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Self-Reciprocal Polynomials of Binomial Type

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Abstract In this paper, we define the self-inverse sequences related to sequences of polynomials of binomial type, and give some interesting results of these sequences. Moreover, we study the self-inverse sequences related to the Laguerre polynomials.

Keywords self-inverse sequences; sequences of binomial type; basic sequences; Laguerre polynomials.

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1. Introduction

The classical binomial inversion formula states that

$$a_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}b_k \iff b_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k.$$

We say that a sequence $\{a_n\}$ of complex numbers is self-inverse or invariant if

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k=a_n,\quad n\ge 0.$$

Sun [1] and Wang [2] studied those self-inverse sequences and gave some results of self-inverse sequences. For general self-inverse pairs

$$a_n=\sum_{k=0}^{n}A(n,k)b_k \iff b_n=\sum_{k=0}^{n}A(n,k)a_k,$$

we have the infinite lower triangle matrix $A=(A(n,k))_{n,k=0}^{\infty}$ satisfies $A^2=I$. A sequence $\{a_n\}$ is called a general self-inverse sequence if it satisfies

$$a_n=\sum_{k=0}^{n}A(n,k)a_k,\quad n\ge 0,$$

where $A=(A(n,k))$ is an infinite lower triangle matrix and $A^2=I$. We denote $A_m=(A(n,k))_{n,k=0}^{m}$, then we have $A_m^2=I_m$. Therefore, we get $|A_m|^2=(\prod_{i=0}^{m}A(i,i))^2=1$ for

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all $m \geq 0$. Thus we see that the diagonal entries of the matrix A are non-zero. Let

$$p_n(x) = \sum_{k=0}^{n} A(n,k)x^k, \quad n \geq 0.$$

Then $p_n(x)$ is exactly a polynomial of degree n for all $n \geq 0$. Following the inverse relation, we get

$$x^n = \sum_{k=0}^{n} A(n,k)p_k(x), \quad n \geq 0.$$

If $q_m(x) = \sum_{k=0}^{m} q_{m,k}x^k$ is another polynomial, we denote $q_m(\mathbf{p}(x)) = \sum_{k=0}^{m} q_{m,k}p_k(x)$. With this notation, we have $p_n(\mathbf{p}(x)) = x^n$ for all $n \geq 0$.

For the self-inverse pair

$$a_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(-1)^k b_k \Longleftrightarrow b_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(-1)^k a_k,$$

we know that

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(-1)^k x^k, \quad n \geq 0$$

is the Laguerre polynomials, and $L_n(\mathbf{L}(x)) = x^n$. The Laguerre polynomials are of binomial type. A sequence of polynomials $p_n(x)$ ($n \geq 0$), where $p_n(x)$ is exactly of degree n for all n, is said to be binomial type if it satisfies the infinite sequence of identities

$$p_n(x+y) = \sum_{k \geq 0} \binom{n}{k} p_k(x)p_{n-k}(y), \quad n = 0, 1, 2, \ldots.$$

Let

$$p_n(x) = \sum_{k=0}^{n} A(n,k)x^k, \quad n \geq 0$$

be a sequence of polynomials of binomial type and $p_n(\mathbf{p}(x)) = x^n$. A sequence $\{a_n\}$ is called a self-inverse sequence related to the sequence of polynomials $p_n(x)$ of binomial type if

$$a_n = \sum_{k=0}^{n} A(n,k)a_k, \quad n \geq 0.$$

Henceforth, we say "$\{a_n\}$ is a self-inverse sequence related to $p_n(x)$" rather than "$\{a_n\}$ is a self-inverse sequence related to the sequence of polynomials $p_n(x)$ of binomial type".

In this paper, we study the self-inverse sequences related to sequences of polynomials of binomial type. In order to render this work self-contained, we list some important results of sequences of polynomials of binomial type in Section 2, but we omit the proofs which can be found in [3]. In Section 3, we give some general results of the self-inverse sequences related to sequences of polynomials of binomial type. Moreover, we study the self-inverse sequences related to the Laguerre polynomials in Section 4.

2. Fundamentals

In this section, we list the main results of polynomials of binomial type which we shall use in next section. These polynomials were studied by Mullin and Rota [3].

