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Positive Solutions for Singular Fourth-Order Integral Boundary-Value Problem with p-Laplacian Operator

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Abstract In this paper, we investigate the existence of positive solutions for singular fourth-order integral boundary-value problem with p-Laplacian operator by using the upper and lower solution method and fixed point theorem. Nonlinear term may be singular at $t = 0$ and/or $t = 1$ and $x = 0$.

Keywords p-Laplacian operator; singular boundary value problems; integral boundary condition; positive solution.

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1. Introduction

Boundary value problems with integral boundary conditions for ordinary differential equations arise in different fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. Moreover, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and nonlocal boundary-value problems as special cases, which have received much attention of many authors (for instance, see [6-9] and references therein). For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [1], Karakostas and Tsamatos [2], Lomtatidze and Malaguti [3] and the references therein.

Recently, Zhang and Feng [4] have studied the existence and nonexistence of symmetric positive solutions for the following nonlinear fourth-order boundary value problems

$$\begin{cases} (\phi_p(x''(t))'' = \omega(t)f(t,x(t)), \quad 0 < t < 1, \\ x(0) = x(1) = \displaystyle\int_0^1 g(s)x(s)\mathrm{d}s, \\ \phi_p(x''(0)) = \phi_p(x''(1)) = \displaystyle\int_0^1 h(s)\phi_p(x''(s))\mathrm{d}s, \end{cases} \tag{$**$}$$

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where $\phi_p(t) = |t|^{p-2} \cdot t$, $p > 1$, $\phi_q = \phi_p^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\omega \in L[0,1]$ is nonnegative, symmetric on the interval $[0,1]$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, g, $h \in L^1[0,1]$ are nonnegative, symmetric on $[0,1]$.

Zhao [10] and Du and Zhao [11] investigated the existence and uniqueness of positive solutions for some $2n$-order two-point and second-order m-point boundary value problems under the assumption that nonlinearity $f(t,x)$ is decreasing with respect to x. To the best of our knowledge, no paper considers the existence of positive solution for integral boundary-value problem with p-Laplacian operator when $f(t,x)$ is decreasing with respect to x. To fill this gap, motivated by above work, in this paper, we investigate the existence of positive solutions for the following singular fourth-order differential equations

$$\begin{cases} (\phi_p(x''(t))'' = f(t,x(t)), \quad 0 < t < 1, \\ x(0) = \displaystyle\int_0^1 g(s)x(s)\mathrm{d}s, \quad x(1) = 0, \\ \phi_p(x''(0)) = \phi_p(x''(1)) = \displaystyle\int_0^1 h(s)\phi_p(x''(s))\mathrm{d}s, \end{cases} \tag{1}$$

where $f \in C[(0,1) \times (0,+\infty), [0,+\infty)]$, g, $h \in L^1[0,1]$ is nonnegative. Nonlinear term $f(t,x)$ may be singular at $x = 0$, $t = 0$ and /or $t = 1$. Let $\sigma_1 = \int_0^1 (1-s)g(s)\mathrm{d}s$, $\sigma_2 = \int_0^1 h(s)\mathrm{d}s$. Throughout this paper, we always assume that $0 \leq \sigma_1$, $\sigma_2 < 1$.

The main features of this paper are as follows. Firstly, we discuss integral boundary value problems with p-Laplacian operator which includes fourth-order two-point, three-point, multi-point and nonlocal boundary value problem as special cases. As pointed in [4], up to now, no paper has considered the fourth-order p-Laplacian equation with integral boundary conditions except for [4]. To the author's knowledge, boundary value problem (1) has not been considered in the literature. Secondly, nonlinear term $f(t,x)$ in this paper may be singular not only at $t = 0$ and/or $t = 1$, but also at $x = 0$ which makes it different from that in [4]. Thirdly, we study the existence of positive solution under the condition that $f(t,x)$ is decreasing in x. This is new, since most results given in the literature are under the assumption that $f(t,x)$ is increasing in x.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and lemmas. The main results are formulated in Section 3 and an example is also given.

2. Preliminaries and several lemmas

In our discussion, the space

$$X = \{x : x, \phi_p(x'') \in C^2[0,1]\}$$

will be the basic space to study BVP (1).

A function $\alpha(t)$ is called a solution of BVP (1) if $\alpha(t) \in C^2[0,1]$ satisfies $\phi_p(\alpha''(t)) \in C^2[0,1]$ and the BVP (1). In addition, $\alpha(t)$ is said to be a positive solution if $\alpha(t) > 0$ for $t \in (0,1)$ and $\alpha(t)$ is a solution of BVP (1).

