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Fuzzy Multi-Objective Semi-Definition Programming

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Abstract This paper first applies the fuzzy set theory to multi-objective semi-definite programming (MSDP), and proposes the fuzzy multi-objective semi-definite programming (FMSDP) model whose optimal efficient solution is defined for the first time, too. By constructing a membership function, the FMSDP is translated to the MSDP. Then we prove that the optimal efficient solution of FMSDP is consistent with the efficient solution of MSDP and present the optimality condition about these programming. At last, we give an algorithm for FMSDP by introducing a new membership function and a series of transformation.

Keywords fuzzy multi-objective semi-definite programming; membership function; optimality efficient solution; efficient solution; optimality condition.

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0. Introduction

In the optimization of project, it is difficult to precisely describe the objective function and the constrained function of many practical problems precisely. Therefore, fuzzy set theory was applied to variety of optimization, which was called fuzzy programming [1–4]. At the same time, in practical issues, many optimization objective functions consist of more than one function, which makes the fuzzy multi-objective optimization become one of the most active areas in recent years.

Tanaka et al. [5, 6] discussed the formulating of the fuzzy multi-objective linear programming problem for the case where the coefficients in the linear objective and linear constraints can be represented by trapezoidal fuzzy numbers. The extension to the nonlinear cases was made by Orlovski [7] who discussed the multi-objective nonlinear programming problem containing fuzzy parameters. Then, Slowiński and Teghem [8] made the comparisons between fuzzy optimization and stochastic optimization for multi-objective programming problems.

In addition, fuzzy mathematical programming offers a powerful means of handling optimization problems with non-stochastic imprecision and vagueness. A detailed survey on fuzzy

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optimization has also been made by Luhangdjula [2]. Several papers [9–12] had considered fuzzy linear programming or fuzzy multi-objective linear programming problems and proposed a series of ideas of translating the original chance constraints into crisp equivalents via possibility theory provided by Zadeh and Dubois and Prade [13, 14]. Furthermore, Chakraborty et al. [15] discussed multi-objective imprecise-chance constraints programming problems. However, with the development of more effective computer and intelligent algorithms, many new complex optimization problems can be processed by digital computers. Thus, Liu and Iwamura [16, 17] presented a framework of nonlinear chance-constrained programming as well as multi-objective case and goal programming with fuzzy coefficients, provided a fuzzy simulation-based genetic algorithm to solve general chance-constrained programming models and illustrated the genetic algorithm.

This paper first applies the fuzzy set theory to multi-objective semi-definition programming (MSDP). In Section 1, we establish the constrained fuzzy multi-objective semi-definite optimization(FMSDP) model, set a precise membership function and translate this model into MSDP whose objective functions are decided by constraint functions by setting a precise membership function. In Section 2, we give the definition of the optimality efficient solution of FMSDP for the first time. In Section 3, we prove that the optimality efficient solution of FMSDP and the efficient solution of MSDP are consistent. In Section 4, we discuss their optimality conditions. In Section 5, we present a theory algorithm of FMSDP.

1. Problem formulation

In this paper, we establish the following forms of FMSDP model:

$$
\text{(FMSDP)} \qquad \begin{array}{l} \min f(X) = (\ f_1(X), f_2(X), \ldots, f_p(X)\ )^{\mathrm{T}} \\ \text{s.t. } g(X) = (\ g_1(X), g_2(X), \ldots, g_m(X)\ )^{\mathrm{T}} \underset{\approx}{\leq} 0 \qquad X \geq 0. \\ h(X) = (h_{m+1}(X), h_{m+2}(X), \ldots, h_n(X)\ )^{\mathrm{T}} \underset{\approx}{=} 0 \end{array}
$$

Denote by S^n the $n \times n$ real symmetric matrix set, and S^n_+ the $n \times n$ real symmetric positive semi-definite matrix set. $X \geq 0$ means that $X \in S^n_+$. And " $\underset{\approx}{\leq}$ " means that "almost less than or equal to", " $\underset{\approx}{=}$ " means that "almost equivalent". $f_i(X)$ $(i = 1, 2, \ldots, p)$, $g_j(X)$ $(j = 1, 2, \ldots, m)$, $h_j(X)$ $(j = m+1, m+2, \ldots, n)$ are inner product of X and the other known matrices.

Let $M = \{X \in\ S^n | g(X) \underset{\approx}{\leq} 0, h(X) \underset{\approx}{\leq} 0, x \geq 0\}$.