First, we give some definitions in the theory of binomial enumeration.

Definition 1 *A linear operator T which commutes with all shift operators is called a shift-invariant operator, i.e., $TE^a = E^aT$.*

Definition 2 *A delta operator usually denoted by the letter Q, is a shift-invariant operator for which Qx is a non-zero constant.*

Delta operators possess many of the properties of the derivative operator D.

Definition 3 *Let Q be a delta operator. A polynomial sequence $p_n(x)$ is called the sequence of basic polynomials for Q if:*

(1) $p_0(x) = 1$;

(2) $p_n(0) = 0$ *whenever* $n > 0$;

(3) $Qp_n(x) = np_{n-1}(x)$.

It is not difficult to show that every delta operator has a unique sequence of basic polynomials associated with it.

Now, we can give some general consequences of polynomials of binomial type as Lemmata. These Lemmata can be found in [3].

Lemma 1 *(a) If $p_n(x)$ is a basic sequence for some delta operator Q, then it is a sequence of polynomials of binomial type. (b) If $p_n(x)$ is a sequence of polynomials of binomial type, then it is a basic sequence for some delta operator.*

Lemma 2 (Expansion Theorem) *Let T be a shift-invariant operator, and let Q be a delta operator with basic set $p_n(x)$. Then*

$$T = \sum_{k\geq 0} \frac{a_k}{k!} Q^k,$$

where $a_k = [Tp_k(x)]_{x=0}$.

Lemma 3 *Let $\mathbf{P}$ be a ring of polynomials over a field $\mathbf{K}$ and Σ be the ring of shift-invariant operators on $\mathbf{P}$. Suppose that Q is a delta operator and F is the ring of formal power series in the variable t over $\mathbf{K}$. Then there exists an isomorphism from F onto Σ, which carries*

$$f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \quad \textit{into} \quad \sum_{k\geq 0} \frac{a_k}{k!} Q^k.$$

Lemma 4 *Let Q be a delta operator with basic polynomials $p_n(x)$, and let $q(D) = Q$. Let $q^{-1}(t)$ be the inverse formal power series. Then*

$$\sum_{n\geq 0} \frac{p_n(x)}{n!} u^n = e^{xq^{-1}(u)}.$$

Lemma 5 *If P and Q are delta operators with basic sequences $p_n(x)$ and $q_n(x)$, and expansions $P = p(D)$ and $Q = q(D)$, then $r_n(x) = p_n(\mathbf{q}(x))$ is the sequence of basic polynomials for the delta operator $R = p(q(D))$.*

Lemma 6 *Let $Q = g(D)$ be a delta operator with basic polynomials*

$$p_n(x) = \sum_{k\geq 1} c_{n,\,k} x^k, \quad n \geq 1.$$

Then $g = f^{-1}$, where

$$f(t) = \sum_{k\geq 1} c_{k,\,1} \frac{t^k}{k!}.$$

3. Main results

Throughout this section, let

$$p_n(x) = \sum_{k=0}^{n} A(n,k) x^k, \quad n \geq 0 \tag{1}$$

be of binomial type and satisfy $p_n(\mathbf{p}(x)) = x^n$ for all n. Using Lemma 1, we see that $p_n(x)$ is a basic sequence for some delta operator Q. By Expansion Theorem and Lemma 3, we shall denote $Q = q(D)$, where

$$q(t) = \sum_{k\geq 0} q_k \frac{t^k}{k!}$$

is a formal power series, and the coefficients $q_k = [Qx^k]_{x=0}$. Obviously, $q_0 = 0$. Following the definition of delta operator, we get $q_1 \neq 0$. Hence, a unique inverse formal power series $q^{-1}(t)$ exists. We use Lemma 6 to get that

$$q^{-1}(t) = \sum_{k\geq 1} A(k,1) \frac{t^k}{k!}. \tag{2}$$

By Lemma 5, we know that $p_n(\mathbf{p}(x)) = x^n$ is a basic sequence for the delta operator $q(q(D))$. However, x^n is the basic sequence for the derivative operator D. Thus we have $q(q(t)) = t$, i.e., $q(t) = q^{-1}(t)$. Using Lemma 4, we get that $p_n(x)$ $(n \geq 0)$ have the following exponential generating function:

$$\sum_{n\geq 0} p_n(x) \frac{t^n}{n!} = e^{xq^{-1}(t)} = e^{xq(t)}. \tag{3}$$

From (1) and (3), we have

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=0}^{n} A(n,k) x^k = \sum_{k\geq 0} x^k \sum_{n\geq k} A(n,k) \frac{t^n}{n!} = \sum_{k\geq 0} x^k \frac{(q(t))^k}{k!} .$$