Definition 1 *A function $\alpha(t) \in X$ is called a lower solution of BVP (1) if $\alpha(t)$ satisfies*

$$\begin{cases} (\phi_p(\alpha''(t))'' \le f(t,\alpha(t)), \quad 0<t<1, \\ \alpha(0) - \displaystyle\int_0^1 g(s)\alpha(s)\mathrm{d}s \le 0, \quad \alpha(1) \le 0, \\ -\Big[\phi(\alpha''(0)) - \displaystyle\int_0^1 h(s)\phi_p(\alpha''(s))\mathrm{d}s\Big] \le 0, \\ -\Big[\phi_p(\alpha''(1)) - \displaystyle\int_0^1 h(s)\phi_p(\alpha''(s))\mathrm{d}s\Big] \le 0. \end{cases}$$

Definition 2 *A function $\beta(t) \in X$ is called an upper solution of BVP (1) if $\beta(t)$ satisfies*

$$\begin{cases} (\phi_p(\beta''(t))'' \ge f(t,\beta(t)), \quad 0<t<1, \\ \beta(0) - \displaystyle\int_0^1 g(s)\beta(s)\mathrm{d}s \ge 0, \quad \beta(1) \ge 0, \\ -\Big[\phi(\beta''(0)) - \displaystyle\int_0^1 h(s)\phi_p(\beta''(s))\mathrm{d}s\Big] \ge 0, \\ -\Big[\phi_p(\beta''(1)) - \displaystyle\int_0^1 h(s)\phi_p(\beta''(s))\mathrm{d}s\Big] \ge 0. \end{cases}$$

If there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of BVP (1) such that $\alpha(t) \le \beta(t)$, then $(\alpha(t), \beta(t))$ is called a couple of lower and upper solutions of BVP (1).

We first state the following lemmas.

Lemma 1 *Let $r_1 \ge 0$ and $r_2 \ge 0$. If $\sigma(t) \in C[0,1]$ and $\sigma(t) \ge 0$, then the following problem*

$$-x''(t) = \sigma(t), \quad t \in (0,1), \tag{2}$$

$$x(0) - \int_0^1 g(s)x(s)\mathrm{d}s = r_1, \quad x(1) = r_2 \tag{3}$$

has a unique solution $x(t)$ such that $x(t) \ge 0, \ t \in [0,1]$.

Proof Integrating (2) from 0 to t, we get

$$x'(t) = x'(0) - \int_0^t \sigma(s)\mathrm{d}s. \tag{4}$$

Integrating again, we obtain

$$x(t) = x(0) + x'(0)t - \int_0^t (t-s)\sigma(s)\mathrm{d}s. \tag{5}$$

Letting $t=1$ in (5), we have

$$\begin{aligned} r_2 = x(1) &= x(0) + x'(0) - \int_0^1 (1-s)\sigma(s)\mathrm{d}s \\ &= r_1 + \int_0^1 g(s)x(s)\mathrm{d}s + x'(0) - \int_0^1 (1-s)\sigma(s)\mathrm{d}s. \end{aligned}$$

Therefore,

$$x'(0) = (r_2 - r_1) + \int_0^1 (1-s)\sigma(s)\mathrm{d}s - \int_0^1 g(s)x(s)\mathrm{d}s. \tag{6}$$

Substituting (6) into (5), we obtain

$$\begin{aligned}x(t) =&r_1 + \int_0^1 g(s)x(s)\mathrm{d}s + t\Big[(r_2 - r_1) + \int_0^1 (1-s)\sigma(s)\mathrm{d}s -\\ &\int_0^1 g(s)x(s)\mathrm{d}s\Big] - \int_0^t (t-s)\sigma(s)\mathrm{d}s\\ =&(1-t)r_1 + tr_2 + \int_0^1 G(t,s)\sigma(s)\mathrm{d}s + (1-t)\int_0^1 g(s)x(s)\mathrm{d}s, \end{aligned} \tag{7}$$

where

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1,\\ s(1-t), & 0 \le s \le t \le 1.\end{cases} \tag{8}$$