Suppose A, $B \in S^n$, then $A \cdot B = \mathrm{tr}(AB)$ means the inner product of A and B. When $A = (a_{ij})_{n \times n} \in S^n_+$,

$$
\mathrm{nvec}(A) = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn})^{\mathrm{T}},
$$

So, $A \cdot B = \mathrm{nvec}(A)^{\mathrm{T}} \cdot \mathrm{nvec}(B)$.

Assume that $t_i > 0$ is the i-th constraint tolerance, then $g(X) \underset{\approx}{\leq} 0$ is a fuzzy set, and its membership function is defined as

$$
\mu_j(X) = \begin{cases} 0, & g_j(X) \leq 0, \\ \frac{g_j(X)}{t_j}, & g_j(X) > 0, \end{cases} \tag{1.1}
$$

where $j = 1, 2, \ldots, m$.

$$\mu_j(X) = \frac{h_j^2(X)}{t_j^2}, \tag{1.2}$$

where $j = m+1, m+2, \ldots, n$. Then (FMSDP) can be transformed into

$$\text{(MSDP)} \quad \begin{array}{l} \min F(X) = (f_1(X), f_2(X), \ldots, f_p(X), \mu_1(X), \ldots, \mu_n(X))^{\mathrm{T}} \\ \text{s.t. } G(X) = (g_1(X), g_2(X), \ldots, g_m(X))^{\mathrm{T}} \overset{\leq}{\approx} a, \\ H(X) = (h_{m+1}^2(X), h_{m+2}^2(X), \ldots, h_n^2(X))^T \overset{\leq}{\approx} b, \end{array} \quad X \geq 0,$$

where $a = (t_1, t_2, \ldots, t_m)^{\mathrm{T}}$, $b = (t_{m+1}^2, t_{m+2}^2, \ldots, t_n^2)^{\mathrm{T}}$.

Let $N = \{X \in S^n | G(X) \leq a, H(X) \leq b, X \geq 0\}$. So, solving (FMSDP) is transformed into solving (MSDP).

2. Preliminaries

Definition 1 *Assume $D \subset S^n$, I and J are finite index set, one of them can be empty set, and $\phi_k(X)$ $(k \in I \cup J)$ is the real function on D. If there exists a solution $X^0 \in D$ for the inequalities*

$$\phi_i(X) < 0, i \in I, \phi_j(X) \leq 0, \quad j \in J, \tag{2.1}$$

we say the inequalities (2.1) are consistent on D, otherwise, (2.1) are said to be inconsistent on D.

Definition 2 *If there does not exist an $X \in M$ such that*

1) $f(X) \leq f(X^*)$, $f(X) \neq f(X^*)$, when $0 < g_j(X) \leq t_j$, we have $g_j(X) \leq g_j(X^*)$ $(j = 1, 2, \ldots, m)$. When $h_j(X) \neq 0$, we have $|h_j(X)| \leq |h_j^*(X)| (j = m+1, m+2, \ldots, n)$.

2) $f(X) \leq f(X^*)$, when $0 < g_j^*(X) \leq t_j$, we have $g(X) \leq g(X^*)$ $(j = 1, 2, \ldots, m)$. When $h_j^*(X) \neq 0$, we have $h_j(X) \neq 0, |h_j(X)| \leq |h_j^*(X)| (j = m+1, m+2, \ldots, n)$, and at least there exists j, such that $g_j(X) < g_j(X^*)$ or $|h_j(X)| \leq |h_j(X^*)|$.

We say X^* is the optimal efficient solution of (FMSDP).

Definition 3 *Assume $X^* \in R$. If there is no $X \in N$ such that $F(X) \leq F(X^*)$, $F(X) \neq F(X^*)$, then X^* is claimed to be the efficient solution of (MSDP).*

Definition 4 *Assume $D \subset R^{m \times n}$, the function $f : D \to R^1, X = (x_{ij})_{m \times n} \in D$. Then we can define $\nabla f(X) = [\frac{\partial f}{\partial x_{ij}}]_{m \times n}$.*

Lemma 1 *Assume that $\phi_i(X)$ $(i \in I)$ and $\varphi_j(X) (j \in J)$ which are defined on the open set $D \subset S^n$ are real functions, which have the first order partial derivatives. If*

$$\begin{cases} \phi_i(X) = 0, & i \in I \\ \varphi_j(X) = c_j, & j \in J \end{cases} \tag{2.2}$$

has the solution X^* on D, and

$$\begin{cases} \phi_i(X) < 0, & i \in I \\ \varphi_j(X) = c_j, & j \in J \end{cases} \tag{2.3}$$

has no solutions on D, *and* $\nabla\varphi_j(X^*)$ *are linearly independent, then*

$$\begin{cases} Y \cdot \nabla\phi_i(X) < 0, & i \in I \\ Y \cdot \nabla\varphi_j(X) = 0, & j \in J \end{cases} \tag{2.4}$$

has no solutions on S^n.

