*Journal of Mathematical Research & Exposition*
May, 2010, Vol. 30, No. 3, pp. 475–486
DOI:10.3770/j.issn:1000-341X.2010.03.012
Http://jmre.dlut.edu.cn

# Triple Positive Solutions of the Multi-Point Boundary Value Problem for Second-Order Differential Equations

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**Abstract** We consider the second-order differential equation

$$u''(t)+q(t)f(t,u(t),u'(t))=0,\quad 0<t<1,$$

subject to three-point boundry condition

$$u(0)=0,\quad u(1)=a_0u(\xi_0),$$

or to $m$-point boundary condition

$$u'(0)=\sum_{i=1}^{m-2}b_iu'(\xi_i),\quad u(1)=\sum_{i=1}^{m-2}a_iu(\xi_i).$$

We show the existence of at least three positive solutions of the above multi-point boundary-value problem by applying a new fixed-point theorem introduced by Avery and Peterson.

**Keywords** ordinary differential equation; triple positive solutions; Multi-point boundary-value problem; fixed point theorem.

**Document code** A
**MR(2000) Subject Classification** 34B10
**Chinese Library Classification** O175

## 1. Introduction

The study of multi-point boundary-value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1]. Since then nonlinear multi-point boundary-value problems have been studied by several authors using the Leray-Schauder continuation, Nonlinear Alternatives of Leray-Schauder, coincidence degree theory, and fixed point theorems in cones. We refer the readers to [2–8] for some existence results of nonlinear multi-point boundary-value problems. Recently, Ma [6] proved the existence of positive solutions for the three-point boundary-value problem

$$u''+b(t)g(u)=0,\quad 0<t<1,$$

Received April 15, 2008; Accepted January 5, 2009
Supported by the "Qing-Lan" Project of Jiangsu Education Committee and the Natural Science Foundation of Jiangsu Education Committee (Grant No. 02KJD460011).
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$$u(0) = 0, \quad u(1) = hu(\tau),$$

by the application of a fixed point theorem in cones. Cao and Ma [7] proved the existence of positive solutions to the boundary-value problem

$$u'' + \lambda a(t) f(u, u') = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} h_i u(\tau_i),$$

by the use of the Leray-Schauder fixed point theorem. Ma [8] proved the existence of at least two positive solutions to multi-point boundary-value problem

$$u'' + \lambda f(t, u) = 0, \quad 0 < t < 1,$$
$$u'(0) = \sum_{i=1}^{m-2} k_i u'(\tau_i), \quad u(1) = \sum_{i=1}^{m-2} h_i u(\tau_i).$$

In this paper, we concentrate on getting three positive solutions for the second-order differential equation

$$u''(t) + q(t) f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to three-point boundary condition

$$u(0) = 0, \quad u(1) = a_0 u(\xi_0) \tag{1.2}$$

or to $m$-point boundary condition

$$u'(0) = \sum_{i=1}^{m-2} b_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{1.3}$$

In this article, we always assume that

($A_1$) $\xi_0 \in (0, 1), a_0 \in (0, \infty)$ satisfy $0 < a_0 \xi_0 < 1$.

($A_2$) $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i, b_i \in [0, \infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$ and $\sum_{i=1}^{m-2} b_i < 1$.

($A_3$) $f : [0, 1] \times [0, \infty) \times \mathbb{R} \longrightarrow [0, \infty)$ is continuous.

($A_4$) $q : [0, 1] \longrightarrow [0, \infty)$ is continuous and there is $t_0 \in [\xi_0, 1]$ such that $q(t_0) > 0$.

($A_4'$) $q : [0, 1] \longrightarrow [0, \infty)$ is continuous and there is $t_1 \in [0, 1]$ such that $q(t_1) > 0$.

By a positive solution of (1.1) with (1.2) or (1.1) with (1.3) we mean a function $u(t)$ which satisfies the differential equation (1.1), the boundary condition (1.2) or (1.3) and $u(t) \geq 0$, $t \in [0, 1]$.

Our main results will depend on an application of a fixed-point theorem due to Avery and Peterson [10] which is a generalization of the fixed-point theorem of Leggett-Williams. The emphasis here is that the nonlinear term $f$ depends on the first-order derivative explicitly. To the best of the authors' knowledge, there are no results for triple positive solutions to the multi-point boundary-value problems.