From (7) we obtain

$$\begin{aligned}\int_0^1 g(s)x(s)\mathrm{d}s =& \int_0^1 g(s)\Big[\int_0^1 G(s,\tau)\sigma(\tau)\mathrm{d}\tau + (1-s)\int_0^1 g(\tau)x(\tau)\mathrm{d}\tau +\\ &(1-s)r_1 + sr_2\Big]\mathrm{d}s\\ =& \int_0^1 g(s)\Big[\int_0^1 G(s,\tau)\sigma(\tau)\mathrm{d}\tau\Big]\mathrm{d}s + \int_0^1 (1-s)g(s)\mathrm{d}s\cdot\int_0^1 g(s)x(s)\mathrm{d}s +\\ &\int_0^1 g(s)[(1-s)r_1 + sr_2]\mathrm{d}s,\end{aligned}$$

therefore

$$\begin{aligned}\int_0^1 g(s)x(s)\mathrm{d}s =& \frac{1}{1-\int_0^1(1-s)g(s)\mathrm{d}s}\Big\{\int_0^1 g(s)\Big[\int_0^1 G(s,\tau)\sigma(\tau)\mathrm{d}\tau\Big]\mathrm{d}s +\\ &\int_0^1 g(s)[(1-s)r_1 + sr_2]\mathrm{d}s\Big\}.\end{aligned} \tag{9}$$

Substituting (9) into (7), we have

$$\begin{aligned}x(t) =&(1-t)r_1 + tr_2 + \int_0^1 H(t,s)\sigma(s)\mathrm{d}s +\\ &\frac{1-t}{1-\sigma_1}\int_0^1 g(s)[(1-s)r_1 + sr_2]\mathrm{d}s,\end{aligned} \tag{10}$$

where

$$H(t,s) = G(t,s) + \frac{1-t}{1-\sigma_1}\int_0^1 G(\tau,s)g(\tau)\mathrm{d}\tau, \tag{11}$$

$$\sigma_1 = \int_0^1 (1-s)g(s)\mathrm{d}s. \tag{12}$$

Obviously, $G(t,s) \ge 0$, $H(t,s) \ge 0$, $\sigma_1 \ge 0$. From (10), it is easily seen that $x(t) \ge 0$ for $t \in [0,1]$. □

Lemma 2 *Let $r_3 \ge 0$ and $r_4 \ge 0$. If $y(t) \in C[0,1]$ and $y(t) \ge 0$, then the following problem*

$$-x''(t) = y(t), \tag{13}$$

$$x(0) - \int_0^1 h(s)x(s)\mathrm{d}s = r_3, \quad x(1) - \int_0^1 h(s)x(s)\mathrm{d}s = r_4 \tag{14}$$

has a unique solution $x(t)$ such that $x(t) \geq 0,\ t \in [0,1]$.

Proof By integration of (13) from 0 to t, we have

$$x'(t) = x'(0) - \int_0^t y(s)\mathrm{d}s.$$

Integrating again, we obtain

$$x(t) = x(0) + x'(0)t - \int_0^t (t-s)y(s)\mathrm{d}s. \tag{15}$$

Letting $t = 1$ in (15), we have

$$x'(0) = (r_4 - r_3) + \int_0^1 (1-s)y(s)\mathrm{d}s. \tag{16}$$

Substituting (14) and (16) into (15), we obtain

$$\begin{aligned} x(t) =& r_3 + \int_0^1 h(s)x(s)\mathrm{d}s + t\Big[(r_4 - r_3) + \int_0^1 (1-s)y(s)\mathrm{d}s\Big] - \int_0^t (t-s)y(s)\mathrm{d}s \\ =& r_3 + t(r_4 - r_3) + \int_0^1 G(t,s)y(s)\mathrm{d}s + \int_0^1 h(s)x(s)\mathrm{d}s, \end{aligned} \tag{17}$$

where $G(t,s)$ is defined as in (8) and

$$\begin{aligned} \int_0^1 h(s)x(s)\mathrm{d}s =& \int_0^1 h(s)\Big[r_3 + s(r_4 - r_3) + \int_0^1 G(s,\tau)y(\tau)\mathrm{d}\tau + \int_0^1 h(s)x(s)\mathrm{d}s\Big]\mathrm{d}s \\ =& r_3\int_0^1 h(s)\mathrm{d}s + (r_4 - r_3)\int_0^1 sh(s)\mathrm{d}s + \int_0^1 h(s)\Big[\int_0^1 G(s,\tau)y(\tau)\mathrm{d}\tau\Big]\mathrm{d}s + \\ & \int_0^1 h(s)\mathrm{d}s \cdot \int_0^1 h(s)x(s)\mathrm{d}s, \end{aligned}$$

therefore

$$\begin{aligned} \int_0^1 h(s)x(s)\mathrm{d}s =& \frac{1}{1 - \int_0^1 h(s)ds}\Big\{r_3\int_0^1 h(s)\mathrm{d}s + (r_4 - r_3)\int_0^1 sh(s)\mathrm{d}s + \\ & \int_0^1 h(s)\Big[\int_0^1 G(s,\tau)y(\tau)\mathrm{d}\tau\Big]\mathrm{d}s\Big\}. \end{aligned} \tag{18}$$