Proof See Lemma 2.2-1 of [18].

3. The consistence of the optimality efficient solution of (FMSDP) and the efficient solution of (MSDP)

Theorem 1 *If* X^* *is the optimal efficient solution of (FMSDP), then it is the efficient solution of (FMSDP).*

The proof can be completed by means of Definition 2.

Theorem 2 X^* *is the optimal efficient solution of (FMSDP) if and only if* X^* *is the efficient solution of (MSDP).*

Proof Sufficiency (Proof by contradiction). Assume that X^* is the optimal efficient solution of (FMSDP), then X^* is the feasible solution of (MSDP) via the construction of (MSDP).

Now we assume $X \in N$, such that $F(X) \leq F(X^*)$, $F(X) \neq F(X^*)$, so X^* is not the efficient solution of (MSDP). Then we have $\mu(X) = (\mu_1(X), \mu_2(X), \ldots, \mu_n(X)) \leq \mu(X^*)$, that is, $\mu_j(X) \leq \mu_j(X^*)$ for all $j = 1, 2, \ldots, n$.

1) If $\mu_j(X) = 0$, when $1 \leq j \leq m$, we have $g_j(X) \leq 0$. When $m+1 \leq j \leq n$, we have $h_j(X) = 0$.

2) If $0 < \mu_j(X) \leq \mu_j(X^*) \leq 1$, when $1 \leq j \leq m$, we have $0 < \frac{g_j(X)}{t_j} \leq \frac{g_j(X^*)}{t_j} \leq 1$, that is, $0 < g_j(X) \leq g_j(X^*) \leq t_j$.

When $m+1 \leq j \leq n$, we have $0 < \frac{|h_j(X)|}{t_j} \leq \frac{|h_j(X^*)|}{t_j} \leq 1$, that is, $0 < |h_j(X)| \leq |h_j(X^*)| \leq t_j$.

From 1) and 2), we can see that $X \in M$.

Then, when $f(X) \leq f(X^*)$, $f(X) \neq f(X^*)$, X^* is not the optimal efficient solution of (FMSDP), which contradicts the assumption.

When $f(X) = f(X^*)$, we have $\mu(X) \leq \mu(X^*), \mu(X) \neq \mu(X^*)$. By Definition 3, we know X^* is not the optimal efficient solution of (FMSDP), which contradicts the assumption too.

Therefore, if X^* is the optimal efficient solution of (FMSDP), it is the efficient solution of (MSDP) certainly.

Necessity (Proof by contradiction). Assume that X^* is the efficient solution of (MSDP). Then X^* is the feasible solution of (FMSDP) via the construction of (MSDP) in Section 1.

Now we assume that there exists $X \in M$, such that

1) $f(X) \leq f(X^*)$, $f(X) \neq f(X^*)$, when $0 < g_j(X) \leq t_j$, we have $g_j(X) \leq g_j(X^*)$ ($j = 1, 2, \ldots, m$). When $h_j(X) \neq 0$, we have $|h_j(X)| \leq |h_j^*(X)| (j = m+1, m+2, \ldots, n)$,

2) $f(X) \leq f(X^*)$, when $0 < g_j^*(X) \leq t_j$, we have $g(X) \leq g(X^*)$ ($j = 1, 2, \ldots, m$). When

$h_j^*(X) \neq 0$, we have $h_j(X) \neq 0, |h_j(X)| \leq |h_j^*(X)| (j = m+1, m+2, \ldots, n)$, and at least there exists j, satisfying one of the conditions mentioned above, such that $g_j(X) < g_j(X^*)$ or $|h_j(X)| \leq |h_j(X^*)|$.

X^* is not the optimal efficient solution of (FMSDP) if either one is established.

