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The Existence and Uniqueness for the Solution of Neutral Stochastic Functional Differential Equations with Infinite Delay

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Abstract In this paper, we will make use of a new method to study the existence and uniqueness for the solution of neutral stochastic functional differential equations with infinite delay (INSFDEs for short) in the phase space $BC((-\infty,0];R^d)$. By constructing a new iterative scheme, the existence and uniqueness for the solution of INSFDEs can be directly obtained only under uniform Lipschitz condition, linear grown condition and contractive condition. Meanwhile, the moment estimate of the solution and the estimate for the error between the approximate solution and the accurate solution can be both given. Compared with the previous results, our method is partially different from the Picard iterative method and our results can complement the earlier publications in the existing literatures.

Keywords existence and uniqueness; neutral stochastic functional differential equations; infinite delay.

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1. Introduction

To the best of our knowledge, many dynamical systems depend not only on present and past states but also involve derivative with delays as well as the functional of the past history. Neutral functional differential equations are often used to describe the following systems:

$$\frac{\mathrm{d}[x(t)-D(x_t)]}{\mathrm{d}t}=f(t,x_t),\quad t\in[0,T]. \tag{1.1}$$

Taking the environmental disturbances into account, we are led to a neutral stochastic functional differential equations with finite delay (NSFDEs for short):

$$\mathrm{d}[x(t)-D(x_t)]=f(t,x_t)\mathrm{d}t+g(t,x_t)\mathrm{d}B(t),\quad t\in[0,T]. \tag{1.2}$$

As for some other dynamical properties of NSFDEs, we can refer to [6–8, 10–13]. By using the well-known Picard iterative method, Mao [13] has discussed the existence and uniqueness for

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the solution of NSFDEs (1.2) under uniform Lipschitz condition, linear grown condition and contractive condition. In [13], Mao firstly studied the existence and uniqueness on every small interval into which the total time interval $[0, T]$ was felicitously subdivided, and then this result was gradually extended to the total time interval $[0, T]$, that is, the existence and uniqueness could indirectly be derived on the total interval $[0, T]$. For the detailed statements, the readers can refer to [13]. And, very recently, when introducing the phase space $BC((-\infty, 0]; R^d)$, the existence and uniqueness for the solution of stochastic functional differential equations with infinite delay can be gained without any great difficulties [1–2]. It should be pointed out that although Zhou et al. [5] have directly obtained the existence and uniqueness for the solution of INSFDEs on the total interval $[0, T]$ by utilizing the Picard iterative method, there are many restrictive conditions to be imposed in their paper and these conditions are of great importance to show their main results. Therefore, a problem to be solved is whether or not we can directly yield the existence and uniqueness to INSFDEs on the total interval $[0, T]$ in the phase space $BC((-\infty, 0]; R^d)$ only under uniform Lipschitz condition, linear grown condition and contractive condition.

In this paper, motivated by the presentations above, we shall further study the existence and uniqueness for the solution of INSFDEs in the phase space $BC((-\infty, 0]; R^d)$ by employing a new iterative method. Firstly, we should emphasize that this method is partially different from that used [13] and the key of this method is how to obtain the next one from the initial value. To overcome this difficulty in the proof, applying the fixed point theorem, we firstly show Lemma 3.4, and then can define the new iterative scheme to consider problem. We find that this method is not only used to deal with the existence and uniqueness for the solution of INSFDEs, but also to investigate the existence and uniqueness for NSFDEs only under uniform Lipschitz condition, linear grown condition and contractive condition. So, the Picard iterative scheme is not the optimal tool to study such problems. And what is more, we do not require any additional restrictive conditions. Satisfactorily, we can also give the moment estimate of the solution and the estimate for the error between the approximate solution and the accurate solution in this paper.

2. Preliminaries

Let $|\cdot|$ denote the Euclidean norm in R^d. If A is a vector or a matrix, its transpose is denoted by A^T; If A is a matrix, for no confusion, its Frobenius norm is also represented by $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. Throughout this paper unless otherwise specified, let $(\Omega,\ \mathcal{F},\ P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Assume that $B(t)$ is an m-dimensional standard Brownian motion defined on this complete probability space, that is, $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))^{\mathrm{T}}$. Let $BC((-\infty, 0]; R^d)$ be the family of bounded and continuous R^d-value function φ defined in $(-\infty, 0]$ with norm $\|\varphi\| = \sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$. And $\mathfrak{M}^2((-\infty, 0]; R^d)$ represents the family of all $\mathcal{F}_0$-measurable, R^d-valued process $\varphi(t) = \varphi(t, \omega)$, $t \in (-\infty, 0]$ such that $E\int_{-\infty}^0 |\varphi(t)|^2 \mathrm{d}t < \infty$.