Substituting (18) into (17), we get

$$\begin{aligned} x(t) =& r_3 + t(r_4 - r_3) + \int_0^1 G(t,s)y(s)\mathrm{d}s + \frac{1}{1 - \int_0^1 h(s)\mathrm{d}s}\int_0^1 h(s)\Big[\int_0^1 G(s,\tau)y(\tau)\mathrm{d}\tau\Big]\mathrm{d}s + \\ & \frac{1}{1 - \int_0^1 h(s)\mathrm{d}s}\Big[r_3\int_0^1 h(s)\mathrm{d}s + (r_4 - r_3)\int_0^1 sh(s)\mathrm{d}s\Big] \\ =& (1-t)r_3 + tr_4 + \frac{1}{1-\sigma_2}\Big[\int_0^1 (1-s)h(s)\mathrm{d}sr_3 + \int_0^1 sh(s)\mathrm{d}sr_4\Big] + \int_0^1 K(t,s)y(s)\mathrm{d}s, \end{aligned} \tag{19}$$

where

$$K(t,s) = G(t,s) + \frac{1}{1-\sigma_2}\int_0^1 G(s,\tau)h(\tau)\mathrm{d}\tau, \tag{20}$$

$$\sigma_2 = \int_0^1 h(s)\mathrm{d}s. \tag{21}$$

Obviously, $G(t,s) \geq 0$, $K(t,s) \geq 0$, $\sigma_2 \geq 0$. From (19), it is easily seen that $x(t) \geq 0$ for $t \in [0,1]$. $\square$

Lemma 3 *Let $y(t) \in C[0,1]$. Then the boundary value problem*

$$\begin{cases} -y''(t) = -f(t, x(t)), \quad t \in (0,1), \\ y(0) = y(1) = \displaystyle\int_0^1 h(s) y(s) \mathrm{d}s \end{cases} \tag{22}$$

has a unique solution

$$y(t) = -\int_0^1 K(t,s) f(s, x(s)) \mathrm{d}s, \tag{23}$$

where $K(t,s)$ is defined as in (20).

Proof The proof is similar to the proof of Lemma 2, we omit the details. $\square$

Lemma 4 *Let $y(t) \in C[0,1]$. Then the boundary value problem*

$$\begin{cases} -x''(t) = -\phi_q(y(t)), \quad t \in (0,1), \\ x(0) = \displaystyle\int_0^1 g(s) x(s) \mathrm{d}s, \quad x(1) = 0 \end{cases} \tag{24}$$

has a unique solution

$$x(t) = -\int_0^1 H(t,s) \phi_q(y(s)) \mathrm{d}s, \tag{25}$$

where $H(t,s)$ is defined as in (11).

Proof The proof is similar to the proof of Lemma 1, we omit the details. $\square$

Suppose that x is a solution of problem (1). Then from Lemma 4, we have

$$x(t) = -\int_0^1 H(t,s) \phi_q(y(s)) \mathrm{d}s. \tag{26}$$

By Lemma 3, we have

$$x(t) = \int_0^1 H(t,s) \phi_q\Big(\int_0^1 K(s,\tau) f(\tau, x(\tau)) \mathrm{d}\tau \Big) \mathrm{d}s.$$

Lemma 5 ([5]) *For $t, s \in [0,1]$, we have*

$$\rho_1 e(t) e(s) \leq H(t,s) \leq \gamma_1 e(s), \tag{27}$$

where

$$e(s) = s(1-s), \ \rho_1 = \frac{\int_0^1 e(\tau) g(\tau) \mathrm{d}\tau}{1 - \sigma_1}, \ \gamma_1 = \frac{1 + \int_0^1 s g(s) \mathrm{d}s}{1 - \sigma_2}.$$

Lemma 6 ([4]) *For $t, s \in [0,1]$, we have*

$$\rho_2 e(s) \leq K(t,s) \leq \gamma_2 e(s), \tag{28}$$

where

$$\rho_2 = \frac{\int_0^1 e(\tau) h(\tau) \mathrm{d}\tau}{1 - \sigma_2}, \ \gamma_2 = \frac{1}{1 - \sigma_2}.$$

