Wide Diameter of Generalized Petersen Graphs

Jun ZHANG\textsuperscript{1,3}, Xi Rong XU\textsuperscript{2,*}, Jun WANG\textsuperscript{1}

1. School of Mathematical Sciences, Dalian University of Technology, Liaoning 116024, P. R. China;
2. Department of Computer Science, Dalian University of Technology, Liaoning 110024, P. R. China;
3. College of Information Technology, Shanghai Ocean University, Shanghai 201306, P. R. China

Abstract Generalized Petersen graphs are commonly used interconnection networks, and wide diameter is an important parameter to measure fault-tolerance and efficiency of parallel processing computer networks. In this paper, we show that the diameter and 3-wide diameter of generalized Petersen graph $P(m,a)$ are both $O\left(\frac{m^2}{a}\right)$, where $a \geq 3$.

Keywords Petersen graph; diameter; wide diameter.

1. Introduction

Let $x$ and $y$ be two distinct vertices of a graph $G$. A set of internally disjoint $(x,y)$-paths in $G$ is called an $(x,y)$-container in $G$, denoted by $C(G; x, y)$. The number of paths in $C(G; x, y)$ is called the width of $C(G; x, y)$. An $(x, y)$-container with width $w$ is denoted by $C_w(G; x, y)$. The length of $C(G; x, y)$, denoted by $l(C(G; x, y))$, is the largest length of paths in $C(G; x, y)$.

Suppose that $G$ is a $w$-connected graph, $w \geq 1$. By Menger’s theorem there exists an $(x,y)$-container $C_w(G; x, y)$ for any pair of two distinct vertices $x$ and $y$ in $G$. The distance with width $w$, wide-distance or $w$-distance for short, from $x$ to $y$, denoted by $d_w(G; x, y)$, is defined as the minimum length over all $(x, y)$-containers $C_w(G; x, y)$. The diameter with width $w$, wide-diameter or $w$-diameter for short, of $G$, denoted by $d_w(G)$, is defined as

$$d_w(G) = \max\{d_w(G; x, y) : x, y \in V(G), x \neq y\}.$$ 

It is desirous that an ideal interconnection network $G$ should be one with connectivity $\kappa(G)$ as large as possible and diameter $d(G)$ as small as possible. The wide-diameter $d_w(G)$ combines connectivity $\kappa(G)$ and diameter $d(G)$, where $1 \leq w \leq \kappa(G)$. Hence $d_w(G)$ is a more suitable parameter than $d_w(G)$ to measure fault-tolerance and efficiency of parallel processing computer networks.
networks. Thus, determining the value of \( d_w(G) \) is of significance for a given graph \( G \) and an integer \( w \). However, Hsu [1] proved that this problem is \( NP \)-complete.

The concept of wide-diameter was proposed by Hsu and Lyuu [2], who used the term “wide-diameter” for the first time. Their study is based on the concept of the container stated above. Flandrin and Li [3] proposed this concept motivated by the property \( P_{l,w} \) by Faudree, Jacobson, Ordman, Schelp, and Tuza. A graph \( G \) has the property \( P_{l,w} \) implies that there exist \( w \) internally disjoint \((x,y)\)-paths in \( G \) of length at most \( l \) for any pair of two distinct vertices \( x \) and \( y \). It is clear that the graph \( G \) has the property that \( P_{l,w} \) is equivalent to that \( d_w(G) \leq l \).

Let \( m \) and \( a \) be integers such that \( m \geq 3 \), \( 1 \leq a < m \) and \( m > 2a \). For such \( m, a \), the generalized Petersen graph \( P(m, a) \) is defined by

\[
V(P(m, a)) = \{u_i, v_i : 0 \leq i \leq m - 1\},
\]

\[
E(P(m, a)) = \{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+a}) : 0 \leq i \leq m - 1\},
\]

where and in the sequel the subscripts of vertices are computed under modulo \( m \). When we draw the graph, we can order the vertices in \( U \) in one circle (called outer circle) and the vertices in \( V \) in one invisible circle (called inner circle).