In this paper, we mainly establish the existence and uniqueness theorem for the following

INSFDEs:

$$\mathrm{d}[x(t) - D(x_t)] = f(t, x_t)\mathrm{d}t + g(t, x_t)\mathrm{d}B(t), \quad t \in [0, T], \tag{2.1}$$

where $x_t = \{x(t+\theta) : (-\infty < \theta \leq 0\}$ can be regarded as a $BC((-\infty, 0]; R^d)$-value stochastic process and $f : [0, T] \times BC((-\infty, 0]; R^d) \to R^d$ and $g : [0, T] \times BC((-\infty, 0]; R^d) \to R^{m\times d}$ are Borel measurable. And the initial value of (2.1) is imposed as follows: $x_0 = \xi = \{\xi(\theta) : -\infty < \theta \leq 0\}$ is an $\mathcal{F}_0$-measurable, $BC((-\infty, 0]; R^d)$-value random variable such that

$$\xi \in \mathfrak{M}^2((-\infty, 0]; R^d). \tag{2.2}$$

Definition 1 ([13]) *An R^d-value stochastic process $x(t)$ defined on $-\infty < t \leq T$ is called the solution of (2.1) with the initial value (2.2), if it has the following properties:*

(i) $x(t)$ is continuous and for all $0 \leq t \leq T$, x_t is $\Im_t$-adapted;

(ii) $\{f(t, x_t)\} \in \mathfrak{L}^1([0, T]; R^d)$ and $\{g(t, x_t)\} \in \mathfrak{L}^2([0, T]; R^{d\times m})$;

(iii) $x_0 = \xi$, for each $0 \leq t \leq T$, $x(t) = D(x_t) + x(0) - D(x_0) + \int_0^t f(s, x_s)\mathrm{d}s + \int_0^t g(s, x_s)\mathrm{d}B(s)$, a.s.

Moreover, a solution $x(t)$ is said to be unique, if any other solution $\widetilde{x}(t)$ is indistinguishable from it, that is, $P(x(t) = \widetilde{x}(t)$, for all $-\infty < t \leq T)=1$.

In order to guarantee the existence and uniqueness for the solution of INSFDEs (2.1) with the initial value (2.2), some required conditions are assumed as follows:

(A1) (uniform Lipschitz condition) For all $\phi,\ \varphi \in BC((-\infty, 0]; R^d)$ and $t \in [0, T]$, it then follows that

$$|f(t, \phi) - f(t, \varphi)|^2 \vee |g(t, \phi) - g(t, \varphi)|^2 \leq K_1\|\phi - \varphi\|^2, \quad K_1 > 0;$$

(A2) (linear grown condition) For all $(t, \varphi) \in [0, T] \times BC((-\infty, 0]; R^d)$, it then follows that

$$|f(t, \varphi)|^2 \vee |g(t, \varphi)|^2 \leq K_2(1 + \|\varphi\|^2), \quad K_2 > 0;$$

(A3) (contractive condition) There exists a positive constant $\kappa \in (0, 1)$ such that

$$|D(\varphi) - D(\phi)| \leq \kappa\|\varphi - \phi\|,$$

for all $\varphi, \phi \in BC((-\infty, 0]; R^d)$.

Here, we define a space B_T which is the set of all functions $\xi(t, \omega) : (-\infty, T] \times \Omega \to R^d$ satisfying the condition: $\xi(t, \omega)$ is measurable in ω for each fixed $t \in (-\infty, T]$ and is bounded and continuous in t for a.e. fixed $\omega \in \Omega$. And with its norm:

$$\|\xi(t, \omega)\|_{B_T} = \{E(\sup_{t\in(-\infty,T]} |\xi(t, \omega)|^2)\}^{\frac{1}{2}}.$$

It is easily verified that the space B_T is a Banach space with this norm $\|\cdot\|_{B_T}$, the reader can refer to [3, 4, 9].

3. Main results

Lemma 3.1 ([13]) *For any $a, b \geq 0$ and $\alpha \in (0, 1)$, we have*

$$(a + b)^2 \leq \frac{a^2}{\alpha} + \frac{b^2}{1 - \alpha}.$$

Lemma 3.2 ([13]) *Let $p \geq 2$ and $g \in \mathfrak{L}^p([0,T]; R^{d\times m})$ such that $E\int_0^T |g(s)|^p ds < +\infty$. Then,*

$$E\Big(\sup_{0\leq t\leq T} |\int_0^t g(s)\mathrm{d}B(s)|^p\Big) \leq \Big(\frac{p^3}{2(p-1)}\Big)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p \mathrm{d}s.$$

In particular, if $p=2$, it follows $E(\sup_{0\leq t\leq T} |\int_0^t g(s)\mathrm{d}B(s)|^2) \leq 4E\int_0^T |g(s)|^2 ds$.