Throughout this paper, we make the following assumptions:

(H_1) $f \in C[(0,1) \times (0,+\infty), [0,+\infty)]$ and $f(t,x)$ is nonincreasing with respect to x;

(H_2) For any $\lambda > 0, f(t,\lambda) \not\equiv 0$ and $\int_0^1 s(1-s)f(s,\lambda s(1-s))\mathrm{d}s < +\infty$;

(H_3) There exists a continuous function $a(t)$ and some fixed positive number k such that $a(t) \geq kt(1-t)$, $t \in [0,1]$ and

$$\int_0^1 H(t,s)\phi_q\Big(\int_0^1 K(s,\tau)f(\tau,a(\tau))\mathrm{d}\tau\Big)\mathrm{d}s = b(t) \geq a(t),$$
$$\int_0^1 H(t,s)\phi_q\Big(\int_0^1 K(s,\tau)f(\tau,b(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \geq a(t).$$

3. Main results

Theorem 1 *Assume that conditions (H_1)–(H_3) are satisfied. Then the boundary value problem (1) has at least one positive solution ω which satisfies $\omega(t) \geq mt(1-t)$ for some $m > 0$.*

Proof Let

$$P = \{x(t) \in C[0,1] : \text{ there exists a positive number } k_x \text{ such that } x(t) \geq k_x e(t),\ t \in [0,1]\}. \quad (29)$$

Obviously, P is not empty since $t(1-t) \in P$. Now, let us define an operator T on X by

$$(Tx)(t) = \int_0^1 H(t,s)\phi_q\Big(\int_0^1 K(s,\tau)f(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s, \forall\ x \in P.$$

In the following, we divide the long proof into four parts.

Firstly, we show that T is well defined on P.

For any $x \in P$, by the definition of P, there exists a positive number k_x such that $x(t) \geq k_x t(1-t),\ t \in [0,1]$. By ($H_1$), ($H_2$), Lemmas 5 and 6, we have

$$\int_0^1 K(s,\tau)f(\tau,x(\tau))\mathrm{d}\tau \leq \gamma_2 \int_0^1 \tau(1-\tau)f(\tau,k_x\tau(1-\tau))\mathrm{d}\tau < +\infty.$$

Therefore

$$\begin{aligned}(Tx)(t) &= \int_0^1 H(t,s)\phi_q\Big(\int_0^1 K(s,\tau)f(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\ &\leq \int_0^1 \gamma_1 s(1-s)\phi_q\Big(\int_0^1 \gamma_2\tau(1-\tau)f(\tau,k_x\tau(1-\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\ &\leq \gamma_1\phi_q(\gamma_2)\int_0^1 s(1-s)\mathrm{d}s \cdot \phi_q\Big(\int_0^1 \tau(1-\tau)f(\tau,k_x\tau(1-\tau))\mathrm{d}\tau\Big) < +\infty. \quad (30)\end{aligned}$$

Let $B = \max_{t\in[0,1]} x(t)$. By ($H_2$) and the continuity of $f(t,x)$, we know that $\int_0^1 e(s)f(s,B)\mathrm{d}s > 0$. Thus,

$$\int_0^1 e(s)f(s,x(s))\mathrm{d}s \geq \int_0^1 e(s)f(s,B)\mathrm{d}s > 0.$$

Therfore,

$$(Tx)(t) = \int_0^1 H(t,s)\phi_q\Big(\int_0^1 K(s,\tau)f(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s$$

$$
\begin{aligned}
&\geq \rho_1 \int_0^1 e(t)e(s)\phi_q\Big(\int_0^1 \rho_2 e(\tau)f(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
&= \rho_1\phi_q(\rho_2)\int_0^1 e(s)\mathrm{d}s\cdot\phi_q\Big(\int_0^1 e(\tau)f(\tau,x(\tau))\mathrm{d}\tau\Big)\cdot e(t)\\
&= \frac{1}{6}\rho_1\phi_q(\rho_2)\cdot\phi_q\Big(\int_0^1 e(\tau)f(\tau,x(\tau))\mathrm{d}\tau\Big)\cdot e(t) = k_{Tx}e(t), \qquad (31)
\end{aligned}
$$

where $k_{Tx} = \frac{1}{6}\rho_1\phi_q(\rho_2)\cdot\phi_q\Big(\int_0^1 e(\tau)f(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s > 0$. It follows from (30) and (31) that T is well defined on P and $T(P)\subset P$.

