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Covering Morphisms in a Pushout-Pullback Diagram

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Abstract In a pushout-pullback diagram, which consists of four morphisms $f : A \to B$, $g : A \to C$, $\alpha : C \to D$ and $\beta : B \to D$, we give some relations among the covers of these four modules. If $\ker f \in I(\mathscr{L})$, then $g : A \to C$ is $\mathscr{L}$-covering if and only if $\beta : B \to D$ is $\mathscr{L}$-covering. If every module has an $\mathscr{L}$-precover and $\ker f \in I(\mathscr{L})$, then A and C have isomorphic $\mathscr{L}$-precovers if and only if B and D have isomorphic $\mathscr{L}$-precovers.

Keywords cover; covering morphism; pushout-pullback diagram.

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1. Introduction

The concepts of envelopes and covers of modules have been investigated under different guise by several authors [1–6]. A good reference for an introduction to the topic of covers and envelopes is [2]. The following pullback or pushout diagram of homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow{\alpha} & D \end{array}$$

Diagram 1 The pullback of α and β or pushout of f and g

is a useful tool in the theory of (pre)envelopes and (pre)covers of modules. Recently, Rothmaler [3] established some interesting relations between flat covers and cotorsion envelopes by way of using the pushout or pullback diagram. Motivated by these, we shall investigate covers of these four modules in the Diagram 1 in this paper. In particular, we prove if $\ker f \in I(\mathscr{L})$, then $g : A \to C$ is $\mathscr{L}$-covering if and only if $\beta : B \to D$ is $\mathscr{L}$-covering. If every module has an $\mathscr{L}$-precover and $\ker f \in I(\mathscr{L})$, then A and C have isomorphic $\mathscr{L}$-precovers if and only if B and D have isomorphic $\mathscr{L}$-precovers. Much of what we claim in this paper will have a dual version.

We recall some definitions and facts needed in the later sections. Let R be a ring and $\mathscr{L}$ a class of right R-modules. We obtain two Ext-orthogonal classes $I(\mathscr{L}) = \{X \in \text{Mod}-R \mid \text{Ext}^1_R(L, X) = 0, \text{ for each } L \in \mathscr{L}\}$, and $P(\mathscr{L}) = \{X \in \text{Mod}-R \mid \text{Ext}^1_R(X, L) = 0, \text{ for each } L \in \mathscr{L}\}$. It is easy to see that $P(\mathscr{L})$ and $I(\mathscr{L})$ are closed under extensions and direct summands [7, 8].

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Recall an important definition given first by Enochs [1, 2, 9]. Let M be a module, $\mathscr{L}$ a class of modules and $F \in \mathscr{L}$. A morphism $\phi : F \to M$ is called an $\mathscr{L}$-precover [1] of M, if $\mathrm{Hom}(G, F) \to \mathrm{Hom}(G, M) \to 0$ is exact for all $G \in \mathscr{L}$. Moreover, if every endomorphism $f : F \to F$ satisfies that $\phi f = \phi$ is an automorphism of F, then $\phi : F \to M$ is an $\mathscr{L}$-cover [1] of M. It is straightforward to verify that if M has an $\mathscr{L}$-cover, it is unique up to isomorphism. The dual notions are called $\mathscr{L}$-preenvelopes, $\mathscr{L}$-envelopes [2, 9].

Throughout this paper, R is an associative ring with identity, all modules are unitary right R-modules, and all classes of modules are assumed to be closed under isomorphisms. For other unexplained concepts and notations, we refer the reader to [2] and [9].

2. Main results

According to [10], if $f : M_1 \to M_2$ is a morphism, $p_1 : P_1 \to M_1$, and $p_2 : P_2 \to M_2$ are $\mathscr{L}$-precovers, then the diagram

$$\begin{tikzcd} P_1 \arrow[r, dotted, "g"] \arrow[d, "p_1"'] & P_2 \arrow[d, "p_2"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

Diagram 2 The lifting of f

can be completed to a commutative diagram. In this situation, g is called a *lifting* of f (relative to the two precovers). It follows from [10, Corollary 1.5] that if M_1 and M_2 have $\mathscr{L}$-covers, then f has some lifting which is an isomorphism if and only if every lifting of f is an isomorphism. Thus a morphism $f : M_1 \to M_2$ is said to be $\mathscr{L}$-covering [10], if M_1 and M_2 have $\mathscr{L}$-covers $p_1 : P_1 \to M_1$ and $p_2 : P_2 \to M_2$ and some (so every) lifting $g : P_1 \to P_2$ is an isomorphism. The morphism is said to be $\mathscr{L}$-enveloping, if the dual situation holds.