Lemma 3.3 *Assume that the conditions (A2) and (A3) hold. If $x(t)$ is a solution of INSFDEs (2.1) with the initial value (2.2), then one yields*

$$E(\sup_{-\infty<t\leq T} |x(t)|^2) \leq \Big[\frac{\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})} E\|\xi\|^2 + \frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big] \exp\Big(\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big). \tag{3.1}$$

In particular, $x(t)$ ($t \in (-\infty, T]$) belongs to B_T.

Proof For every integer $n \geq 1$, define the stopping time:

$$\tau_n = T \wedge \inf\{t \in [0,T]: \ \|x_t\| \geq n\}.$$

Obviously, as $n \to +\infty$, $\tau_n \uparrow T$ a.s. Let $x^n(t) = x(t\wedge\tau_n)$, for $t \in (-\infty, T]$. Then, for $t \in [0,T]$, $x^n(t)$ satisfy the following equation:

$$x^n(t) = D(x_t^n) + x(0) - D(x_0) + \int_0^t f(s, x_s^n)1_{[[0,\tau_n]]}(s)\mathrm{d}s + \int_0^t g(s, x_s^n)1_{[[0,\tau_n]]}(s)\mathrm{d}B(s).$$

From Lemma 3.1 and the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we obtain

$$\begin{aligned}
|x^n(t)|^2 =& |D(x_t^n) + x(0) - D(x_0) + \int_0^t f(s, x_s^n)1_{[[0,\tau_n]]}(s)\mathrm{d}s + \int_0^t g(s, x_s^n)1_{[[0,\tau_n]]}(s)\mathrm{d}B(s)|^2 \\
\leq& \frac{1}{\kappa}|D(x_t^n) - D(x_0)|^2 + \frac{3}{1-\kappa}|x(0)|^2 + \frac{3t}{1-\kappa}\int_0^t |f(s, x_s^n)1_{[[0,\tau_n]]}(s)|^2\mathrm{d}s + \\
& \frac{3}{1-\kappa}|\int_0^t g(s, x_s^n)1_{[[0,\tau_n]]}(s)\mathrm{d}B(s)|^2 \\
\leq& \sqrt{\kappa}\|x_t^n\|^2 + \frac{\kappa}{1-\sqrt{\kappa}}\|x_0\|^2 + \frac{3}{1-\kappa}|x(0)|^2 + \frac{3t}{1-\kappa}\int_0^t |f(s, x_s^n)1_{[[0,\tau_n]]}(s)|^2\mathrm{d}s + \\
& \frac{3}{1-\kappa}|\int_0^t g(s, x_s^n)1_{[[0,\tau_n]]}(s)\mathrm{d}B(s)|^2.
\end{aligned}$$

By Lemma 3.2, it is easily shown that

$$\begin{aligned}
E(\sup_{0\leq s\leq t} |x^n(s)|^2) \leq& \sqrt{\kappa}E(\sup_{-\infty<s\leq t} |x^n(s)|^2) + \frac{\kappa+\kappa\sqrt{\kappa}+3}{1-\kappa}E\|\xi\|^2 + \\
& \frac{3K_2(T+4)}{1-\kappa}\int_0^t (1 + E(\sup_{-\infty<r\leq s} |x^n(r)|^2))\mathrm{d}s.
\end{aligned}$$

Noticing that $E(\sup_{-\infty<s\leq t} |x^n(s)|^2) \leq E\|\xi\|^2 + E(\sup_{0\leq s\leq t} |x^n(s)|^2)$, we have

$$\begin{aligned}
E(\sup_{-\infty<s\leq t} |x^n(s)|^2) \leq& \sqrt{\kappa}E(\sup_{-\infty<s\leq t} |x^n(s)|^2) + \frac{\kappa\sqrt{\kappa}+4}{1-\kappa}E\|\xi\|^2 + \frac{3K_2(T+4)T}{1-\kappa} + \\
& \frac{3K_2(T+4)}{1-\kappa}\int_0^t E(\sup_{-\infty<r\leq s} |x^n(r)|^2)\mathrm{d}s.
\end{aligned}$$

$$
\begin{aligned}
E(\sup_{-\infty<s\leq t}|x^n(s)|^2)\leq & \frac{\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})}E\|\xi\|^2+\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}+\\
& \frac{3K_2(T+4)}{(1-\kappa)(1-\sqrt{\kappa})}\int_0^t E(\sup_{-\infty<r\leq s}|x^n(r)|^2)\mathrm{d}s.
\end{aligned}
$$

From Gronwall inequality, we have

$$
E(\sup_{-\infty<s\leq t}|x(s)|^2)\leq\Big[\frac{\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})}E\|\xi\|^2+\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big]\exp\Big(\frac{3K_2(T+4)}{(1-\kappa)(1-\sqrt{\kappa})}t\Big).
$$