## 2. Background materials and definitions

**Definition 1** *The map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ provided that $\alpha : P \to [0,\infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

*for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ provided that $\beta : P \to [0,\infty)$ is continuous and*

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

*for all $x, y \in P$ and $0 \leq t \leq 1$.*

Let $P$ be a cone in a real Banach space E, $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$. Then for positive real numbers $c$, $d$, $l$ and $R$, we define the following convex sets:

$$P(\gamma; R) = \{x \in P | \gamma(x) < R\},$$
$$P(\gamma, \alpha; d, R) = \{x \in P | d \leq \alpha(x), \gamma(x) \leq R\},$$
$$P(\gamma, \theta, \alpha; d, l, R) = \{x \in P | d \leq \alpha(x), \theta(x) \leq l, \gamma(x) \leq R\},$$

and a closed set

$$Q(\gamma, \psi; c, R) = \{x \in P | c \leq \psi(x), \gamma(x) \leq R\}.$$

The following fixed-point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

**Lemma 1** ([10]) *Let $P$ be a cone in a real Banach space $E$. Let $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$ satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers $M_0$ and $R$*

$$\alpha(x) \leq \psi(x) \text{ and } \| x \| \leq M_0\gamma(x) \tag{2.1}$$

*for all $x \in \overline{P(\gamma, R)}$. Suppose $T : \overline{P(\gamma, R)} \to \overline{P(\gamma, R)}$ is completely continuous and there exist positive numbers $c$, $d$ and $l$ with $c < d$ such that*

*($S_1$)* $\{x \in P(\gamma, \theta, \alpha; d, l, R) | \alpha(x) > d\} \neq \emptyset$ *and* $\alpha(Tx) > d$ *for all* $x \in P(\gamma, \theta, \alpha; d, l, R)$;

*($S_2$)* $\alpha(Tx) > d$ *for* $x \in P(\gamma, \alpha; d, R)$ *with* $\theta(Tx) > l$;

*($S_3$)* $0 \notin Q(\gamma, \psi; c, R)$, *and* $\psi(Tx) < c$ *for* $x \in Q(\gamma, \psi; c, R)$ *with* $\psi(x) = c$.

*Then $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, R)}$, such that*

$$\gamma(x_i) \leq R \text{ for } i = 1, 2, 3;\ d < \alpha(x_1);$$

$$c < \psi(x_2) \text{ with } \alpha(x_2) < d;\ \psi(x_3) < c.$$

## 3. Existence of triple positive solutions

In this section, we impose growth conditions on $f$ which allow us to apply Lemma 1 to establish the existence of triple positive solutions of Problem (1.1), (1.2) and (1.1), (1.3).

We first deal with the problem (1.1) with three-point boundary-value conditon (1.2). Let $X = C^1[0,1]$ be endowed with the maximum norm

$$\|u\| = \max\{\max_{0\le t\le 1}|u(t)|, \max_{0\le t\le 1}|u'(t)|\}, \quad u \in X.$$

Define the cone $P \subset X$ by

$$P = \{u \in X | u(t) \ge 0, u(0) = 0, u(1) = a_0 u(\xi_0), \quad u(t) \text{ is concave on } [0,1]\}.$$

**Lemma 2** ([6]) *Under the assumption ($A_1$), if $u \in P$, then* $\min_{\xi_0\le t\le 1} u(t) \ge \varepsilon_0 \cdot \max_{0\le t\le 1} u(t)$, *where*

$$\varepsilon_0 = \min\Big\{a_0\xi_0, \frac{a_0(1-\xi_0)}{1-a_0\xi_0}, \xi_0\Big\}.$$

Let the nonnegative continuous concave functional $\alpha$, the nonnegative continuous convex functional $\theta, \gamma$, and the nonnegative continuous functional $\psi$ be defined on the cone $P$ by

$$\alpha(u) = \min_{\xi_0\le t\le 1} u(t),\ \theta(u) = \psi(u) = \max_{0\le t\le 1} u(t),\ \gamma(u) = \max_{0\le t\le 1}|u'(t)|.$$

By Lemma 2, the functionals defined above satisfy

$$\varepsilon_0\theta(u) \le \alpha(u) \le \theta(u) = \psi(u),\ \|u\| = \max\{\theta(u), \gamma(u)\} = \gamma(u), \tag{3.1}$$

for all $u \in P$. Therefore, Condition (2.1) is satisfied.

Let $k(t,s) : [0,1] \times [0,1] \to [0,\infty)$ be defined by

$$k(t,s) = \begin{cases} \dfrac{t(1-s)}{1-a_0\xi_0} - \dfrac{a_0 t(\xi_0 - s)}{1-a_0\xi_0} - (t-s), & \text{for } 0 \le s \le t \le 1 \text{ and } s \le \xi_0; \\[2ex] \dfrac{t(1-s)}{1-a_0\xi_0} - \dfrac{a_0 t(\xi_0 - s)}{1-a_0\xi_0}, & \text{for } 0 \le t \le s \le \xi_0; \\[2ex] \dfrac{t(1-s)}{1-a_0\xi_0}, & \text{for } 0 \le t \le s \le 1 \text{ and } \xi_0 \le s; \\[2ex] \dfrac{t(1-s)}{1-a_0\xi_0} - (t-s), & \text{for } \xi_0 \le s \le t \le 1. \end{cases}.$$