According to [11], a morphism $f : M_1 \to M_2$ is said to be left (right) minimal if any endomorphism $g \in \mathrm{End}_R(M_2)$ (any $g \in \mathrm{End}_R(M_1)$) satisfies that $gf = f$ ($fg = f$) is an automorphism. It was proved in [11, Proposition 2.5] that every $\mathscr{L}$-covering homomorphism is right minimal. We first obtain a relation of being right minimal between two morphisms in the Diagram 1 as follows.

Proposition 1 *Assume that the commutative Diagram 1 is a pullback of α and β such that $f : A \to B$ is an $\mathscr{L}$-envelope. If g is right minimal, then β is right minimal.*

Proof Suppose that σ is an endomorphism of B such that $\beta\sigma = \beta$. We want to show σ is an automorphism. Note that $\alpha g = \beta f = \beta\sigma f$. Consider the pair of maps $\sigma f : A \to B$ and $g : A \to C$, by the property of the pullback diagrams there is a linear map $h : A \to A$ such that $\sigma f = fh$ and $g = gh$.

$$\begin{tikzcd} A \arrow[dr, dotted, "h"] \arrow[drr, bend left, "\sigma f"] \arrow[ddr, bend right, "g"'] & & \\ & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\ & C \arrow[r, "\alpha"] & D \end{tikzcd}$$

Diagram 3 The pullback and σf

Since g is right minimal, it follows that h is an automorphism of A. Consider the map $fh^{-1} : A \to B$. There is a map $\overline{\sigma}$ such that $\overline{\sigma} f = fh^{-1}$ since $f : A \to B$ is an $\mathscr{L}$-envelope. Thus $f = \overline{\sigma}\sigma f$ and $f = \sigma\overline{\sigma} f$. So $\overline{\sigma}\sigma$ and $\sigma\overline{\sigma}$ are automorphisms, which implies that σ is an automorphism. Therefore β is right minimal.

Theorem 2 *Assume that the commutative Diagram 1 is a pullback of α and β, and $p_2 : P_2 \to B$, $p_4 : P_4 \to D$ are $\mathscr{L}$-precovers such that there is an isomorphism $\phi_2 : P_2 \to P_4$ with $\beta p_2 = p_4\phi_2$, that is, the following diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"'] & P_2 \arrow[l, "p_2"'] \arrow[d, "\phi_2"] \\
C \arrow[r, "\alpha"] & D & P_4 \arrow[l, "p_4"']
\end{tikzcd}$$

Diagram 4 Precovers and ϕ_2

(1) If $p_1 : P_1 \to A$ is an $\mathscr{L}$-precover, then $gp_1 : P_1 \to C$ is an $\mathscr{L}$-precover. Moreover, if p_1 is an $\mathscr{L}$-cover and β is monomorphic, then gp_1 is an $\mathscr{L}$-cover;

(2) If $p_3 : P_3 \to C$ is an $\mathscr{L}$-precover and β is monomorphic, then there is an $\mathscr{L}$-precover $p_1 : P_3 \to A$ with $gp_1 = p_3$. Moreover, if p_3 is an $\mathscr{L}$-cover, then p_1 is an $\mathscr{L}$-cover.

Proof (1) (i) Suppose that $p_1 : P_1 \to A$ is an $\mathscr{L}$-precover. Then $gp_1 \in \mathrm{Hom}_R(P_1, C)$. Let $q : Q \to C$ be a morphism with $Q \in \mathscr{L}$. Since $p_4 : P_4 \to D$ is an $\mathscr{L}$-precover, there is a morphism $t : Q \to P_4$ with $p_4 t = \alpha q$, that is, the following diagram commutes.

$$\begin{tikzcd}
C \arrow[r, "\alpha"] & D \\
Q \arrow[u, "q"] \arrow[r, "t"] & P_4 \arrow[u, "p_4"']
\end{tikzcd}$$