Secondly, we are in position to determine a couple of lower and upper solutions of BVP (1). In fact, by direct computation, we obtain

$$[\phi_p(Tx)''(t)]'' = f(t,x(t)), \quad t\in(0,1), \tag{32}$$

and

$$(Tx)(0) = \int_0^1 g(s)(Tx)(s)\mathrm{d}s, \quad (Tx)(1) = 0, \tag{33}$$

$$\phi_p((Tx)''(0)) = \phi_p((Tx)''(1)) = \int_0^1 h(s)\phi_p((Tx)''(s))\mathrm{d}s. \tag{34}$$

Let $b(t) = (Ta)(t)$. Then by (H_1) and (H_3), we have

$$a(t)\leq (Ta)(t) = b(t), \quad b(t) = (Ta)(t)\geq (Tb)(t), \quad t\in[0,1]. \tag{35}$$

Since $a(t)\in P$, by (31), we get that $(Ta)(t), (Tb)(t)\in P$. Thus, it follows from (32)–(35) that

$$[\phi_p(Tb)''(t)]'' - f(t,(Tb)(t)) \leq [\phi_p(Tb)''(t)]'' - f(t,b(t)) = 0, \tag{36}$$

$$[\phi_p(Ta)''(t)]'' - f(t,(Ta)(t)) \geq [\phi_p(Ta)''(t)]'' - f(t,a(t)) = 0. \tag{37}$$

Let $\alpha(t) = (Tb)(t)$, $\beta(t) = (Ta)(t)$. (33) and (34) imply that $(Ta)(t)$, $(Tb)(t)\in X$ and satisfy boundary condition. It is clear, $\alpha(t)\leq\beta(t)$. From (35)–(37), we know that $\alpha(t)$ and $\beta(t)$ are a couple of lower and upper solutions of BVP (1).

Thirdly, we shall show that the following boundary value problem

$$
\begin{cases}
(\phi_p(x''(t))'' = F(t,x(t)), \quad 0<t<1,\\
x(0) = \int_0^1 g(s)x(s)\mathrm{d}s, \quad x(1) = 0,\\
\phi_p(x''(0)) = \phi_p(x''(1)) = \int_0^1 h(s)\phi_p(x''(s))\mathrm{d}s
\end{cases} \tag{38}
$$

has a positive solution, where

$$
F(t,x(t)) = \begin{cases}
f(t,\alpha(t)), & \text{if } x(t)<\alpha(t),\\
f(t,x(t)), & \text{if } \alpha(t)\leq x(t)\leq\beta(t),\\
f(t,\beta(t)), & \text{if } x(t)>\beta(t).
\end{cases} \tag{39}
$$

To this end, we consider the operator $A: C[0,1]\to C[0,1]$ defined as follows:

$$(Ax)(t) = \int_0^1 H(t,s)\phi_q\Big(\int_0^1 K(s,\tau)F(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s.$$

It is clear that a fixed-point of the operator A is a solution of the boundary value problem (38).

First, A is continuous since F is continuous. Next, since $\alpha(t) \in P$, there exists a positive number k_α such that $\alpha(t) \geq k_\alpha t(1-t)$, $t \in [0,1]$. By (H_2), we have

$$\begin{aligned}\int_0^1 s(1-s)F(s,\alpha(s))\mathrm{d}s &\leq \int_0^1 s(1-s)f(s,\alpha(s))\mathrm{d}s \\ &\leq \int_0^1 s(1-s)f(s,k_\alpha s(1-s))\mathrm{d}s < +\infty. \end{aligned} \tag{40}$$

As a consequence, for any $x(t) \in C[0,1]$, by (39) and (40), we get

$$\begin{aligned}(Ax)(t) &= \int_0^1 H(t,s)\phi_q\Big(\int_0^1 K(s,\tau)F(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leq \int_0^1 \gamma_1 s(1-s)\phi_q\Big(\int_0^1 \gamma_2\tau(1-\tau)F(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leq \gamma_1\phi_q(\gamma_2)\int_0^1 s(1-s)\mathrm{d}s\cdot\phi_q\Big(\int_0^1 \tau(1-\tau)F(\tau,\alpha(\tau))\mathrm{d}\tau\Big) < +\infty, \end{aligned} \tag{41}$$

which implies that the operator A is uniformly bounded.