Let $t=T$. Then it follows that

$$
E(\sup_{-\infty<t\leq T}|x(s)|^2)\leq\Big[\frac{\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})}E\|\xi\|^2+\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big]\exp\Big(\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big).
$$

Consequently,

$$
E(\sup_{-\infty<t\leq \tau_n}|x(t)|^2)\leq\Big[\frac{\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})}E\|\xi\|^2+\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big]\exp\Big(\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big).
$$

Finally, the required inequality (3.1) follows by letting $n\to+\infty$. The proof is completed. □

To derive our main results in this paper, firstly, we define an operator $\Pi: B_T\to B_T$, that is,

$$
(\Pi x)(t):=\begin{cases} D(x_t)+x(0)-D(x_0)+\int_0^t f(s)\mathrm{d}s+\int_0^t g(s)\mathrm{d}B(s),\ t\in[0,T],\\ \xi\in BC((-\infty,0];R^d),\quad t\in(-\infty,0].\end{cases}
$$

Lemma 3.4 *Assume that the condition (A3) holds, then the operator $\Pi:\ B_T\to B_T$ has a unique fixed point.*

Proof We first verify the mean square continuity of the operator Π on $[0,T]$. Let $x\in B_T$, $t'\in(0,T)$, and $|h|$ be sufficiently small. Applying Lemma 3.1 and Hölder inequality, we obtain

$$
\begin{aligned}
E(|(\Pi x)(t'+h)-(\Pi x)(t')|^2)=&E|D(x_{t'+h})-D(x_{t'})+\int_{t'}^{t'+h}f(s)\mathrm{d}s+\int_{t'}^{t'+h}g(s)\mathrm{d}B(s)|^2\\
\leq&\kappa E(\sup_{-\infty<\theta\leq 0}|x(t'+h+\theta)-x(t'+\theta)|^2)+\\
&\frac{2h}{1-\kappa}\int_{t'}^{t'+h}E|f(s)|^2\mathrm{d}s+\frac{2}{1-\kappa}E|\int_{t'}^{t'+h}g(s)\mathrm{d}B(s)|^2.
\end{aligned}
$$

Owing to the fact that $E|\int_{t'}^{t'+h}g(s)\mathrm{d}B(s)|^2=\int_{t'}^{t'+h}E|g(s)|^2\mathrm{d}s$, it follows

$$
\begin{aligned}
E(|(\Pi x)(t'+h)-(\Pi x)(t')|^2)\leq&\kappa E(\sup_{-\infty<\theta\leq 0}|x(t'+h+\theta)-x(t'+\theta)|^2)+\\
&\frac{2h}{1-\kappa}\int_{t'}^{t'+h}E|f(s)|^2ds+\frac{2}{1-\kappa}\int_{t'}^{t'+h}E|g(s)|^2\mathrm{d}s\to 0,
\end{aligned}
$$

as $|h|\to 0$, the operator Π is indeed the mean square continuous on $[0,T]$.

Next, we show that $\Pi(B_T)\subset B_T$. Let $x\in B_T$. From (3.1), Lemmas 3.1 and 3.2, we have

$$
\begin{aligned}
E(\sup_{-\infty<t\leq T}|(\Pi x)(t)|^2)\leq&E(\sup_{0\leq t\leq T}|(\Pi x)(t)|^2)+E(\sup_{-\infty<\theta\leq 0}|(\Pi x)(\theta)|^2)\\
\leq&\sqrt{\kappa}E(\sup_{-\infty<t\leq T}|x(t)|^2)+\frac{\kappa\sqrt{\kappa}+4}{1-\kappa}E\|\xi\|^2+
\end{aligned}
$$

$$\frac{3T}{1-\kappa}\int_0^T E|f(t)|^2\mathrm{d}t+\frac{12}{1-\kappa}\int_0^T E|g(t)|^2\mathrm{d}t<+\infty.$$

Therefore, it implies that $\Pi(B_T)\subset B_T$.

Finally, we prove that Π is a contractive map. In fact, for any $x,\ y\in B_T$, throughout the standard computation, we yield

$$E(\sup_{-\infty<t\le T}|(\Pi x)(t)-(\Pi y)(t)|^2)\le k^2E(\sup_{-\infty<t\le T}|x(t)-y(t)|^2).$$

That is, the operator Π is contractive on B_T. Thus, the operator Π has a unique fixed point. The proof is completed. $\Box$

Up to now, we can define the following iterative scheme:

$$\begin{aligned}&x^n(t)=\xi(t),\quad t\in(-\infty,0],\ n=0,1,2,\ldots,\\&x^0(t)=\xi(0),\quad t\in[0,T],\\&x^n(t)=D(x_t^n)+x(0)-D(x_0)+\int_0^t f(s,x_s^{n-1})\mathrm{d}s+\int_0^t g(s,x_s^{n-1})\mathrm{d}B(s),\\&\qquad t\in[0,T],\ n=1,2,\ldots.\end{aligned}\tag{3.2}$$