Diagram 5 The precover p_4

Note that $\beta p_2 = p_4\phi_2$ and ϕ_2 is isomorphic, thus $\alpha q = \beta p_2\phi_2^{-1} t$ and $p_2\phi_2^{-1} t \in \mathrm{Hom}_R(Q, B)$. By the property of the pullback diagram, there exists a morphism $k : Q \to A$ such that $gk = q$ and $fk = p_2\phi_2^{-1} t$.

$$\begin{tikzcd}
Q \arrow[dr, dotted, "k"] \arrow[drr, bend left, "p_2\phi_2^{-1}t"] \arrow[ddr, bend right, "q"'] & & \\
& A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\
& C \arrow[r, "\alpha"] & D
\end{tikzcd}$$

Diagram 6 The pullback and q

Since $p_1 : P_1 \to A$ is an l-precover, there is a morphism $s \in \mathrm{Hom}_R(Q, P_1)$ with $p_1 s = k$. So $gp_1 s = gk = q$, that is, $gp_1 : P_1 \to C$ is an $\mathscr{L}$-precover.

(ii) Let $s \in End(P_1)$ with $gp_1 s = gp_1$. By hypothesis, β is monomorphic, so is g [12]. Thus $p_1 s = p_1$. Since $p_1 : P_1 \to A$ is an $\mathscr{L}$-cover, it follows that s is an automorphism. So

$gp_1 : P_1 \to C$ is an $\mathscr{L}$-cover.

(2) (i) Suppose that $p_3 : P_3 \to C$ is an $\mathscr{L}$-precover. Since $p_4 : P_4 \to D$ is an $\mathscr{L}$-precover, there is a morphism $t : P_3 \to P_4$ with $p_4 t = \alpha p_3$, that is, the following diagram commutes.

$$\begin{tikzcd} C \arrow[r, "\alpha"] & D \\ P_3 \arrow[u, "p_3"] \arrow[r, "t"] & P_4 \arrow[u, "p_4"'] \end{tikzcd}$$

Diagram 7 Precovers p_3 and p_4

Note that $\beta p_2 = p_4 \phi_2$ and ϕ_2 is isomorphic, thus $\alpha p_3 = \beta p_2 \phi_2^{-1} t$ and $p_2 \phi_2^{-1} t \in \mathrm{Hom}_R(P_3, B)$. By the property of the pullback diagram, there exists a morphism $p_1 : P_3 \to A$ such that $gp_1 = p_3$.

$$\begin{tikzcd} P_3 \arrow[dr, dotted, "p_1"] \arrow[drr, bend left, "p_2\phi_2^{-1}t"] \arrow[ddr, bend right, "p_3"'] & & \\ & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\ & C \arrow[r, "\alpha"] & D \end{tikzcd}$$

Diagram 8 The pullback and p_3

We claim that $p_1 : P_3 \to A$ is an $\mathscr{L}$-precover. Suppose that $q : Q \to A$ is a morphism with $Q \in \mathscr{L}$, thus $gq \in Hom_R(Q, C)$. Since $p_3 : P_3 \to C$ is an $\mathscr{L}$-precover, there exists a morphism $h : Q \to P_3$ with $p_3 h = gq$, that is, the following diagram commutes.

$$\begin{tikzcd} Q \arrow[r, "q"] \arrow[d, "h"'] & A \arrow[d, "g"] \\ P_3 \arrow[r, "p_3"] & C \end{tikzcd}$$

Diagram 9 The precover p_3 and g

Hence $gq = gp_1 h$. Since β is monomorphic, which implies g is monomorphic, it follows that $q = p_1 h$, so $p_1 : P_3 \to A$ is an $\mathscr{L}$-precover.

(ii) Let $s \in End(P_3)$ with $p_1 s = p_1$. Thus $gp_1 s = gp_1$, so $p_3 s = p_3$. Since p_3 is an $\mathscr{L}$-cover, it follows that s is an automorphism, therefore $p_1 : P_3 \to A$ is an $\mathscr{L}$-cover.

Corollary 3 *Assume that every right R-module has an $\mathscr{L}$-precover, and the commutative Diagram 1 is a pullback of α and β. If B and D have isomorphic $\mathscr{L}$-precovers, then A and C have isomorphic $\mathscr{L}$-precovers.*

Corollary 4 *Assume that the commutative Diagram 1 is a pullback of α and β, and β is monomorphic. If $\beta : B \to D$ is $\mathscr{L}$-covering, then $g : A \to C$ is $\mathscr{L}$-covering.*

Corollary 5 *Assume that the commutative Diagram 1 is a pullback of α and β, and β is monomorphic. If $\beta : B \to D$ is an $\mathscr{L}$-precover and $A \in \mathscr{L}$, then $g : A \to C$ is an $\mathscr{L}$-cover.*

As an application, we have the following property on $\mathscr{L}$-precovers.