Then, we have $F(X) \leq F(X^*)$, $F(X) \neq F(X^*)$. That is to say, X^* is not the efficient solution of (MSDP), which contradicts the topic assumptions.

When $g_j(X) \leq 0$ $(1 \leq j \leq m)$, via the construction of membership in Section 1, we have $\mu_j(X) = 0$. When $0 < g_j(X) \leq t_j$, if $g_j(X) \leq g_j(X^*)$, then $0 < \frac{g_j(X)}{t_j} \leq \frac{g_j(X^*)}{t_j} \leq 1$, so $\mu_j(X) \leq \mu_j(X^*)$.

When $h_j(X^*) = 0$ $(m+1 \leq j \leq n)$, via the construction of membership in Section 1, we have $\mu_j(X) = 0$. When $h_j(X) \neq 0$, if $|h_j(X)| \leq |h_j(X^*)|$, then $0 < \frac{|h_j(X)|}{t_j} \leq \frac{|h_j(X^*)|}{t_j} \leq 1$, so $\mu_j(X) \leq \mu_j(X^*)$.

For the situation 1), based on the above analysis, we can see that $X \in N$, and $f(X) \leq f(X^*)$, $f(X) \neq f(X^*)$, and $\mu_j(X) \leq \mu_j(X^*)$.

For the situation 2), based on the above analysis, we can see that $X \in N$, and $f(X) \leq f(X^*)$, $\mu_j(X) \leq \mu_j(X^*)$, and $\mu_j(X) \neq \mu_j(X^*)$.

In conclusion, we can see that $X \in N$, $F(X) \leq F(X^*)$ and $F(X) \neq F(X^*)$. So X^* is not the efficient solution of (MSDP), which contradicts the topic assumptions. □

By Theorems 1 and 2, we can get:

Theorem 3 *If X^* is the efficient solution of (MSDP), then X^* is the efficient solution of (FMSDP).*

4. The optimality conditions of (FMSDP)

By the analysis of Section 3, we know that solving the efficient optimal solution of (FMSDP) can be transformed into solving the effective solution of (MSDP). So we discuss the optimality conditions of (FMSDP) by discussions of the optimality conditions of (MSDP).

Theorem 4 *For (MSDP), if X^* is the efficient solution of (MSDP), and $f(X), g(X), h(X)$ are differentiable on $X^* \in N$, then there exists $\lambda \in R_p, u \in R_m, \nu, \omega \in R_{n-m}$ such that*

$$\begin{cases} \nabla_x L(X^*, \lambda, u) = \sum_{i=1}^{p} \lambda_i \nabla f_i(X^*) + \sum_{j=m+1}^{n} \omega_j \nabla \mu_j(X^*) + \sum_{j=1}^{m} u_j \nabla g_j(X^*) + \\ \qquad \sum_{j=m+1}^{n} \nu_j \nabla h_j(X^*) = 0, \\ (\lambda, \omega, u, \nu) \neq 0, \lambda \geq 0, u \geq 0, \omega \geq 0. \end{cases} \tag{4.1}$$

Proof We order $J = \{1 \leq j \leq m | g_j(X^*) \neq 0\}$, $I = \{m+1 \leq j \leq n | h_j(X^*) \neq 0\}$, $L = \{m+1 \leq j \leq n | h_j(X^*) = 0\}$, $D = \{X \in S^n | g_j(X) < 0, j \notin J, j \notin I, 1 \leq j \leq m\}$, then we have $\mu_j(X) = 0$ on D.

From the topic assumption, we can see that

$$\begin{cases} f_i(X) - f_i(X^*) = 0, & i = 1, 2, \ldots, p, \\ \mu_j(X) - \mu_j(X^*) = 0, & j \in I \bigcup J, \\ g_j(X) = \mu_j(X) t_j, & j \in J, \\ h_j^2(X) = \mu_j(X) t_j^*, & j = m+1, m+2, \ldots, n \end{cases} \tag{4.2}$$

has solution X^* on D. And from the definition of (MSDP), we know

$$h_j^2(X^*) = \mu_j(X^*) t_j^2 (j = m+1, m+2, \ldots, n), \quad g_j(X^*) = \mu_j(X^*) t_j \ (j \in J).$$

Then (4.2) is equivalent to

$$\begin{cases} f_i(X) - f_i(X^*) = 0, & i = 1, 2, \ldots, p, \\ \mu_j(X) - \mu_j(X^*) = 0, & j \in I, \\ g_j(X) - g_j(X^*) = 0, & j \in J, \\ h_j(X) = 0, & j \in L. \end{cases} \tag{4.3}$$

