. Otherwise, suppose there is a vertex $v \in G - V(M)$, $v \neq u$ and $v \in V(H_e)$. We must have $u, v \in V(P_e)$ and $uv \in E(G^3)$, a contradiction to the fact that the maximum matchings of G and G^3 have the same cardinality. We have the following two subcases.

Case 2.1 $u \in V(P_e) \setminus \{x, y\}$. Then by Corollary 1, $(H_e)^3 - \{x, y\}$ has a near perfect matching. Let M_e be a near perfect matching in $(H_e)^3 - \{x, y\}$.

Case 2.2 $u = x$ or $u = y$. Without loss of generality, suppose that $u = x$. Then $(H_e)^3 - u$ has a perfect matching. Let $M_e{}'$ be a perfect matching in $(H_e)^3 - u$ and suppose that $e' = yt \in M_e{}'$. Let $M_e = M_e{}' - e'$.

It can be seen that $(M \setminus (\cup_{e \in N} E(H_e))) \cup (\cup_{e \in N} M_e) \cup N$ is a maximum matching in G^3.

This completes the proof. $\square$

From Theorem 1, we can easily have

Theorem 2 *If G is a connected graph and has a near perfect matching, then G^3 is nearly IM-extendable.*

Lemma 2 ([11]) *If G is a connected claw-free graph with even number of vertices, then G has a perfect matching.*

Theorem 3 *If G is a connected claw-free graph, then G^3 is generalized induced matching extendable.*

Proof We distinguish the following two cases.

Case 1 $V(G)$ is even. By Lemma 2, G has a perfect matching and so G^3 also has a perfect matching. By Theorem 1, G^3 is generalized induced matching extendable.

Case 2 $V(G)$ is odd. We have the following two subcases.

Case 2.1 G is 2-connected. Then G has no cut vertex. For any vertex $u \in V(G)$, $G - u$ is connected and $|V(G - u)|$ is even. It is easy to see that $G - u$ is also claw-free. By Lemma 2, $G - u$ has a perfect matching, and so G has a near perfect matching. By Theorem 2, G^3 is nearly IM-extendable.

Case 2.2 G has a cut vertex v. From the fact that G is claw-free, we know that $G - v$ has exactly two components G_1 and G_2. If both $|V(G_1)|$ and $|V(G_2)|$ are even, from Lemma 2, G_1 and G_2 have perfect matchings M_1 and M_2, respectively. So $M_1 \cup M_2$ is a near perfect matching of G. By Theorem 2, G^3 is nearly IM-extendable. If both $|V(G_1)|$ and $|V(G_2)|$ are odd, then $G_2 + v$ has a perfect matching N_2. Repeat the above analysis on G_1, we will finally deduce that G_1 has a near perfect matching N_1. So $N_1 \cup N_2$ is a near perfect matching of G. By Theorem 2, G^3 is nearly IM-extendable.

This completes the proof. $\square$

3. Examples

Our result in Theorem 2 is best possible in three aspects. Firstly, there is a k-connected $(k \geq 2)$ graph G having a near perfect matching such that G^2 is not nearly IM-extendable. This will be shown in Example 1. Secondly, there is a connected graph H with $|V(H)|$ odd such that H^3 is not nearly IM-extendable. This will be shown in Example 2. Thirdly, there is a connected graph D having a near perfect matching such that D^3 is not strongly nearly IM-extendable. This will be shown in Example 3. Note that the nearly IM-extendable graph is a special case of the generalized IM-extendable graph, we can also use these three Examples to explain the best possibility of the result in Theorem 1.

Example 1 Let $k \geq 2$ be an integer. Let G_1, G_2, G_3 and G_4 be four complete graphs with $|V(G_1)| = |V(G_2)| - 1 = |V(G_3)| = |V(G_4)|$ and such that $|V(G_1)| \geq k^2$ and $|V(G_1)|$ is odd. Let $(V_1, V_2, \ldots, V_k)$ be a k-partition of $V(G_1)$, $(U_1, U_2, \ldots, U_k)$ be a k-partition of $V(G_2)$, $(R_1, R_2, \ldots, R_k)$ be a k-partition of $V(G_3)$ and $(S_1, S_2, \ldots, S_k)$ be a k-partition of $V(G_4)$ such that $|V_i|, |U_i|, |R_i|, |S_i| \geq k$ for $1 \leq i \leq k$. Let M be the set of $2k$ edges with $M = \{v_i {u^1}_i, {r^2}_i s_i : 1 \leq i \leq k\}$ and let M_1 be the set of k edges with $M_1 = \{{u^2}_i {r^1}_i : 1 \leq i \leq k\}$, where $v_i, {u^1}_i, {u^2}_i, {r^1}_i, {r^2}_i, s_i \notin V(G_1) \cup V(G_2) \cup V(G_3) \cup V(G_4)$ for $1 \leq i \leq k$. The graph G is constructed as follows.

$$V(G) = V(G_1) \cup V(G_2) \cup V(G_3) \cup V(G_4) \cup V(M) \cup V(M_1),$$

$$E(G) = E(G_1) \cup E(G_2) \cup E(G_3) \cup E(G_4) \cup M \cup M_1 \cup$$

$$(\cup_{1 \leq i \leq k} \{v_i v : v \in V_i\}) \cup (\cup_{1 \leq i \leq k} \{{u^1}_i u : u \in U_i\}) \cup (\cup_{1 \leq i \leq k} \{{u^2}_i u : u \in U_i\}) \cup$$

$$(\cup_{1 \leq i \leq k} \{{r^1}_i r : r \in R_i\}) \cup (\cup_{1 \leq i \leq k} \{{r^2}_i r : r \in R_i\}) \cup (\cup_{1 \leq i \leq k} \{s_i s : s \in S_i\}).$$

M is an induced matching of G. It is easy to check that G is a k-connected graph and has a near perfect matching. Now, M is still an induced matching in G^2. But $G^2 - V(M)$ has three odd components. Hence, G^2 is not nearly IM-extendable.