**Lemma 3** ([5]) *Under the assumption ($A_1$), $k(t,s) \le \Phi(s)$, for $(t,s) \in [0,1]\times[0,1]$, where*

$$\Phi(s) = \max\{1, a_0\} \cdot \frac{s(1-s)}{1-a_0\xi_0}.$$

Let

$$M = \int_0^1 q(s)\mathrm{d}s + \frac{a_0}{1-a_0\xi_0}\int_0^{\xi_0}(\xi_0 - s)q(s)\mathrm{d}s + \frac{1}{1-a_0\xi_0}\int_0^1 (1-s)q(s)\mathrm{d}s,$$

$$N = \int_0^1 \Phi(s)q(s)\mathrm{d}s.$$

Choose $\delta > 0$, $d > 0$ such that

$$0 < \delta < \min\{1, a_0\} \cdot \frac{\xi_0}{1-a_0\xi_0}\int_{\xi_0}^1 (1-s)q(s)\mathrm{d}s,$$

$$(d+1)\cdot \max\Big\{\frac{(1-a_0\xi_0^2)^2}{4(1-a_0\xi_0)^2}, \frac{a_0\xi_0 - a_0\xi_0^2}{1-a_0\xi_0}\Big\} > d.$$

Let

$$d_0 = \frac{d+1}{\varepsilon_0} \cdot \max\Big\{\frac{1-a_0\xi_0^2}{1-a_0\xi_0}, \frac{(1-a_0\xi_0^2)^2}{4(1-a_0\xi_0)^2}\Big\}.$$

To present our main results, we assume that there exist constants $c > 0$, $l > 0$, $R > 0$ satisfying $0 < c < d < d_0 < l < R$ and $\frac{R}{M} > \frac{d}{\delta}$, such that

(H$_1$) $f(t,u,v) \le \frac{R}{M}$, for $(t,u,v) \in [0,1] \times [0,R] \times [-R,R]$;

(H$_2$) $f(t,u,v) \ge \frac{d}{\delta}$, for $(t,u,v) \in [\xi_0,1] \times [d,l] \times [-R,R]$;

(H$_3$) $f(t,u,v) < \dfrac{c}{N}$, for $(t,u,v) \in [0,1] \times [0,c] \times [-R,R]$.

**Theorem 1** *Assume that (A$_1$), (A$_3$), (A$_4$) and (H$_1$)–(H$_3$) hold. Then the problem (1.1) with (1.2) has at least three positive solutions $u_1$, $u_2$ and $u_3$ satisfying*

$$\begin{aligned}
&\max_{0\le t\le 1} |u_i'(t)| \le R, \ \textit{for } i = 1,2,3;\\
&d < \min_{\xi_0 \le t \le 1} u_1(t);\\
&c < \max_{0\le t\le 1} u_2(t), \ \textit{with} \ \min_{\xi_0\le t\le 1} u_2(t) < d;\\
&\max_{0\le t\le 1} u_3(t) < c.
\end{aligned} \tag{3.2}$$

**Proof** The problem (1.1) with (1.2) is equivalent to the integral equation

$$\begin{aligned}
u(t) = &-\int_0^t (t-s)q(s)f(s,u(s),u'(s))\mathrm{d}s - \frac{a_0 t}{1-a_0\xi_0}\int_0^{\xi_0}(\xi_0 - s)q(s)f(s,u(s),u'(s))\mathrm{d}s +\\
&\frac{t}{1-a_0\xi_0}\int_0^1 (1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s\\
=&\int_0^1 k(t,s)q(s)f(s,u(s),u'(s))\mathrm{d}s \overset{\text{def}}{=} Tu(t).
\end{aligned}$$

For $u \in P$, it is easy to check that $(Tu)(0) = 0$, $(Tu)(1) = a_0(Tu)(\xi_0)$ and $(Tu)''(t) = -q(t)f(t,u(t),u'(t)) \le 0$. Hence, $Tu$ is concave on [0,1] and $Tu \in P$. Moreover, it is well known that this operator $T : P \to P$ is completely continuous and fixed points of $T$ are solutions of (1.1), (1.2). We now show that all conditions of Lemma 1 are satisfied.

If $u \in \overline{P(\gamma,R)}$, then $\gamma(u) = \max_{0\le t\le 1}|u'(t)| \le R$, so $\max_{0\le t\le 1} u(t) \le R$ and the assumption (H$_1$) implies $f(t,u(t),u'(t)) \le \frac{R}{M}$. On the other hand, for $u \in P$, we have $Tu \in P$. Because of the concavity of $Tu$ on $[0,1]$, we have $\max_{0\le t\le 1}|(Tu)'(t)| = \max\{|(Tu)'(0)|, |(Tu)'(1)|\}$, where