Because X^* is the efficient solution of (MSDP), it is the optimal efficient solution of (FMSDP). So the inequalities

$$\begin{cases} f_i(X) - f_i(X^*) < 0, & i = 1, 2, \ldots, p, \\ \mu_j(X) - \mu_j(X^*) < 0, & j \in I, \\ g_j(X) = \mu_j(X) t_j, & j \in J, \\ h_j^2(X) = \mu_j(X) t_j^2, & j = m+1, m+2, \ldots, n \end{cases} \tag{4.4}$$

have no solution on D. As above, (4.4) can be transformed into:

$$\begin{cases} f_i(X) - f_i(X^*) < 0, & i = 1, 2, \ldots, p, \\ \mu_j(X) - \mu_j(X^*) < 0, & j \in I, \\ g_j(X) - g_j(X^*) < 0, & j \in J, \\ h_j(X) = 0, & j \in L \end{cases} \tag{4.5}$$

which have no solution on D.

1) If $\nabla h_j(X^*)$ are linearly dependent, then there are $\nu_{m+1}, \nu_{m+2}, \ldots, \nu_n$ which are not all in 0, such that $\sum_{j=m+1}^{n} \nu_j \nabla h_j(X^*) = 0$. Let $\lambda = 0 \in R^p$, $\omega = u = 0 \in R^{n-m}$. Then $(\lambda, \omega, u, \nu)$ meet (4.1).

2) If $\nabla h_j(X^*)$ are linearly independent, then from (4.3), (4.5) and Lemma 1, we have the inequalities

$$\begin{cases} Y \cdot \nabla f_i(X^*) < 0, & i = 1, 2, \ldots, p, \\ Y \cdot \nabla \mu_j(X^*) < 0, & j \in I, \\ Y \cdot \nabla g_j(X^*) < 0, & j \in J, \\ Y \cdot \nabla h_j(X^*) = 0, & j \in L \end{cases} \tag{4.6}$$

which have no solution on S^n. By Motzkin theorem [19, Theorem 2.4.2], we know there are $\lambda \geq 0$, $\lambda \neq 0$, $u_j \geq 0$ $(j \in J)$, $\omega_j \geq 0$ $(j \in I)$ and $\nu_j \geq 0$ $(j \in L)$, where u_j, ω_j, ν_j respectively are not all 0, such that

$$\sum_{i=1}^{p}\lambda_i\nabla f_i(X^*)+\sum_{j\in J}\omega_j\nabla\mu_j(X^*)+\sum_{j\in J}u_j\nabla g_j(X^*)+\sum_{j=m+1}^{n}\nu_j\nabla\ h_j(X^*)=0. \tag{4.7}$$

When $j\in\{1\le j\le m|j\notin J\}$, taking $u_j=0$, when $j\in\{m+1\le j\le n|j\notin I\}$, taking $\omega_j=0$, then we have

$$\sum_{i=1}^{p}\lambda_i\nabla f_i(X^*)+\sum_{j=m+1}^{n}\omega_j\nabla\mu_j(X^*)+\sum_{j=1}^{m}u_j\nabla g_j(X^*)+\sum_{j=m+1}^{n}\nu_j\nabla\ h_j(X^*)=0.$$

So, such $(\lambda,\omega,u,\nu)\neq 0$, $\lambda\ge 0$, $u\ge 0$, $\omega\ge 0$ meet (4.1). □

Theorem 5 *For (FMSDP), if X^* is the optimal efficient solution of (FMSDP), and $f(X)$, $g(X)$, $h(X)$ are differentiable on $X^*\in N$, then there are $\lambda\in R_p$, $u\in R_m$, $\nu,\omega\in R_{n-m}$, such that*

$$\begin{cases}\nabla_x L(X^*,\lambda,u)=\displaystyle\sum_{i=1}^{p}\lambda_i\nabla f_i(X^*)+\sum_{j=1}^{m}u_j\nabla g_j(X^*)+\\ \qquad\qquad\qquad\displaystyle\sum_{j=m+1}^{n}(\nu_j+2\omega_j h_j(X^*))\nabla\ h_j(X^*)=0,\\ (\lambda,\omega,u,\nu)\neq 0,\ \lambda\ge 0,\ u\ge 0,\ \omega\ge 0.\end{cases} \tag{4.8}$$

Proof By Theorems 2 and 5, there are $\lambda\in R_p$, $u\in R_m$, $\nu,\omega\in R_{n-m}$ which meet (4.1), where $\nabla\mu_j(X^*)=2h_j(X^*)\cdot\nabla h_j(X_*)$ $(j=m+1,m+2,\ldots,n)$. Substituting it into (4.1), we can get (4.8). □

5. The algorithm of (FMSDP)

In engineering case, we only need one of the optimal efficient solutions, instead of solving the optimal efficient solution set of (FMSDP).