Theorem 3.5 *Assume that the conditions: (A1)–(A3) hold, then there exists a unique solution $x(t)$ to INSFDEs (2.1) with the initial value (2.2). Moreover, the solution $x(t)$ ($t\in(-\infty,T]$) belongs to B_T.*

Proof Existence. Obviously, $x^0(t)\in B_T$ ($t\in(-\infty,T]$). Moreover, we easily show that $x^n(t)\in B_T$, for $t\in(-\infty,T]$ and $n=1,2,\ldots$. In fact, from (3.2), Lemmas 3.1 and 3.2, it is easily obtained that

$$\begin{aligned}|x^n(t)|^2\le&\sqrt{\kappa}\|x_t^n\|^2+\frac{\kappa}{1-\sqrt{\kappa}}\|x_0\|^2+\frac{3}{1-\kappa}|x(0)|^2+\frac{3K_2T\int_0^t(1+\|x_s^{n-1}\|^2)\mathrm{d}s}{1-\kappa}+\\&\frac{3|\int_0^t g(s,x_s^{n-1})\mathrm{d}B(s)|^2}{1-\kappa}.\end{aligned}$$

$$\begin{aligned}E(\sup_{0\le s\le t}|x^n(s)|^2)\le&\sqrt{\kappa}E(\sup_{-\infty<s\le t}|x^n(s)|^2)+\frac{\kappa+\kappa\sqrt{\kappa}+3}{1-\kappa}E\|\xi\|^2+\\&\frac{3K_2(T+4)}{1-\kappa}\int_0^t(1+E(\sup_{-\infty<r\le s}|x^{n-1}(r)|^2))\mathrm{d}s.\end{aligned}$$

In view of $E(\sup_{-\infty<s\le t}|x^n(s)|^2)\le E\|\xi\|^2+E(\sup_{0\le s\le t}|x^n(s)|^2)$, we have

$$\begin{aligned}E(\sup_{-\infty<s\le t}|x^n(s)|^2)\le&\sqrt{\kappa}E(\sup_{-\infty<s\le t}|x^n(s)|^2)+\frac{\kappa\sqrt{\kappa}+4}{1-\kappa}E\|\xi\|^2+\\&\frac{3K_2(T+4)T}{1-\kappa}+\frac{3K_2(T+4)}{1-\kappa}\int_0^t E(\sup_{-\infty<r\le s}|x^{n-1}(r)|^2)\mathrm{d}s.\end{aligned}$$

$$\begin{aligned}E(\sup_{-\infty<s\le t}|x^n(s)|^2)\le&\frac{\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})}E\|\xi\|^2+\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}+\\&\frac{3K_2(T+4)}{(1-\kappa)(1-\sqrt{\kappa})}\int_0^t E(\sup_{-\infty<r\le s}|x^{n-1}(r)|^2)\mathrm{d}s.\end{aligned}$$

For any $N \geq 1$, one yields

$$\max_{1\leq n\leq N} E(\sup_{-\infty<s\leq t} |x^n(s)|^2) \leq \frac{3K_2(T+4)T+\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})} E\|\xi\|^2 + \frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})} + \frac{3K_2(T+4)}{(1-\kappa)(1-\sqrt{\kappa})} \int_0^t \max_{1\leq n\leq N} E(\sup_{-\infty<r\leq s} |x^n(r)|^2)) \mathrm{d}s.$$

Applying Gronwall inequality gives

$$\max_{1\leq n\leq N} E(\sup_{-\infty<s\leq t} |x^n(s)|^2) \leq \Big(\frac{3K_2(T+4)T+\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})} E\|\xi\|^2 + \frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big) \times \exp\Big(\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big).$$

Since N is arbitrary, we deduce

$$E(\sup_{-\infty<s\leq t} |x^n(s)|^2) \leq \Big(\frac{3K_2(T+4)T+\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})} E\|\xi\|^2 + \frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big) \exp\Big(\frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}\Big)$$

for all $t \in [0,T]$ and $n = 1,2,\ldots$. Hence, $x^n(t) \in B_T$, for all $t \in (-\infty, T]$ and $n = 0,1,2,\ldots$.