On the other hand, since $H(t,s)$ is continuous on $C[0,1]\times C[0,1]$, it is uniformly continuous on $C[0,1]\times C[0,1]$ as well. Thus, for fixed $s \in [0,1]$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| \leq \delta$,

$$|H(t_1,s) - H(t_2,s)| < \frac{\varepsilon}{\phi_q(\gamma_2)\cdot\phi_q(\int_0^1 \tau(1-\tau)f(\tau,k_\alpha\tau(1-\tau))\mathrm{d}\tau)}.$$

In addition, for all $x(t) \in C[0,1]$, we have

$$\begin{aligned}|(Ax)(t_1) - (Ax)(t_2)| &\leq \int_0^1 |H(t_1,s)-H(t_2,s)|\phi_q\Big(\int_0^1 K(s,\tau)F(\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leq \int_0^1 |H(t_1,s)-H(t_2,s)|\phi_q\Big(\int_0^1 \gamma_2\tau(1-\tau)f(\tau,\alpha(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leq \int_0^1 |H(t_1,s)-H(t_2,s)|\phi_q(\gamma_2)\phi_q\Big(\int_0^1 \tau(1-\tau)f(\tau,k_\alpha\tau(1-\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &< \varepsilon, \end{aligned}$$

which implies that the operator A is equicontinuous. Thus, the Ascoli-Arzela theorem guarantees that A is a compact operator. By Schauder's fixed point theorem, A has a fixed point ω, i.e., $\omega = A\omega$. So, the boundary value problem (38) has a solution.

Finally, we prove that boundary value problem (1) has at least one positive solution. To see this, we need only to prove that $\alpha(t) \leq \omega(t) \leq \beta(t)$, $t \in [0,1]$.

Since ω is a solution of (38), this means

$$\omega(0) = \int_0^1 g(s)\omega(s)\mathrm{d}s, \quad \omega(1) = 0, \tag{42}$$

$$\phi_p(\omega''(0)) = \phi_p(\omega''(1)) = \int_0^1 h(s)\phi_p(\omega''(s))\mathrm{d}s. \tag{43}$$

In addition, since $f(t,x)$ is nonincreasing in x, we have

$$f(t,\beta(t)) \le F(t,\omega(t)) \le f(t,\alpha(t)), \quad t\in[0,1]. \tag{44}$$

By (35) and (H_3), we get

$$f(t,b(t)) \le F(t,\omega(t)) \le f(t,a(t)), \quad t\in[0,1]. \tag{45}$$

Since $a(t)\in P$, by (32), we have

$$[\phi_p(\beta''(t))]'' = [\phi_p((Ta)''(t))]'' = f(t,a(t)), \quad t\in[0,1],$$

which together with (33)–(35), (42)–(44) implies that

$$\begin{cases} [\phi_p(\beta''(t))]'' - [\phi_p(\omega''(t))]'' = f(t,a(t)) - F(t,\omega(t)) \ge 0, \quad t\in[0,1], \\ (\beta-\omega)(0) = \displaystyle\int_0^1 g(s)(\beta(s)-\omega(s))\mathrm{d}s, \ (\beta-\omega)(1)=0, \\ \phi_p(\beta''(0)) - \phi_p(\omega''(0)) = \displaystyle\int_0^1 h(s)[\phi_p(\beta''(s)) - \phi_p(\omega''(s))]\mathrm{d}s, \\ \phi_p(\beta''(1)) - \phi_p(\omega''(1)) = \displaystyle\int_0^1 h(s)[\phi_p(\beta''(s)) - \phi_p(\omega''(s))]\mathrm{d}s. \end{cases} \tag{46}$$

Let $z=\phi_p(\beta'')-\phi_p(\omega'')$. Then z is twice differentiable in $[0,1]$ and by (46) we have

$$\begin{cases} z''(t)\ge 0, \quad 0<t<1, \\ z(0) - \displaystyle\int_0^1 h(s)z(s)\mathrm{d}s = 0, \quad z(1) - \displaystyle\int_0^1 h(s)z(s)\mathrm{d}s = 0. \end{cases} \tag{47}$$

By Lemma 2, we know $z(t)\le 0$, $t\in[0,1]$, that is, $\phi_p(\beta''(t))\le\phi_p(\omega''(t))$ on $[0,1]$. Since ϕ_p is monotone increasing, we have

$$\beta''(t)\le\omega''(t), \quad \text{for } t\in[0,1],$$

i.e.,

$$(\beta-\omega)''(t)\le 0, \quad \text{for } t\in[0,1].$$

Set $m=\beta-\omega$. Then we have

$$\begin{cases} m''(t)\le 0, \quad 0<t<1, \\ m(0) - \displaystyle\int_0^1 g(s)m(s)\mathrm{d}s = 0, \quad m(1)=0. \end{cases} \tag{48}$$