At first, solve the following (MSDP):

$$\begin{aligned}&\min f(X)\\ &\text{s.t. } G(X)\le 0,\\ &\qquad H(X)=0,\\ &\qquad X\ge 0.\end{aligned} \tag{5.1}$$

$$\begin{aligned}&\min f(X)\\ &\text{s.t. } G(X)\le a,\\ &\qquad H(X)\le b,\\ &\qquad X\ge 0.\end{aligned} \tag{5.2}$$

Assume that X_1 and X_2 are the efficient solution of (5.1) and (5.2). Let

$$\rho_i=\min\{f_i(X_1),f_i(X_2)\},\quad \sigma_i=\max\{f_i(X_1),f_i(X_2)\}\ (1\le i\le p).$$

Then construct the membership function as follows:

$$\tau_i(X) = \begin{cases} 0, & f_i(X) \le \rho_i, \\ \frac{f_i(X)-\rho_i}{\sigma_i-\rho_i}, & \rho_i < f_i(X) < \sigma_i, \\ 1, & f_i(X) \ge \sigma_i. \end{cases} \tag{5.3}$$

Then we can transform (MSDP) into:

$$\begin{aligned} &\min\max\{\tau_1(X), \tau_2(X), \ldots, \tau_p(X), \mu_1(X), \mu_2(X), \ldots, \mu_n(X)\} \\ &\text{s.t. } X \ge 0. \end{aligned} \tag{5.4}$$

That is:

$$\begin{aligned} &\min \alpha \\ &\text{s.t. } \tau_i(X) \le \alpha,\ 1 \le i \le p, \\ &\quad\ \ \mu_j(X) \le \alpha,\ 1 \le j \le n, \\ &\quad\ \ X \ge 0. \end{aligned} \tag{5.5}$$

Then solving (MSDP) can be transformed into solving

$$\begin{aligned} &\min \alpha \\ &\text{s.t. } f_i(X) \le \rho_i + \alpha(\sigma_i - \rho_i),\ 1 \le i \le p, \\ &\quad\ \ G(X) \le \alpha a, \\ &\quad\ \ H(X) \le \alpha b, \\ &\quad\ \ X \ge 0, \\ &\quad\ \ \alpha \ge 0. \end{aligned} \tag{5.6}$$

Theorem 6 *If (5.1) has an efficient solution X_1, there are always X_1, X_2 which respectively are the optimal efficient solution of (FMSDP) and efficient solution of (5.2) satisfying $f(X_2) \le f(X) \le f(X_1)$.*

Proof If X_1 is the efficient solution of (5.1), then it is the feasible solution of (FMSDP). For (FMSDP), X_1 is the optimal efficient solution or not. If it is, we have $X = X_1$, $f(X) \le f(X_1)$.

If it is not, there must exist X^1, such that $X^1 \in M$, and $f(X^1) \le f(X_1)$, $f(X^1) \ne f(X_1)$. Then for (FMSDP), X^1 is the optimal efficient solution or not. If it is, we make $X = X^1$, then $f(X) \le f(X_1)$. If it is not, there must exist X^2, such that $X^2 \in M$, and $f(X^2) \le f(X^1) \le f(X_1)$, $f(X^2) \ne f(X_1)$. For (FMSDP), X^2 is the optimal efficient solution or not. If it is, we make $X = X^2$, then $f(X) \le f(X^1) \le f(X_1)$, $f(X) \ne f(X_1)$. If it is not, we analogize and always can find out X which is the optimal efficient solution of (FMSDP) and such that $f(X) \le f(X_1)$.

If X is the optimal efficient solution of (FMSDP), through Theorem 2, we can see that X is the efficient solution of (MSDP), then it is the feasible solution of (5.2). As above, we can find X_2 which is the efficient solution of (5.2), satisfying $f(X_2) \le f(X)$.