$$\begin{aligned}
|(Tu)'(0)| =&\Big| -\frac{a_0}{1-a_0\xi_0}\int_0^{\xi_0}(\xi_0-s)q(s)f(s,u(s),u'(s))\mathrm{d}s +\\
&\frac{1}{1-a_0\xi_0}\int_0^1 (1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s\Big|\\
\le&\frac{R}{M}\Big(\frac{a_0}{1-a_0\xi_0}\int_0^{\xi_0}(\xi_0-s)q(s)\mathrm{d}s + \frac{1}{1-a_0\xi_0}\int_0^1(1-s)q(s)\mathrm{d}s\Big)\\
<&\frac{R}{M}\cdot M = R,\\
|(Tu)'(1)| =&\Big| -\int_0^1 q(s)f(s,u(s),u'(s))\mathrm{d}s -
\end{aligned}$$

$$
\begin{aligned}
&\frac{a_0}{1-a_0\xi_0}\int_0^{\xi_0}(\xi_0-s)q(s)f(s,u(s),u'(s))\mathrm{d}s+\\
&\frac{1}{1-a_0\xi_0}\int_0^{1}(1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s\Big|\\
\le&\frac{R}{M}\Big(\int_0^1 q(s)\mathrm{d}s+\frac{a_0}{1-a_0\xi_0}\int_0^{\xi_0}(\xi_0-s)q(s)\mathrm{d}s+\frac{1}{1-a_0\xi_0}\int_0^1(1-s)q(s)\mathrm{d}s\Big)\\
=&\frac{R}{M}\cdot M=R.
\end{aligned}
$$

So $\gamma(Tu)=\max_{0\le t\le 1}|(Tu)'(t)|\le R$. Hence, $T:\overline{P(\gamma,R)}\to\overline{P(\gamma,R)}$.

To check condition $(S_1)$ of Lemma 1, we choose $u_0(t)=\frac{d+1}{\varepsilon_0}(-t^2+\frac{1-a_0\xi_0^2}{1-a_0\xi_0}t), t\in[0,1]$. It is easy to see that $u_0\in P$. By (3.1) and the choice of $u_0$, $d$, $l$, $R$, we have

$$
\begin{aligned}
&\theta(u_0)=\max_{0\le t\le 1}|u_0(t)|=\frac{d+1}{\varepsilon_0}\cdot\max\Big\{\frac{a_0\xi_0-a_0\xi_0^2}{1-a_0\xi_0},\frac{(1-a_0\xi_0^2)^2}{4(1-a_0\xi_0)^2}\Big\}\le d_0<l,\\
&\gamma(u_0)=\max_{0\le t\le 1}|u_0'(t)|=\frac{d+1}{\varepsilon_0}\cdot\frac{1-a_0\xi_0^2}{1-a_0\xi_0}\le d_0<R,\\
&\alpha(u_0)\ge\varepsilon_0\theta(u_0)=(d+1)\cdot\max\Big\{\frac{a_0\xi_0-a_0\xi_0^2}{1-a_0\xi_0},\frac{(1-a_0\xi_0^2)^2}{4(1-a_0\xi_0)^2}\Big\}>d.
\end{aligned}
$$

So $u_0\in P(\gamma,\theta,\alpha;d,l,R)$ and $\alpha(u_0)>d$, i.e., $\{u\in P(\gamma,\theta,\alpha;d,l,R)|\alpha(u)>d\}\neq\emptyset$. If $u\in P(\gamma,\theta,\alpha;d,l,R)$, then $d\le u(t)\le l, |u'(t)|\le R$ for $\xi_0\le t\le 1$. From the assumption $(\mathrm{H}_2)$ we have $f(t,u(t),u'(t))\ge\frac{d}{\delta}$ for $\xi_0\le t\le 1$, and by the definition of $\alpha$ and the cone $P$, we have to distinguish two cases: (i) $\alpha(Tu)=(Tu)(\xi_0)$ and (ii) $\alpha(Tu)=(Tu)(1)$.

In case (i), by $0<\xi_0<1$ we have

$$
\begin{aligned}
(Tu)(\xi_0)=&-\int_0^{\xi_0}(\xi_0-s)q(s)f(s,u(s),u'(s))\mathrm{d}s-\\
&\frac{a_0\xi_0}{1-a_0\xi_0}\int_0^{\xi_0}(\xi_0-s)q(s)f(s,u(s),u'(s))\mathrm{d}s+\\
&\frac{\xi_0}{1-a_0\xi_0}\int_0^1(1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s\\
=&\frac{1}{1-a_0\xi_0}\int_0^{\xi_0}sq(s)f(s,u(s),u'(s))\mathrm{d}s+\\
&\frac{\xi_0}{1-a_0\xi_0}\int_{\xi_0}^1 q(s)f(s,u(s),u'(s))\mathrm{d}s-\\
&\frac{\xi_0}{1-a_0\xi_0}\int_0^1 sq(s)f(s,u(s),u'(s))\mathrm{d}s\\
\ge&\frac{\xi_0}{1-a_0\xi_0}\Big(\int_0^{\xi_0}sq(s)f(s,u(s),u'(s))\mathrm{d}s+\\
&\int_{\xi_0}^1 q(s)f(s,u(s),u'(s))\mathrm{d}s-\int_0^1 sq(s)f(s,u(s),u'(s))\mathrm{d}s\Big)\\
=&\frac{\xi_0}{1-a_0\xi_0}\int_{\xi_0}^1(1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s
\end{aligned}
$$

$$\geq \frac{d}{\delta} \cdot \frac{\xi_0}{1-a_0\xi_0} \int_{\xi_0}^{1} (1-s)q(s)\mathrm{d}s$$
$$> \frac{d}{\delta} \cdot \delta = d.$$