Since \( P(m, a) \)'s form an important class of 3-regular and 3-connected graphs with \( 2m \) vertices and \( 3m \) edges, which are commonly used interconnection networks and have been studied by various researchers. It is desirable to determine \( P(m, a) \)'s wide diameter. Recently, Liaw and Chang [4] gave the wide diameter for many specific classes of networks. Krishnamoorth and Krishnamurthy [5] gave the fault diameter for generalized Petersen graph. Hou [6] gave the \( w \)-diameter for \( P(m, 2) \). In this paper, we give the 3-wide diameter of \( P(m, a) \) \((a \geq 3)\).

For two integers \( a \) and \( b \) with \( a \leq b \), by \([a, b]\) we denote the set \( \{a, a+1, \ldots, b\} \).

2. Statement of the main result

Lemma 1 The diameter of \( P(m, a) \) is \( O\left(\frac{m}{a}\right) \).

Proof The case of \( a = 1 \) is trivial, and the case of \( a = 2 \) has been proved in [6]. Now we consider the case of \( a = 3 \). Given an integer \( i \) \((\leq \frac{m}{3})\). When \( i \) is large enough, we consider the distances from \( u_0 \) to \( u_i \), \( u_0 \) to \( v_i \) and \( v_0 \) to \( v_i \). Since the vertices in inner circle jump in steps of 3, we will walk in the inner circle as much as possible. When \( i \equiv 0 \mod 3 \), we enter in the inner circle from vertex \( v_0 \); when \( i \equiv 1 \mod 3 \), we enter in the inner circle from vertex \( v_1 \); when \( i \equiv 2 \mod 3 \), we enter in the inner circle from vertex \( v_2 \), and leave the circle from vertex \( v_1 \). For example,

\[
u_0 - u_1 - u_2 - v_2 - v_3 - \cdots - v_{3k+2} - u_{3k+2},
\]

\[
v_0 - u_0 - u_1 - v_1 - v_4 - \cdots - v_{3k+1},
\]

where \( k \) is an integer. So for any two distinct vertices \( x \) and \( y \), \( \text{dist}(x, y) \leq \frac{m}{6} + \alpha(3) \), where \( \alpha(3) = 4 \). Given an integer \( s \leq \frac{m}{6} \), the subscript of vertex that any path with length \( s \) starting from \( u_0 \) can reach belongs to \([-3s, 3s]\), where \( \lceil \frac{m}{6} \rceil \) does not. So we have that the diameter of
$P(m, 3)$ is $O(\frac{m}{a})$. When $a > 3$, by the similar method, we yield $d(P(m, a)) = O(\frac{m}{a^2})$.

**Theorem 2** The 3-wide diameter of $P(m, a)$ is $O(\frac{m}{a^2})$, where $a \geq 3$.

**Proof** First we consider the case of $a = 3$. Let $x, y$ be any two distinct vertices of $P(m, 3)$. We will exhibit the path strategies when $x$ and $y$ are in different cases.

**Case 1** $x = u_0$ and $y = u_i$, $3 \leq i \leq \frac{m}{2}$. We consider three subcases:

**Case 1.1** $i \equiv 0 \pmod{3}$. We have three vertex-disjoint paths from $x$ to $y$,

$$P_1 : u_0 - v_0 - v_3 - \cdots - v_i - u_i,$$

$$P_2 : u_0 - u_1 - v_1 - v_4 - \cdots - v_1 + i - u_1 + i - u_i$$

and

$$P_3 : u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_{i-1} - u_{i-1} - u_i$$

with lengths $|P_1| = \frac{i}{3} + 2$, $|P_2| = \frac{i}{3} + 4$ and $|P_3| = \frac{i}{3} + 4$. And the container has length $\frac{i}{3} + 4$.