To sum up, X_1 and X_2 are always the optimal efficient solution of (FMSDP) and efficient solution of (5.2), respectively, satisfying $f(X_2) \le f(X) \le f(X_1)$. □

Theorem 7 *If α is the optimal solution of (5.6), then it is the optimal solution of (5.5).*

Proof If $f_i(X) \leq \rho_i + \alpha(\sigma_i - \rho_i)$, when $1 \leq \alpha \leq 1$. By (5.3) we can see that

$$\tau_i(X) = \frac{f_i(X) - \rho_i}{\sigma_i - \rho_i} \leq \alpha \ (1 \leq i \leq p).$$

When $\alpha \geq 0$, by (5.3) we can see that

$$\tau_i(X) = 0 \leq \alpha \ (1 \leq i \leq p).$$

If $G(X) \leq \alpha a$, $H(X) \leq \alpha b$, then by (1.1) and (1.2), we have $\mu_j(X) = 0 \leq \alpha \ (1 \leq j \leq n)$, or $\mu_j(X) = \frac{g_j(X)}{t_j} = \alpha \ (1 \leq j \leq m)$, $\mu_j(X) = \frac{h_j^2(X)}{t_j^2} = \alpha \ (m+1 \leq j \leq n)$, then α is the feasible solution of (5.5).

Now we assume α is not the optimal solution of (5.5). There exists β satisfying $\tau_i(X) \leq \beta < \alpha \ (1 \leq i \leq p)$ and $\mu_j(X) \leq \beta < \alpha \ (1 \leq j \leq n)$.

Then $\tau_i(X) = \frac{f_i(X)-\rho_i}{\sigma_i-\rho_i} \leq \beta < \alpha \ (1 \leq i \leq p,\ 0 \leq \beta \leq 1)$ or $\tau_i(X) = 1 \leq \beta < \alpha \ (1 \leq i \leq p,\ \beta \geq 1)$, that is to say, $f_i(X) \leq \rho_i + \beta(\sigma_i - \rho_i)$. $\mu_j(X) = 0 \leq \beta < \alpha \ (1 \leq j \leq n)$ or $\mu_j(X) = \frac{g_j(X)}{t_j} \leq \beta < \alpha \ (1 \leq j \leq m)$, $\mu_j(X) = \frac{h_j^2(X)}{t_j} \leq \beta < \alpha \ (m+1 \leq j \leq n)$, that is, $g_j(X) \leq \beta t_j$, $h_j^2 \leq \beta t_j^2$, that is, $\mu_j(X) \leq \beta < \alpha \ (i \leq j \leq n)$.

Then we can see that α is not the optimal solution of (5.6), which has contradiction with the topic assumption. So, α is the optimal solution of (5.5). □

Theorem 8 *If α is the optimal solution of (5.5), it is the optimal solution value of (5.4).*

Proof Now we assume that α is not the optimal solution value of (5.4).

That is to say, $\exists X^* \leq 0$, such that $\forall X \leq 0$,

$$\begin{aligned}\alpha \neq \beta &= \min\max\{\tau_1(X^*), \tau_2(X^*), \dots, \tau_p(X^*), \mu_1(X^*), \mu_2(X^*), \dots, \mu_n(X^*)\}\\ &\leq \min\max\{\tau_1(X), \tau_2(X), \dots, \tau_p(X), \mu_1(X), \mu_2(X), \dots, \mu_n(X)\},\end{aligned}$$

then $\tau_i(X^*) \leq \beta \ (1 \leq i \leq p)$, $\mu_j(X^*) \leq \beta \ (1 \leq j \leq n)$.

Therefore, β is the feasible solution of (5.5).

Since $\alpha \neq \beta$, we have $\alpha < \beta$. Then $\exists X > 0$, such that $\tau_i(X) \leq \alpha < \beta \ (1 \leq i \leq p)$, $\mu_j(X) \leq \alpha \leq \beta \ (1 \leq j \leq n)$. And we can see

$$\begin{aligned}\beta &= \min\max\{\tau_1(X^*), \tau_2(X^*), \dots, \tau_p(X^*), \mu_1(X^*), \mu_2(X^*), \dots, \mu_n(X^*)\}\\ &> \alpha \geq \min\max\{\tau_1(X), \tau_2(X), \dots, \tau_p(X), \mu_1(X), \mu_2(X), \dots, \mu_n(X)\},\end{aligned}$$

which is in contradiction with the topic assumption. Thus, α is the optimal solution value of (5.4). □