Corollary 6 *Assume that $\mathscr{L}$ is closed under extensions and $0 \to L \to M \to N \to 0$ is exact with $L \in \mathscr{L}$. If N has an $\mathscr{L}$-precover, then M has an $\mathscr{L}$-precover.*

Proof Suppose that $c : C \to N$ is an $\mathscr{L}$-precover. Consider the pullback diagram of α and c as follows.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r, "\alpha"] & N \arrow[r] & 0
\end{tikzcd}$$

Diagram 10 The pullback of α and c

Since $\mathscr{L}$ is closed under extensions and $L, C \in \mathscr{L}$, it follows that $P \in \mathscr{L}$. By Theorem 2(1) $P \to M$ is an $\mathscr{L}$-precover.

Example 7 Assume $\mathscr{L}$ is closed under extensions (e.g. $\mathscr{L}$=the class of all flat right R-modules), $\alpha : C \to D$ is an epimorphic $\mathscr{L}$-cover and $\beta : B \to D$ is a monomorphic $\mathscr{L}$-injective precover. We consider the pullback of α and β, and obtain the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & K \arrow[r] & C \arrow[r, "\alpha"] & D \arrow[r] & 0
\end{tikzcd}$$

Diagram 11 The pullback of $\mathscr{L}$-cover and $\mathscr{L}$-injective precover

Since α is an $\mathscr{L}$-cover, it follows that $K \in I(\mathscr{L})$ by [9, Lemma 2.1.2]. Note that $I(\mathscr{L})$ is closed under extensions, thus $A \in I(\mathscr{L})$. So it follows from Corollary 5 that $g : A \to C$ is an $\mathscr{L}$-injective cover.

According to [3], if the Diagram 1 is a pullback such that α is epimorphic or a pushout such that g is monomorphic, then it is both a pullback and a pushout, thus it is called a pushout-pullback diagram.

Now we turn to investigate the converses of Theorem 2 and Corollary 5 in the following full pushout-pullback diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & K \arrow[r] & C \arrow[r, "\alpha"] \arrow[d, "\pi_1"'] & D \arrow[r] \arrow[d, "\pi_2"] & 0 \\
 & & I \arrow[r, equal] \arrow[d] & I \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \qquad (\divideontimes\divideontimes)$$

Diagram 12 The full pushout-pullback

Theorem 8 *Assume that the Diagram 12 is a pushout-pullback with $K \in I(\mathscr{L})$, and $p_1 : P_1 \to A$, $p_3 : P_3 \to C$ are $\mathscr{L}$-precovers such that there is an isomorphism $\phi_1 : P_1 \to P_3$ with $gp_1 = p_3\phi_1$, that is, the following diagram commutes.*

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] \arrow[d, "\phi_1"'] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\ P_3 \arrow[r, "p_3"'] & C \arrow[r, "\alpha"] & D \end{tikzcd}$$

Diagram 13 Precovers p_1 and p_3

(1) If $p_2 : P_2 \to B$ is an $\mathscr{L}$-precover, then $\beta p_2 : P_2 \to D$ is an $\mathscr{L}$-precover. Moreover, if p_2 is an $\mathscr{L}$-cover, then βp_2 is an $\mathscr{L}$-cover.

(2) If $p_4 : P_4 \to D$ is an $\mathscr{L}$-precover, then there is an $\mathscr{L}$-precover $p_2 : P_4 \to B$ with $\beta p_2 = p_4$. Moreover, if p_4 is an $\mathscr{L}$-cover, then p_2 is an $\mathscr{L}$-cover.