By Lemma 1, we obtain

$$\omega(t)\le\beta(t), \quad \text{for } t\in[0,1].$$

Similarly, we can obtain that $\omega(t)\ge\alpha(t)$ on $[0,1]$. Therefore, $\omega(t)$ is a positive solution of the boundary value problem (1). In addition, $\alpha(t)\in P$ implies that there exists a positive constant $m>0$ such that $\omega(t)\ge\alpha(t)\ge mt(1-t)$, $t\in[0,1]$. The proof is completed. □

If $f(t,x)$ is nonsingular at $x=0$, then for all $x\ge 0$, $f(t,x)\le f(t,0)$, $t\in(0,1)$. Thus, we have the following

Theorem 2 *If (H_1) holds, and for any $\mu > 0$, $f(t,\mu) \not\equiv 0$ and satisfies*

$$0 < \int_0^1 s(1-s)f(s,0)\mathrm{d}s < +\infty.$$

Then the boundary value problem (1) has at least one positive solution $\omega(t)$ which satisfies $\omega(t) \geq mt(1-t)$ for some $m > 0$.

Proof We need only to replace P by

$$P_1 = \{x(t) \in E : x(t) \geq 0,\ t \in [0,1]\}$$

in Theorem 1 and let $a(t) \equiv 0$. The rest of proof is similar to that in Theorem 1. □

If $f(t,x)$ is nonsingular, then the conclusion of Theorem 1 can be strengthened as the following

Theorem 3 *If $f(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, decreasing in x and $f(t,\lambda) \not\equiv 0$ for any $\lambda > 0$, then the boundary value problem (1) has at least one positive solution $\omega(t)$ which satisfies $\omega(t) \geq mt(1-t)$ for some $m > 0$.*

To show the application of our main results, we present an example.

Example 1 Consider the singular fourth-order integral boundary-value problem with p-Laplacian operator

$$\begin{cases} (\phi_p(x''(t))'' = a_0(t) + \sum\limits_{i=1}^{n} a_i(t)x^{-\alpha_i}, \quad 0 < t < 1, \\ x(0) = \displaystyle\int_0^1 \frac{1}{2}x(s)\mathrm{d}s, \quad x(1) = 0, \\ \phi_p(x''(0)) = \phi_p(x''(1)) = 0, \end{cases} \tag{49}$$

where $\phi_p(t) = |t|^{p-2}t$, $p > 1$, and $a_0(t)$, $a_i(t)$ are nonnegative and continuous on $(0,1)$, $0 < \alpha_i < 1$ $(i = 1,2,\ldots,n)$.

If $\sum_{i=1}^{n} a_i(t) \not\equiv 0$ on $[0,1]$, and

$$\int_0^1 t(1-t)\Big(a_0(t) + \sum_{i=1}^{n} a_i(t)t^{-\alpha_i}(1-t)^{-\alpha_i}\Big)\mathrm{d}t < +\infty, \tag{50}$$

then the fourth-order boundary value problem (49) has a positive solution $\omega(t)$ such that $\omega(t) \geq mt(1-t)$ for some $m > 0$.

Proof Let $f(t,x) = a_0(t) + \sum_{i=1}^{n} a_i(t)x^{-\alpha_i}$, $t \in (0,1)$, $g(t) = \frac{1}{2}$, $h(t) = 0$. It is not difficult to verify that conditions (H_1) and (H_2) are satisfied in Theorem 1 under the condition (50). Set $\mu = \max_{1 \leq i \leq n}\{\alpha_i\}$. Then we have $f(t,x) \leq f(t,ru) \leq r^{-\mu}f(t,x)$ holds for all positive numbers $r < 1$. Since $e(t) = t(1-t) \in P$, by (31), we know $Te \in P, T^2e \in P$ which implies that there exist positive numbers k,l such that $Te \geq ke,\ T^2e \geq le$. Take a positive number

$$r_0 \leq \min\{1,\ k,\ l^{1/(1-\mu^2)}\}.$$

Then,

$$T(r_0e) \geq Te \geq ke \geq r_0e, \quad T^2(r_0e) \geq r_0^{\mu^2}T^2e \geq r_0^{\mu^2}le \geq r_0e.$$

If we take $a(t) = r_0 t(1-t)$, then it is easy to check that (H_3) is satisfied. So, our conclusion follows from Theorem 1. $\square$

Remark 1 Example 1 shows that there exist a large number of functions that satisfy condition (H_3) and (H_3) is natural and easy to be verified.

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