By Theorem 6, we can make X_1 and X_2 be the efficient solutions of (5.1) and (5.2), respectively, satisfying $f(X_2) \leq f(X_1)$. Let

$$\rho = f(X_2), \quad \sigma = f(X_1).$$

So, (5.3) can be transformed into:

$$\tau_i(X) = \begin{cases} 0, & f_i(X) \leq f_i(X_2), \\ \frac{f_i(X)-f_i(X_2)}{f_i(X_1)-f_i(X_2)}, & f_i(X_2) < f_i(X) < f_i(X_1), \\ 1, & f_i(X) \geq f_i(X_1). \end{cases} \tag{5.7}$$

Theorem 9 *If X^* is the optimal solution of (5.4), then it is the efficient solution of (MSDP).*

Proof At first, we prove (5.4) only has one optimal solution.

Assume $\forall X_1, X_2 \leq 0$, $0 < \lambda_1, \lambda_2 < 1$, we have

$$f_i(\lambda_1 X_1 + \lambda_2 X_2) = \lambda_1^n f_i(X_1) + \lambda_2^n f_i(X_2) < \lambda_1 f_i(X_1) + \lambda_2 f_i(X_2) \ (i = 1, 2, \ldots, p),$$

$$\mu_j(\lambda_1 X_1 + \lambda_2 X_2) = \lambda_1^n \mu_j(X_1) + \lambda_2^n \mu_j(X_2) < \lambda_1 \mu_j(X_1) + \lambda_2 \mu_j(X_2) \ (j = 1, 2, \ldots, n).$$

Then, $\max\{\tau_1(\lambda_1X_1+\lambda_2X_2), \tau_2(\lambda_1X_1+\lambda_2X_2), \ldots, \tau_p(\lambda_1X_1+\lambda_2X_2), \mu_1(\lambda_1X_1+\lambda_2X_2), \mu_2(\lambda_1X_1+\lambda_2X_2), \ldots, \mu_n(\lambda_1X_1+\lambda_2X_2)\} < \lambda_1 \max\{\tau_1(X_1), \tau_2(X_1), \ldots, \tau_p(X_1), \mu_1(X_1), \mu_2(X_1), \ldots, \mu_n(X_1)\} + \lambda_2 \max\{\tau_1(X_2), \tau_2(X_2), \ldots, \tau_p(X_2), \mu_1(X_2), \mu_2(X_2), \ldots, \mu_n(X_2)\}$.

We can see that (5.4) only has one optimal solution by the nature of convex programming.

Now we assume X^* is the optimal solution of (5.4). By (5.7), we can see that (5.1) and (5.2) all have efficient solutions, which are respectively X_1, X_2, and $f(X_2) \leq f(X_1)$, X_1, X_2 are the feasible solutions of (5.4). Then:

$$\begin{aligned} &\max\{\ tau_1(X^*), \tau_2(X^*), \ldots, \tau_p(X^*), \mu_1(X^*), \mu_2(X^*), \ldots, \mu_n(X^*)\} \\ &\quad \leq \max\{\tau_1(X_1), \tau_2(X_1), \ldots, \tau_p(X_1), \mu_1(X_1), \mu_2(X_1), \ldots, \mu_n(X_1)\} \leq 1. \end{aligned}$$

Therefore, X^* is the feasible solution of (MSDP). Now we assume that $X \neq X^*$ is the efficient solution of (MSDP), and $F(X) \leq F(X^*)$, $F(X) \neq F(X^*)$, then $f(X_2) \leq f(X) \leq f(X^*)$, that is $\tau_i(X) \leq \tau_i(X^*)$ $(i = 1, 2, \ldots, p)$, $\mu_j(X) \leq \mu_j(X^*)$ $(j = 1, 2, \ldots, n)$.

So,

$$\begin{aligned} &\min\max\{\tau_1(X^*), \tau_2(X^*), \ldots, \tau_p(X^*), \mu_1(X^*), \mu_2(X^*), \ldots, \mu_n(X^*)\} \\ &\quad \leq \min\max\{\tau_1(X_1), \tau_2(X_1), \ldots, \tau_p(X_1), \mu_1(X_1), \mu_2(X_1), \ldots, \mu_n(X_1)\}, \end{aligned}$$

which is in contradiction with (5.4). Therefore there exists one and only one optimal solution. □

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