In case (ii), we have

$$(Tu)(1) = a_0(Tu)(\xi_0) \geq \frac{d}{\delta} \cdot \frac{a_0\xi_0}{1-a_0\xi_0} \int_{\xi_0}^{1} (1-s)q(s)\mathrm{d}s > \frac{d}{\delta} \cdot \delta = d.$$

So, combining the cases (i) and (ii), we have $\alpha(Tu) > d$, for all $u \in P(\gamma, \theta, \alpha; d, l, R)$. This shows that the condition $(S_1)$ of Lemma 1 is satisfied.

Secondly, because of $T(P) \subset P$ and (3.1), noting the choice of $d_0$, $d$ and $l$, we have

$$\begin{aligned}|\alpha(Tu)| &\geq \varepsilon_0\theta(Tu) > \varepsilon_0 l > \varepsilon_0 d_0 \\ &= (d+1) \cdot \max\Big\{\frac{1-a_0\xi_0^2}{1-a_0\xi_0}, \frac{(1-a_0\xi_0^2)^2}{4(1-a_0\xi_0)^2}\Big\} \\ &\geq (d+1) \cdot \frac{1-a_0\xi_0^2}{1-a_0\xi_0} > d+1 > d,\end{aligned}$$

for all $u \in P(\gamma, \alpha; d, R)$ with $\theta(Tu) > l$. Thus, the condition $(S_2)$ of Lemma 1 is satisfied.

Finally, we show that $(S_3)$ of Lemma 1 also holds. Clearly, as $\psi(0) = 0 < c$, there holds that $0 \notin Q(\gamma, \psi; c, R)$. Suppose that $u \in Q(\gamma, \psi; c, R)$ with $\psi(u) = c$, then $0 \leq u(t) \leq c$, $|u'(t)| \leq R$ for $0 \leq t \leq 1$. Then, by the definition of the operator $T$, Lemma 3 and the assumption $(H_3)$, we have

$$\begin{aligned}\psi(Tu) &= \max_{0\leq t\leq 1}(Tu)(t) = \max_{0\leq t\leq 1}\int_0^1 k(t,s)q(s)f(s,u(s),u'(s))\mathrm{d}s \\ &\leq \int_0^1 \Phi(s)q(s)f(s,u(s),u'(s))\mathrm{d}s < \frac{c}{N}\int_0^1 \Phi(s)q(s)\mathrm{d}s = \frac{c}{N}\cdot N = c.\end{aligned}$$

So $(S_3)$ of Lemma 1 is satisfied. Therefore, an application of Lemma 1 implies that the problem (1.1) with (1.2) has at least three positive solutions $u_1$, $u_2$, and $u_3$ satisfying (3.2). The proof is completed. □

Now we deal with the problem (1.1) with $m$-point boundary-value condition (1.3). The method is just similar to what we have done above.

Define the cone $P_1 \subset X = C^1[0,1]$ by

$$P_1 = \left\{u \in X \,\middle|\, \begin{array}{l} u(t) \geq 0, u'(0) = \sum_{i=1}^{m-2} b_i u'(\xi_i), u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ u(t) \text{ is concave on } [0,1]. \end{array}\right\}$$

**Lemma 4** ([8]) *Under the assumption $(A_2)$, if $u \in P_1$, then $u(t)$ is non-increasing on $[0,1]$ and satisfies $\min_{0\leq t\leq 1} u(t) \geq \eta_0 \cdot \max_{0\leq t\leq 1} u(t)$, where*

$$\eta_0 = \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i\xi_i}.$$

**Lemma 5** ([8]) *Under the assumption $(A_2)$, then for $y \in C[0,1]$ with $y(t) \geq 0$ for $t \in [0,1]$, the* *problem*

$$u'' + y(t) = 0, \quad 0 < t < 1,$$

$$u'(0) = \sum_{i=1}^{m-2} b_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$$

*has a unique solution* $u \in P_1$. *Moreover,*

$$u(t) = -\int_0^t (t-s)y(s)\mathrm{d}s + At + B,$$

*where*

$$\begin{aligned} A =& \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} y(s)\mathrm{d}s}{\sum_{i=1}^{m-2} b_i - 1}, \\ B =& \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big( \int_0^1 (1-s)y(s)\mathrm{d}s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)\mathrm{d}s - \\ & \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} y(s)\mathrm{d}s}{\sum_{i=1}^{m-2} b_i - 1}(1 - \sum_{i=1}^{m-2} a_i\xi_i)\Big). \end{aligned}$$