From (3.2), Lemmas 3.1 and 3.2 again, we get

$$|x^1(t)|^2 \leq \sqrt{\kappa} \sup_{-\infty<s\leq t} |x^1(s)|^2 + \frac{\kappa+\kappa\sqrt{\kappa}+3}{1-\kappa}\|\xi\|^2 + \frac{3T}{1-\kappa}\int_0^t |f(s,x_s^0)|^2 \mathrm{d}s + \frac{3}{1-\kappa}\Big|\int_0^t g(s,x_s^0)\mathrm{d}B(s)\Big|^2,$$

$$\sup_{0\leq s\leq t} |x^1(s)|^2 \leq \sqrt{\kappa} \sup_{-\infty<s\leq t} |x^1(s)|^2 + \frac{\kappa+\kappa\sqrt{\kappa}+3}{1-\kappa}\|\xi\|^2 + \frac{3K_2T}{1-\kappa}\int_0^t (1+\|x_s^0\|^2) \mathrm{d}s + \frac{3}{1-\kappa}\sup_{0\leq s\leq t}\Big|\int_0^t g(s,x_s^0)\mathrm{d}B(s)\Big|^2.$$

Hence, it is easily shown that

$$E(\sup_{-\infty<t\leq T} |x^1(s)|^2) \leq \frac{\kappa\sqrt{\kappa}+4}{(1-\kappa)(1-\sqrt{\kappa})} E\|\xi\|^2 + \frac{3K_2(T+4)T}{(1-\kappa)(1-\sqrt{\kappa})}(1+E\|\xi\|^2) \equiv C_1.$$

Next, we note that

$$\begin{aligned} |x^1(t)-x^0(t)|^2 \leq& \kappa \sup_{-\infty<\theta\leq 0} |x^1(t+\theta)-x(\theta)|^2 + \frac{2K_2T^2}{1-\kappa}(1+\|\xi\|^2) + \frac{2}{1-\kappa}\Big|\int_0^t g(s,x_s^0)\mathrm{d}B(s)\Big|^2 \\ \leq& \sqrt{\kappa}\sup_{-\infty<s\leq t}|x^1(s)|^2 + \frac{\kappa\sqrt{\kappa}+\kappa}{1-\kappa}\|\xi\|^2 + \frac{2K_2T^2}{1-\kappa}(1+\|\xi\|^2) + \frac{2}{1-\kappa}\Big|\int_0^t g(s,x_s^0)\mathrm{d}B(s)\Big|^2. \end{aligned}$$

$$E(\sup_{0\leq s\leq t} |x^1(s)-x^0(s)|^2) \leq \sqrt{\kappa}E(\sup_{-\infty<s\leq t}|x^1(s)|^2) + \frac{\kappa\sqrt{\kappa}+\kappa}{1-\kappa}E\|\xi\|^2 + \frac{2K_2(T+4)T}{1-\kappa}(1+E\|\xi\|^2).$$

$$
\begin{aligned}
E(\sup_{-\infty<t\leq T}|x^1(t)-x^0(t)|^2) =& E(\sup_{0\leq t\leq T}|x^1(t)-x^0(t)|^2)\\
\leq& \sqrt{\kappa}C_1+\frac{\kappa\sqrt{\kappa}+\kappa}{1-\kappa}E\|\xi\|^2+\frac{2K_2(T+4)T}{1-\kappa}(1+E\|\xi\|^2)\\
\equiv& C. \qquad (3.3)
\end{aligned}
$$

Now, for all $n\geq 0$ and $t\in[0,T]$, we claim that

$$
E(\sup_{-\infty<r\leq t}|x^{n+1}(r)-x^n(r)|^2)\leq C\Big[\frac{2K_2(T+4)}{(1-\kappa)^2}\Big]^n\frac{t^n}{n!}, \qquad (3.4)
$$

where the constant C is defined in (3.3). We shall show this by induction. In view of (3.3), it is easily seen that (3.4) holds when $n=0$. Under the inductive assumption that (3.4) holds for some $n\geq 0$. We shall show that (3.4) still holds for $n+1$. Note that

$$
\begin{aligned}
|x^{n+2}(t)-x^{n+1}(t)|^2 =& |D(x_t^{n+2})-D(x_t^{n+1})+\int_0^t(f(s,x_s^{n+1})-f(s,x_s^n))\mathrm{d}s+\\
&\int_0^t(g(s,x_s^{n+1})-g(s,x_s^n))\mathrm{d}B(s)|^2\\
\leq& \kappa\sup_{-\infty<s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2+\frac{2t}{1-\kappa}\int_0^t|f(s,x_s^{n+1})-f(s,x_s^n)|^2\mathrm{d}s+\\
&\frac{2}{1-\kappa}|\int_0^t(g(s,x_s^{n+1})-g(s,x_s^n))\mathrm{d}B(s)|^2.
\end{aligned}
$$