**Case 1.2** $i \equiv 1 \pmod{3}$. We have three vertex-disjoint paths from $x$ to $y$,

$$P_1 : u_0 - v_0 - v_3 - \cdots - v_{i-1} - u_{i-1},$$

$$P_2 : u_0 - u_1 - v_1 - v_4 - \cdots - v_i - u_i$$

and

$$P_3 : u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_{1+i} - u_{1+i} - u_i$$

with lengths $|P_1| = \frac{i-1}{3} + 3$, $|P_2| = \frac{i-1}{3} + 3$ and $|P_3| = \frac{i+1}{3} + 5$. And the container has length $\frac{i-1}{3} + 5$.

**Case 1.3** $i \equiv 2 \pmod{3}$. We have three vertex-disjoint paths from $x$ to $y$,

$$P_1 : u_0 - v_0 - v_3 - \cdots - v_{i+1} - u_{i+1},$$

$$P_2 : u_0 - u_1 - v_1 - v_4 - \cdots - v_{i-1} - u_{i-1} - u_i$$

and

$$P_3 : u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_i - u_i$$

with lengths $|P_1| = \frac{i-2}{3} + 4$, $|P_2| = \frac{i-2}{3} + 4$ and $|P_3| = \frac{i-2}{3} + 4$. And the container has length $\frac{i-2}{3} + 4$.

**Case 2** $x = u_0$ and $y = v_i$, $3 \leq i \leq \frac{m}{2}$. We consider three subcases:

**Case 2.1** $i \equiv 0 \pmod{3}$. We have three vertex-disjoint paths from $x$ to $y$,

$$P_1 : u_0 - v_0 - v_3 - \cdots - v_{i-3} - v_i,$$

$$P_2 : u_0 - u_1 - v_1 - v_4 - \cdots - v_{i-2} - u_{i-2} - u_{i-1} - u_i - v_i$$
and
\[ P_3 : u_0 - u_{i-1} - v_{i-1} - v_2 - \cdots - v_{i+2} - u_{i+2} - u_{i+3} - v_i \]
with lengths \(|P_1| = \frac{i-1}{3} + 1, |P_2| = \frac{i-1}{3} + 5\) and \(|P_3| = \frac{i}{3} + 7\). And the container has length \(\frac{i}{3} + 7\).

**Case 2.2** \(i \equiv 1 \pmod{3}\). We have three vertex-disjoint paths from \(x\) to \(y\),
\[ P_1 : u_0 - v_0 - v_3 - \cdots - v_{i-1} - u_{i-1} - u_i - v_i, \]
\[ P_2 : u_0 - u_1 - v_1 - v_4 - \cdots - v_{i-3} - v_i \]
and
\[ P_3 : u_0 - u_{i-1} - v_{i-1} - v_2 - \cdots - v_{i+1} - u_{i+1} - u_{i+2} - u_{i+3} - v_i \]
with lengths \(|P_1| = \frac{i-1}{3} + 4, |P_2| = \frac{i-1}{3} + 2\) and \(|P_3| = \frac{i-1}{3} + 8\). And the container has length \(\frac{i-1}{3} + 8\).

**Case 2.3** \(i \equiv 2 \pmod{3}\). We have three vertex-disjoint paths from \(x\) to \(y\),
\[ P_1 : u_0 - v_0 - v_3 - \cdots - v_{i-2} - u_{i-2} - u_{i-1} - u_i - v_i, \]
\[ P_2 : u_0 - u_1 - v_1 - v_4 - \cdots - v_{i+2} - u_{i+2} - u_{i+3} - v_{i+3} - v_i \]
and
\[ P_3 : u_0 - u_{i-1} - v_{i-1} - v_2 - \cdots - v_{i-3} - v_i \]
with lengths \(|P_1| = \frac{i-2}{3} + 5, |P_2| = \frac{i-2}{3} + 7\) and \(|P_3| = \frac{i-2}{3} + 3\). And the container has length \(\frac{i-2}{3} + 7\).