Let the nonnegative continuous concave functional $\alpha_1$, the nonnegative continuous convex functional $\theta_1, \gamma_1$, and the nonnegative continuous functional $\psi_1$ be defined on the cone $P_1$ respectively by

$$\alpha_1(u) = \min_{0\le t\le 1} |u(t)| = u(1), \quad \theta_1(u) = \psi_1(u) = \max_{0\le t\le 1} |u(t)| = u(0),$$

$$\gamma_1(u) = \max\{\max_{0\le t\le 1} |u(t)|, \max_{0\le t\le 1} |u'(t)|\} = \max\{u(0), |u'(1)|\}$$

for $u \in P_1$. By Lemma 4, the functionals defined above satisfy

$$\eta_0\theta_1(u) \le \alpha_1(u) \le \theta_1(u) = \psi_1(u), \quad \|u\| = \gamma_1(u) \tag{3.3}$$

for all $u \in P_1$. Therefore, the condition (2.1) is satisfied.

Let

$$\begin{aligned} M_1 =& \max\Big\{ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\big( \int_0^1 (1-s)q(s)\mathrm{d}s + \frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} q(s)\mathrm{d}s\big), \\ & \int_0^1 q(t)\mathrm{d}s + \frac{1}{1-\sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} q(s)\mathrm{d}s\Big\}, \\ N_1 =& \frac{1}{1-\sum_{i=1}^{m-2} a_i}\big( \int_0^1 (1-s)q(s)\mathrm{d}s + \frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} q(s)\mathrm{d}s\big). \end{aligned}$$

Choose $\delta_1 > 0, d_1 > 0, d^* > 0$, such that

$$0 < \delta_1 < \eta_0 \sum_{i=1}^{m-2} a_i\big( \int_0^{\xi_i} (1-\xi_i)q(s)\mathrm{d}s + \int_{\xi_i}^1 (1-s)q(s)\mathrm{d}s\big),$$

$$(d_1+1)w(0) > d_1, \quad d^* = \frac{d_1+1}{\eta_0}\max\{w(0), |w'(1)|\},$$

where $w(t)$ is the unique solution of the problem

$$u'' + 1 = 0, \quad 0 < t < 1, \tag{3.4}$$

$$u'(0) = \sum_{i=1}^{m-2} b_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{3.5}$$

i.e.,

$$\begin{aligned} w(t) =& -\frac{1}{2}t^2 - \frac{\sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}t + \\ & \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\frac{1}{2}(1-\sum_{i=1}^{m-2} a_i\xi_i^2) + \frac{\sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}(1-\sum_{i=1}^{m-2} a_i\xi_i)\Big), \qquad (3.6) \\ w(0) =& \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\frac{1}{2}(1-\sum_{i=1}^{m-2} a_i\xi_i^2) + \frac{\sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}(1-\sum_{i=1}^{m-2} a_i\xi_i)\Big), \\ |w'(1)| =& 1 + \frac{\sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}. \end{aligned}$$

Suppose that there exist constants $c_1 > 0$, $l_1 > 0$, $R_1 > 0$ with $0 < c_1 < d_1 < d^* < l_1 < R_1$, $\frac{R_1}{M_1} > \frac{d_1}{\delta_1}$, such that

($H_4$) $f(t,u,v) \le \frac{R_1}{M_1}$, for $(t,u,v) \in [0,1] \times [0,R_1] \times [-R_1,R_1]$;

($H_5$) $f(t,u,v) \ge \frac{d_1}{\delta_1}$, for $(t,u,v) \in [0,1] \times [d_1,l_1] \times [-R_1,R_1]$;

($H_3$) $f(t,u,v) < \frac{c_1}{N_1}$, for $(t,u,v) \in [0,1] \times [0,c_1] \times [-R_1,R_1]$.

**Theorem 2** *Assume that ($A_2$), ($A_3$), ($A_4'$) and ($H_4$)–($H_6$) hold. Then the problem (1.1) with (1.3) has at least three positive solutions $u_1$, $u_2$ and $u_3$ satisfying*

$$\begin{aligned} & \max\{\max_{0\le t\le 1} u_i(t), \ \max_{0\le t\le 1} |u_i'(t)|\} \le R_1, \quad \textit{for} \ \ i = 1,2,3; \\ & \min_{0\le t\le 1} u_1(t) > d_1; \\ & c_1 < \max_{0\le t\le 1} u_2(t) \ \ \textit{with} \ \min_{0\le t\le 1} u_2(t) < d_1; \\ & \max_{0\le t\le 1} u_3(t) < c_1. \end{aligned} \tag{3.7}$$