Using Lemma 3.2, we have

$$
\begin{aligned}
E(\sup_{-\infty<s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2) =& E(\sup_{0\leq s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2)\\
\leq& \kappa E(\sup_{-\infty<s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2)+\\
&\frac{2K_1t}{1-\kappa}\int_0^t E(\sup_{-\infty<r\leq s}|x^{n+1}(r)-x^n(r)|^2)\mathrm{d}s+\\
&\frac{8K_1}{1-\kappa}\int_0^t E(\sup_{-\infty<r\leq s}|x^{n+1}(r)-x^n(r)|^2)\mathrm{d}s,\\
E(\sup_{-\infty<s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2) \leq& \frac{2K_1(T+4)}{(1-\kappa)^2}\int_0^t E(\sup_{-\infty<r\leq s}|x^{n+1}(r)-x^n(r)|^2)\mathrm{d}s\\
\leq& C\Big[\frac{2K_2(T+4)}{(1-\kappa)^2}\Big]^{n+1}\frac{t^{n+1}}{(n+1)!}.
\end{aligned}
$$

That is, (3.4) holds for $n+1$. Hence, by induction, (3.4) holds for all $n\geq 0$.

For any $m>n\geq 1$, we obtain

$$
\begin{aligned}
\|x^m-x^n\|_{B_T} =& \Big[E(\sup_{-\infty<t\leq T}|x^m(t)-x^n(t)|^2)\Big]^{\frac{1}{2}}\\
\leq& \sum_{k=n}^{+\infty}\Big[E(\sup_{-\infty<t\leq T}|x^{k+1}(t)-x^k(t)|^2)\Big]^{\frac{1}{2}}\\
\leq& \sum_{k=n}^{+\infty}\Big[C\Big(\frac{2K_1(T+4)}{(1-\kappa)^2}\Big)^k\frac{T^k}{k!}\Big]^{\frac{1}{2}}\to 0,
\end{aligned}
$$

as $n \to +\infty$. Thus, $\{x^n(t)\}_{n\geq 1}$ $(t \in (-\infty, T])$ is a Cauchy sequence in Banach space B_T. Denote the limit by $x(t) \in B_T$ $(t \in (-\infty, T])$. Now, letting $n \to +\infty$ in (3.2), we can derive the solution of INSFDEs (2.1) with the initial value (2.2). In the other words, we have shown the existence.

Uniqueness. Let $x_1(t)$ and $x_2(t)$ $(t \in (-\infty, T])$ be both the solution of INSFDEs (2.1) with the initial value (2.2). Then, by the similar computation, we yield

$$\begin{aligned} E(\sup_{-\infty<s\leq t} |x_1(s) - x_2(s)|^2) =& E(\sup_{0\leq s\leq t} |x_1(s) - x_2(s)|^2) \\ \leq& \frac{2K_1(T+4)}{(1-\kappa)^2} \int_0^t E(\sup_{-\infty<r\leq s} |x_1(r) - x_2(r)|^2)\mathrm{d}s, \quad \text{for } t \in [0, T]. \end{aligned}$$

From Gronwall inequality, it follows

$$E(\sup_{-\infty<s\leq t} |x_1(s) - x_2(s)|^2) = 0, \quad \text{for all } t \in [0, T].$$

Hence,

$$\|x_1 - x_2\|_{B_T}^2 = E(\sup_{-\infty<t\leq T} |x_1(t) - x_2(t)|^2) = 0,$$

that is, the uniqueness is also proved. The proof is completed. □

Theorem 3.6 *Assume that the conditions of Theorem 3.5 hold. Let $x(t)$ $(t \in (-\infty, T])$ be the unique solution of INSFDEs (2.1) with the initial value (2.2) and $\{x^n(t)\}_{n\geq 1}$ $(t \in (-\infty, T])$ be the iterative sequence defined by (3.2). Then, for $n \geq 1$,*

$$E(\sup_{-\infty<t\leq T} |x^n(t) - x(t)|^2) \leq C \exp\Big(\frac{4K_1(T+4)T}{(1-\kappa)^2}\Big)\Big(\frac{2K_1(T+4)}{(1-\kappa)^2}\Big)^n \frac{T^n}{n!},$$

where the constant C is defined in Theorem 3.5.

Proof For $n \geq 1$, similarly, we can compute

$$E(\sup_{-\infty<s\leq t} |x^{n+1}(s) - x(s)|^2) \leq \frac{2K_1(T+4)}{(1-\kappa)^2} \int_0^t E(\sup_{-\infty<r\leq s} |x^n(r) - x(r)|^2)\mathrm{d}s. \tag{3.5}$$

By virtue of (3.4) and (3.5), we deduce

$$\begin{aligned} E(\sup_{-\infty<s\leq t} |x^n(s) - x(s)|^2) =& E(\sup_{-\infty<s\leq t} |x^n(s) - x^{n+1}(s) + x^{n+1}(s) - x(s)|^2) \\ \leq& 2E(\sup_{-\infty<t\leq T} |x^n(t) - x^{n+1}(t)|^2) + 2E(\sup_{-\infty<s\leq t} |x^{n+1}(s) - x(s)|^2) \\ \leq& C\Big(\frac{2K_1(T+4)}{(1-\kappa)^2}\Big)^n \frac{T^n}{n!} + \frac{4K_1(T+4)}{(1-\kappa)^2} \times \\ & \int_0^t E(\sup_{-\infty<r\leq s} |x^n(r) - x(r)|^2)\mathrm{d}s. \end{aligned}$$