Proof (1) Suppose that $p_2 : P_2 \to B$ is an $\mathscr{L}$-precover. Now we want to show that $\beta p_2 : P_2 \to D$ is an $\mathscr{L}$-precover. Let $p_4 : P_4 \to D$ be a morphism with $P_4 \in \mathscr{L}$. Since $K \in I(\mathscr{L})$, there is a morphism $t_1 : P_4 \to C$ such that $p_4 = \alpha t_1$ by [7,Theorem 1]. Thus there exists a morphism $t_2 : P_4 \to P_3$ with $t_1 = p_3 t_2$ for p_3 is an $\mathscr{L}$-precover.

$$\begin{tikzcd} & & P_3 \arrow[d, "p_3"'] & P_4 \arrow[l, dotted, "t_2"'] \arrow[d, "p_4"] \arrow[dl, dotted, "t_1"'] & \\ 0 \arrow[r] & K \arrow[r] & C \arrow[r, "\alpha"'] & D \arrow[r] & 0 \end{tikzcd}$$

Diagram 14 The precover p_3 and α

Set $k = fp_1\phi_1^{-1}t_2$, then $k \in \mathrm{Hom}_R(P_4, B)$. Since p_2 is an $\mathscr{L}$-precover, there is a morphism $\sigma : P \to P_2$ with $p_2\sigma = k$. Hence

$$p_4 = \alpha t_1 = \alpha p_3 t_2 = \alpha p_3 \phi_1 \phi_1^{-1} t_2 = \alpha g p_1 \phi_1^{-1} t_2 = \beta f p_1 \phi_1^{-1} t_2 = \beta p_2 \sigma,$$

that is, βp_2 is an $\mathscr{L}$-precover.

Let $s \in \mathrm{End}(P_2)$ with $\beta p_2 s = \beta p_2$. Since β is monomorphic, it follows that $p_2 s = p_2$, thus s is an automorphism for p_2 is an $\mathscr{L}$-cover. Therefore $\beta p_2 : P_2 \to D$ is an $\mathscr{L}$-cover.

(2) Suppose that $p_4 : P_4 \to D$ is an $\mathscr{L}$-precover. Let $p_2 = fp_1\phi_1^{-1}t_2$, then $p_2 \in \mathrm{Hom}_R(P_4, B)$ and $\beta p_2 = p_4$. We claim that $p_2 : P_4 \to B$ is an $\mathscr{L}$-precover.

Let $p : P \to B$ be a morphism with $P \in \mathscr{L}$. Thus $\beta p \in \mathrm{Hom}_R(P, D)$, there is a morphism $\delta : P \to P_4$ such that $p_4\delta = \beta p$, and so $\beta p_2 \delta = \beta p$.

$$\begin{tikzcd} B \arrow[d, "\beta"'] & P \arrow[l, "p"'] \arrow[d, dotted, "\delta"] \\ D & P_4 \arrow[l, "p_4"] \arrow[ul, "p_2"] \end{tikzcd}$$

Diagram 15 The precover p_4

Since β is monomorphic, it follows that $p_2\delta = p$, that is, p_2 is an $\mathscr{L}$-precover.

Let $s \in End(P_4)$ with $p_2 s = p_2$. Thus $\beta p_2 s = \beta p_2$, i.e., $p_4 s = p_4$. Hence s is an automorphism of P_4. Therefore $p_2 : P_4 \to B$ is an $\mathscr{L}$-cover.

Corollary 9 *Assume that every right R-module has an $\mathscr{L}$-precover, and the Diagram 12 is a pushout-pullback with $K \in I(\mathscr{L})$. If A and C have isomorphic $\mathscr{L}$-precovers, then B and D have isomorphic $\mathscr{L}$-precovers.*

Corollary 10 *Assume that the Diagram 12 is a pushout-pullback with $K \in I(\mathscr{L})$. If $g : A \to C$ is $\mathscr{L}$-covering, then $\beta : B \to D$ is $\mathscr{L}$-covering.*

Corollary 11 *Assume that the Diagram 12 is a pushout-pullback with $K \in I(\mathscr{L})$. If $g : A \to C$ is an $\mathscr{L}$-precover and $B \in \mathscr{L}$, then $\beta : B \to D$ is an $\mathscr{L}$-cover.*

The results of same type hold for $\mathscr{L}$-(pre)envelopes. Following the similar arguments, we get the dual versions on $\mathscr{L}$-envelopes.

Theorem 11 *Assume that the Diagram 12 is a pushout-pullback with $I \in P(\mathscr{L})$.*

(1) Then $\alpha : C \to D$ is $\mathscr{L}$-enveloping if and only if $f : A \to B$ is $\mathscr{L}$-enveloping.

(2) If $B, D \in \mathscr{L}$, then $\alpha : C \to D$ is an $\mathscr{L}$-envelope if and only if $f : A \to B$ is an $\mathscr{L}$-envelope.

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