**Proof** It comes from Lemma 5 that the problem (1.1) with (1.3) is equivalent to the integral equation

$$\begin{aligned} u(t) =& -\int_0^t (t-s)q(s)f(s,u(s),u'(s))\mathrm{d}s + \\ & \frac{t}{\sum_{i=1}^{m-2} b_i - 1}\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} q(s)f(s,u(s),u'(s))\mathrm{d}s + \\ & \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\int_0^1 (1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s - \\ & \quad \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)q(s)f(s,u(s),u'(s))\mathrm{d}s - \end{aligned}$$

$$\frac{1-\sum_{i=1}^{m-2}a_i\xi_i}{\sum_{i=1}^{m-2}b_i-1}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}q(s)f(s,u(s),u'(s))\mathrm{d}s\Big]$$

$$\stackrel{\text{def}}{=}T_1u(t),$$

and the operator $T_1 : P_1 \to P_1$ is completely continuous. Now we show that all the conditions of Lemma 1 are satisfied.

If $u \in \overline{P_1(\gamma_1, R_1)}$, then

$$\gamma_1(u) = \max\{\max_{0\le t\le 1}|u(t)|, \max_{0\le t\le 1}|u'(t)|\} = \max\{u(0), |u'(1)|\} \le R_1,$$

so $0 \le u(t) \le R_1, |u'(t)| \le R_1$ for $0 \le t \le 1$, and the assumption ($\mathrm{H}_4$) implies $f(t, u(t), u'(t)) \le \frac{R_1}{M_1}$ for $0 \le t \le 1$. On the other hand, for $u \in P_1$, then $T_1u \in P_1$ and

$$\gamma_1(T_1u) = \max\{(T_1u)(0), |(T_1u)'(1)|\},$$

where

$$\begin{aligned}(T_1u)(0) =&\frac{1}{1-\sum_{i=1}^{m-2}a_i}\Big[\int_0^1(1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s-\\&\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)q(s)f(s,u(s),u'(s))\mathrm{d}s+\\&\frac{1-\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}b_i}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}q(s)f(s,u(s),u'(s))\mathrm{d}s\Big]\\ \le&\frac{R_1}{M_1}\cdot\frac{1}{1-\sum_{i=1}^{m-2}a_i}\Big(\int_0^1(1-s)q(s)\mathrm{d}s+\frac{1-\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}b_i}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}q(s)\mathrm{d}s\Big)\\ \le&\frac{R_1}{M_1}\cdot M_1=R_1,\\ |(T_1u)'(1)|=&\int_0^1q(s)f(s,u(s),u'(s))\mathrm{d}s+\\&\frac{1}{1-\sum_{i=1}^{m-2}b_i}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}q(s)f(s,u(s),u'(s))\mathrm{d}s\\ \le&\frac{R_1}{M_1}\Big(\int_0^1q(s)\mathrm{d}s+\frac{1}{1-\sum_{i=1}^{m-2}b_i}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}q(s)\mathrm{d}s\Big)\\ \le&\frac{R_1}{M_1}\cdot M_1=R_1.\end{aligned}$$

Therefore, $\gamma_1(T_1u) \le R_1$, i.e., $T_1 : \overline{P_1(\gamma_1, R_1)} \to \overline{P_1(\gamma_1, R_1)}$.

We choose $u_0(t) = \frac{d_1+1}{\eta_0}w(t)$, where $w(t)$ is the unique solution of the problem (3.4), (3.5), i.e., $w(t)$ is given by (3.6). Then $u_0 \in P_1$. From (3.3), and the choice of $d^*$, $d_1$, $l_1$ and $R_1$, we have

$$\begin{aligned}&\theta_1(u_0)=u_0(0)=\frac{d_1+1}{\eta_0}w(0)\le d^*<l_1,\\&\gamma_1(u_0)=\frac{d_1+1}{\eta_0}\gamma_1(w)=\frac{d_1+1}{\eta_0}\max\{w(0),|w'(1)|\}=d^*<R_1,\end{aligned}$$

$$\alpha_1(u_0) \geq \eta_0\theta_1(u_0) = (d_1+1)w(0) > d_1.$$

So $u_0 \in P_1(\gamma_1, \theta_1, \alpha_1; d_1, l_1, R_1)$ and $\alpha_1(u_0) > d_1$, hence $\{u \in P_1(\gamma_1, \theta_1, \alpha_1; d_1, l_1, R_1)|\alpha_1(u) > d_1\} \neq \emptyset$. If $u \in P_1(\gamma_1, \theta_1, \alpha_1; d_1, l_1, R_1)$, then $d_1 \leq u(t) \leq l_1$, $|u'(t)| \leq R_1$ for $0 \leq t \leq 1$. From the assumption ($\mathrm{H}_5$), we have $f(t, u(t), u'(t)) \geq \frac{d_1}{\delta_1}$ for $0 \leq t \leq 1$. Hence, by $T_1 u \in P_1$, (3.3) and ($\mathrm{A}_2$), we have