**Case 3** \(x = v_0\) and \(y = v_i, 3 \leq i \leq \frac{m}{2}\). We consider three subcases:

**Case 3.1** \(i \equiv 0 \pmod{3}\). We have three vertex-disjoint paths from \(x\) to \(y\),
\[ P_1 : v_0 - v_3 - \cdots - v_{i-3} - v_i, \]
\[ P_2 : v_0 - u_0 - u_1 - u_2 - v_2 - \cdots - v_{i+2} - u_{i+2} - u_{i+3} - v_{i+3} - v_i \]
and
\[ P_3 : v_0 - v_3 - u_3 - u_2 - v_2 - \cdots - v_{i-2} - u_{i-2} - u_{i-1} - u_i - v_i \]
with lengths \(|P_1| = \frac{1}{3}, |P_2| = \frac{4}{3} + 8\) and \(|P_3| = \frac{4}{3} + 8\). And the container has length \(\frac{4}{3} + 8\).

**Case 3.2** \(i \equiv 1 \pmod{3}\). We have three vertex-disjoint paths from \(x\) to \(y\),
\[ P_1 : v_0 - v_3 - \cdots - v_{i-1} - u_{i-1} - u_i - v_i, \]
\[ P_2 : v_0 - u_0 - u_1 - u_2 - v_2 - \cdots - v_{i+1} - u_{i+1} - u_{i+2} - u_{i+3} - v_{i+3} - v_i \]
and
\[ P_3 : v_0 - v_3 - u_3 - u_2 - v_2 - \cdots - v_i \]
with lengths \(|P_1| = \frac{i-1}{3} + 3, |P_2| = \frac{i-1}{3} + 9\) and \(|P_3| = \frac{i-1}{3} + 5\). And the container has length \(\frac{i-1}{3} + 9\).
Case 3.3 $i \equiv 2 \pmod{3}$. We have three vertex-disjoint paths from $x$ to $y$,

$$P_1 : v_0 - v_3 - \cdots - v_{i-2} - u_{i-2} - u_{i-1} - u_i - v_i,$$

$$P_2 : v_0 - u_0 - u_1 - v_4 - \cdots - v_{i+2} - u_{i+2} - u_{i+3} - v_{i+3} - v_i$$

and

$$P_3 : v_0 - v_{-3} - u_{-3} - u_{-2} - u_{-1} - v_{-1} - v_2 - \cdots - v_{i-3} - v_i$$

with lengths $|P_1| = \frac{i-2}{3} + 4$, $|P_2| = \frac{i-2}{3} + 8$ and $|P_3| = \frac{i-2}{3} + 6$. And the container has length $\frac{i-2}{3} + 8$.

So we get that $d_3(P(m, 3)) \leq \lfloor \frac{m}{6} \rfloor + 9$. Since $d(P(m, 3)) = O\left(\frac{m}{6}\right)$, we have that $d_3(P(m, 3)) = O\left(\frac{m}{6}\right)$.

Now we consider the general case, let $a > 3$. When $0 \leq k < a$, set $V_k = \{v_i : i \equiv k \pmod{a}, 0 \leq i \leq \frac{m}{2}\}$. When $i \equiv j \pmod{a}, 0 \leq j < a, 0 \leq i \leq \frac{m}{2}$, by employing $V_{j-1}, V_j, V_{j+1}$-paths, we can construct $(u_0, u_i), (u_0, v_i), (v_0, v_i)$-containers, similar to the case of $a = 3$. For example, when $a = 4, i \equiv 0 \pmod{4}$, we construct a $(v_0, v_i)$-container as follows:

$$P_1 : v_0 - v_4 - \cdots - v_{i-4} - v_i,$$

$$P_2 : v_0 - u_0 - u_1 - v_5 - \cdots - v_{i+1} - u_{i+1} - u_i - v_i$$

and

$$P_3 : v_0 - v_{-4} - u_{-4} - u_{-5} - v_{-5} - v_{-1} - v_3 - \cdots - v_{i+3} - u_{i+3} - u_{i+4} - v_{i+4} - v_i.$$ 

And in the similar way, we yield that $d_3(P(m, a)) = O\left(\frac{m}{2a}\right)$.

References