Example 2 Let $P = x_1x_2x_3x_4x_5$, $Q = y_1y_2y_3y_4y_5$, $R = z_1z_2z_3z_4z_5$ and $S = w_1w_2w_3w_4w_5$ be four 5-pathes. Let v be a vertex which is different from x_i, y_i, z_i, w_i, $1 \leq i \leq 5$. The graph H is constructed as follows.

$$V(H) = \{v\} \cup V(P) \cup V(Q) \cup V(R) \cup V(S),$$

$$E(H) = E(P) \cup E(Q) \cup E(R) \cup E(S) \cup \{vx_3, vy_3, vz_3, vw_3\}.$$

Let $M = \{x_2x_4, y_2y_4, z_2z_4, w_2w_4\}$. It is easy to see that M is an induced matching of H^3. For a vertex $u \notin V(H)$, $\{x_1, x_5, y_1, y_5, z_1, z_5, w_1, w_5\}$ is an independent set in H^3 and $H^3 \vee u$. This means that $H^3 \vee u - V(M) - \{u, v, x_3, y_3, z_3, w_3\}$ has eight odd components. By Tutte's Theorem, $H^3 \vee u - V(M)$ has no perfect matching. Hence, H^3 is not nearly IM-extendable.

Example 3 Let $P = x_1x_2x_3x_4x_5x_6x_7x_8$, $Q = y_1y_2y_3y_4y_5y_6y_7y_8$, $R = z_1z_2z_3z_4z_5z_6z_7z_8$ and $S = w_1w_2w_3w_4w_5w_6w_7w_8$ be four 8-pathes. Let v be a vertex which is different from x_i, y_i, z_i, w_i, $1 \leq i \leq 8$. The graph D is constructed as follows.

$$V(D) = \{v\} \cup V(P) \cup V(Q) \cup V(R) \cup V(S),$$

$$E(D) = E(P) \cup E(Q) \cup E(R) \cup E(S) \cup \{vx_3, vy_3, vz_3, vw_3\}.$$

Let $M = \{x_2x_4, y_2y_4, z_2z_4, w_2w_4, x_8y_8, z_8w_8\}$. For a vertex $u \notin V(D)$, it is easy to see that M is an induced matching of $D^3 + x_8y_8 + z_8w_8$ and $(D^3 + x_8y_8 + z_8w_8) \vee u$. But $(D^3 + x_8y_8 + z_8w_8) \vee u - V(M) - \{u, v, x_3, y_3, z_3, w_3\}$ has eight odd components. By Tutte's Theorem, $(D^3 + x_8y_8 + z_8w_8) \vee u - V(M)$ has no perfect matching. Hence $D^3 + x_8y_8 + z_8w_8$ is not nearly IM-extendable, and so, D^3 is not strongly nearly IM-extendable.

References

[1] CAMERON K. *Induced matchings* [J]. Discrete Appl. Math., 1989, **24**(1-3): 97–102.

[2] FAUDREE R J, GYÁRFÁS A, SCHELP R H. et al. *Induced matchings in bipartite graphs* [J]. Discrete Math., 1989, **78**(1-2): 83–87.

[3] YUAN Jinjiang. *Induced matching extendable graphs* [J]. J. Graph Theory, 1998, **28**(4): 203–213.

[4] BONDY J A, MURTY U S R. *Graph theory with applications.* [M]. American Elsevier Publishing Co., Inc., New York, 1976.

[5] TUTTE W T. *The factorization of linear graphs* [J]. J. London Math. Soc., 1947, **22**: 107–111.

[6] YUAN Jinjiang. *Independent-set-deletable factor-critical power graphs* [J]. Acta Math. Sci. Ser. B Engl. Ed., 2006, **26**(4): 577–584.

[7] QIAN Jianguo. *Induced matching extendable graph power* [J]. Graphs Combin., 2006, **22**(3): 391–398.

[8] YUAN Jinjiang, WANG Qin. *Induced matching extendability of* G^3 [J]. Graph Theory Notes N. Y., 2002, **43**: 16–19.

[9] YUAN Jinjiang, WANG Qin, QIAN Jianguo. *IM-extendability of the square of the graphs without independent vertex cut* [J]. J. Xinjiang Univ., 2006, **23**(suppl.): 31–35. (in Chinese)

[10] YUAN Jinjiang. *Induced matching extendable graphs - A survey* [C]. Invited Report of the 6th Conference on Operations Research of China. (Changsha), 2000.

[11] LOVÁSZ L, PLUMMER M D. *Matching Theory* [M]. North-Holland Publishing Co., Amsterdam, 1986.