$$\begin{aligned}
\alpha_1(T_1u) \geq & \eta_0\theta_1(T_1u) = \eta_0(T_1u)(0) \\
= & \eta_0 \cdot \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s - \\
& \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)q(s)f(s,u(s),u'(s))\mathrm{d}s + \\
& \frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} q(s)f(s,u(s),u'(s))\mathrm{d}s\Big) \\
\geq & \eta_0\Big(\sum_{i=1}^{m-2} a_i \int_0^1 (1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s - \\
& \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)q(s)f(s,u(s),u'(s))\mathrm{d}s\Big) \\
= & \eta_0 \sum_{i=1}^{m-2} a_i\Big(\int_0^{\xi_i} (1-\xi_i)q(s)f(s,u(s),u'(s))\mathrm{d}s + \\
& \int_{\xi_i}^1 (1-s)q(s)f(s,u(s),u'(s))\mathrm{d}s\Big) \\
\geq & \frac{d_1}{\delta_1} \cdot \eta_0 \sum_{i=1}^{m-2} a_i\Big(\int_0^{\xi_i} (1-\xi_i)q(s)\mathrm{d}s + \int_{\xi_i}^1 (1-s)q(s)\mathrm{d}s\Big) \\
> & \frac{d_1}{\delta_1} \cdot \delta_1 = d_1.
\end{aligned}$$

So,

$$\alpha_1(T_1u) > d_1 \text{ for all } u \in P_1(\gamma_1, \theta_1, \alpha_1; d_1, l_1, R_1).$$

This shows that the condition ($\mathrm{S}_1$) of Lemma 1 is satisfied.

Secondly, from the choice of $d^*$, $d_1$, $l_1$, $R_1$ and $N_1$, by the assumption ($\mathrm{H}_6$) it is easy to check that the conditions ($\mathrm{S}_2$) and ($\mathrm{S}_3$) of Lemma 1 are satisfied, and hence we omit it. Therefore, by Lemma 1, the problem (1.1) with (1.3) has at least three positive solutions $u_1$, $u_2$ and $u_3$ satisfying (3.7). This completes the proof. □

**Example** Consider the three-point boundary-value problem

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \tag{3.8}$$

$$u(0) = 0, \quad \frac{3}{2}u(\frac{1}{2}) = u(1), \tag{3.9}$$

where

$$f(t,u,v)=\begin{cases}\frac{1}{16}e^t+\frac{1}{2}u^5+(\frac{v}{4(18^5+1)})^4, & \text{for } 0\le u\le 16,\\ \frac{1}{16}e^t+\frac{1}{2}(17-u)u^5+(\frac{v}{4(18^5+1)})^4, & \text{for } 16<u\le 17,\\ \frac{1}{16}e^t+\frac{1}{2}(u-17)u^5+(\frac{v}{4(18^5+1)})^4, & \text{for } 17<u\le 18,\\ \frac{1}{16}e^t+\frac{18^5}{2}+(\frac{v}{4(18^5+1)})^4, & \text{for } u>18.\end{cases}.$$

Clearly, $\xi_0=\frac{1}{2}$, $a_0=\frac{3}{2}$, $0<a_0\xi_0=\frac{3}{4}<1$, $q(t)\equiv 1$, and ($A_1$), ($A_3$) and ($A_4$) hold. Choose $c=1$, $d=2$, $l=16$, $R=2(18^5+1)$, $\delta=\frac{1}{8}$. We note $M=\frac{15}{4}$, $N=1$. Consequently, $f(t,u,v)$ satisfies

$$\begin{aligned}&f(t,u,v)\le\frac{R}{M}=\frac{8}{15}(18^5+1),\\ &\qquad \text{for } 0\le t\le 1,\quad 0\le u\le 2(18^5+1),\quad -2(18^5+1)\le v\le 2(18^5+1);\\ &f(t,u,v)\ge\frac{d}{\delta}=16,\quad \text{for}\quad \frac{1}{2}\le t\le 1,\quad 2\le u\le 16, -2(18^5+1)\le v\le 2(18^5+1);\\ &f(t,u,v)\le\frac{c}{N}=1,\quad \text{for}\quad 0\le t\le 1,\quad 0\le u\le 1,\quad -2(18^5+1)\le v\le 2(18^5+1).\end{aligned}$$

Then all conditions of Theorem 1 hold. Thus, with Theorem 1, the problem (3.8) with (3.9) has at least three positive solutions $u_1$, $u_2$, $u_3$ such that

$$\max_{0\le t\le 1}|u_i'(t)|\le 2(18^5+1),\quad \text{for } i=1,2,3;\quad 2<\min_{\frac{1}{2}\le t\le 1}u_1(t),$$

$$1<\max_{0\le t\le 1}u_2(t),\quad \text{with}\quad \min_{\frac{1}{2}\le t\le 1}u_2(t)<2,\quad \max_{0\le t\le 1}u_3(t)<1.$$

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