Utilizing Gronwall inequality, one yields that

$$E(\sup_{-\infty<s\leq t} |x^n(s) - x(s)|^2) \leq C \exp\Big(\frac{4K_1(T+4)T}{(1-\kappa)^2}\Big)\Big(\frac{2K_1(T+4)}{(1-\kappa)^2}\Big)^n \frac{T^n}{n!}.$$

Hence, letting $t = T$, we have

$$E(\sup_{-\infty<t\leq T} |x^n(t) - x(t)|^2) \leq C \exp\Big(\frac{4K_1(T+4)T}{(1-\kappa)^2}\Big)\Big(\frac{2K_1(T+4)}{(1-\kappa)^2}\Big)^n \frac{T^n}{n!}.$$

The proof is completed. □

Remark 3.1 From the whole argument above, only under uniform Lipschitz condition, linear grown condition and contractive condition, by constructing a new iterative scheme, we can directly obtain the existence and uniqueness for the solution of INSFDEs (2.1) with the initial value (2.2) on the total interval $[0, T]$ in the phase space $BC((-\infty, 0]; R^d)$. Thus, we improve the results in [5]. Besides this, we can also give the moment estimate of the solution and the estimate for the error between the approximate solution and the accurate solution in this paper.

Remark 3.2 Similarly to [1, 13], we could also investigate the existence and uniqueness for the solution of INSFDEs in the phase space $BC((-\infty, 0]; R^d)$ on the entire interval $[0, +\infty)$.

Remark 3.3 In contrast to [13], we can also directly study the existence and uniqueness for the solution of NSFDEs by employing this new iterative method. Hence, this method can complement the shortage of Picard iterative method in [13].

Remark 3.4 We could also discuss the existence and uniqueness for the solution of INSFDEs at the phase space $BC((-\infty, 0]; R^d)$ under the non-Lipschitz condition used in [2, 3].

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References

[1] WEI Fengying, WANG Ke. *The existence and uniqueness of the solution for stochastic functional differential equations with infinite delay* [J]. J. Math. Anal. Appl., 2007, **331**(1): 516–531.
[2] REN Yong, LU Shiping, XIA Ningmao. *Remarks on the existence and uniqueness of the solutions to stochastic functional differential equations with infinite delay* [J]. J. Comput. Appl. Math., 2008, **220**(1-2): 364–372.
[3] TANIGUCHI T. *Successive approximations to solutions of stochastic differential equations* [J]. J. Differential Equations, 1992, **96**(1): 152–169.
[4] RODKINA A E. *On existence and uniqueness of solution of stochastic differential equations with heredity* [J]. Stochastics, 1984, **12**(3-4): 187–200.
[5] ZHOU Shaobo, XUE Minggao. *The existence and uniqueness of the solutions for neutral stochastic functional differential equations with infinite delay* [J]. Math. Appl. (Wuhan), 2008, **21**(1): 75–83.
[6] MAO Xuerong. *Asymptotic properties of neutral stochastic differential delay equations* [J]. Stochastics Stochastics Rep., 2000, **68**(3-4): 273–295.
[7] MAO Xuerong. *Razumikhin-type theorems on exponential stability of neutral stochastic functional-differential equations* [J]. SIAM J. Math. Anal., 1997, **28**(2): 389–401.
[8] MAO Xuerong. *Exponential stability in mean square of neutral stochastic differential-functional equations* [J]. Systems Control Lett., 1995, **26**(4): 245–251.
[9] XU Daoyi, YANG Zhiguo, HUANG Yumei. *Existence-uniqueness and continuation theorems for stochastic functional differential equations* [J]. J. Differential Equations, 2008, **245**(6): 1681–1703.
[10] LUO Qi, MAO Xuerong, SHEN Yi. *New criteria on exponential stability of neutral stochastic differential delay equations* [J]. Systems Control Lett., 2006, **55**(10): 826–834.
[11] LIU Kai, XIA Xuewen. *On the exponential stability in mean square of neutral stochastic functional differential equations* [J]. Systems Control Lett., 1999, **37**(4): 207–215.
[12] WU Fuke, MAO Xuerong. *Numerical solutions of neutral stochastic functional differential equations* [J]. SIAM J. Numer. Anal., 2008, **46**(4): 1821–1841.
[13] MAO Xuerong. *Stochastic Differential Equations and Applications* [M]. Horwood Publishing Limited, Chichester, 1997.
[14] MOHAMMED S E A. *Stochastic Functional Differential Equations* [M]. Pitman, Boston, MA, 1984.