Therefore, $A(n,k)$ have a "vertical" generating function:

$$\varphi_k(t) = \sum_{n\geq k} A(n,k) \frac{t^n}{n!} = \frac{1}{k!} (q(t))^k. \tag{4}$$

Now we can give the main results of the self-inverse sequences related to sequences of polynomials of binomial type.

For any sequence $\{a_n\}$, let the exponential generating function of $\{a_n\}$ be

$$A(x)=\sum_{n\geq 0}a_n\frac{x^n}{n!}.$$

We have:

Theorem 1 *$\{a_n\}$ is a self-inverse sequence related to $p_n(x)$ if and only if $A(x)=A(q(x))$.*

Proof Using the exponential generating function of $\{a_n\}$, we have

$$a_n=\sum_{k=0}^{n}A(n,k)a_k \iff$$
$$A(x)=\sum_{n\geq 0}\frac{x^n}{n!}\sum_{k=0}^{n}A(n,k)a_k=\sum_{k\geq 0}a_k\sum_{n\geq k}A(n,k)\frac{x^n}{n!}=\sum_{k\geq 0}a_k\frac{(q(x))^k}{k!}=A(q(x)).$$

Corollary 1 *If $\{a_n\}$ is a self-inverse sequence related to $p_n(x)$, then*

$$a_{m+1}=\sum_{l=0}^{m}\binom{m}{l}A(m-l+1,1)\sum_{n=0}^{l}a_{n+1}A(l,n),\quad m\geq 0. \tag{5}$$

Proof Differentiating the exponential generating function of $\{a_n\}$, we get

$$A'(x)=\sum_{m\geq 0}a_{m+1}\frac{x^m}{m!}.$$

On the other hand,

$$\begin{aligned}(A(q(x)))'&=\sum_{n\geq 0}a_{n+1}\frac{(q(x))^n}{n!}q'(x)=\sum_{n\geq 0}a_{n+1}\sum_{l\geq 0}A(l,n)\frac{x^l}{l!}q'(x)\\&=\Big(\sum_{l\geq 0}\frac{x^l}{l!}\sum_{l=0}^{n}a_{n+1}A(l,n)\Big)\Big(\sum_{k\geq 0}A(k+1,1)\frac{x^k}{k!}\Big)\\&=\sum_{m\geq 0}\frac{x^m}{m!}\sum_{l=0}^{m}\binom{m}{l}A(m-l+1,1)\sum_{n=0}^{l}a_{n+1}A(l,n).\end{aligned}$$

The result leads to Theorem 1, immediately. □

Obviously, if $\{a_n\}$ and $\{b_n\}$ are self-inverse sequences related to $p_n(x)$, then $\{\alpha a_n+\beta b_n\}$ is a self-inverse sequence related to $p_n(x)$, where α and β are arbitrary constants.

Sun [1] gave some transformation formulas for self-inverse sequences. Similarly, we have the following theorem.

Theorem 2 *Let $\{a_n\}$ be a self-inverse sequence related to $p_n(x)$, and $c_n=\sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k}$. Then $\{c_n\}$ is a self-inverse sequence related to $p_n(x)$ if and only if $\{b_n\}$ is a self-inverse sequence related to $p_n(x)$.*

Proof Let $C(x)=\sum_{n\ge0}c_n\frac{x^n}{n!}$ and $B(x)=\sum_{n\ge0}b_n\frac{x^n}{n!}$. Then

$$C(x)=\sum_{n\ge0}\frac{x^n}{n!}\sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k}=\sum_{k\ge0}a_k\frac{x^k}{k!}\sum_{n\ge k}b_{n-k}\frac{x^{n-k}}{(n-k)!}=A(x)B(x).$$

Because $A(x)=A(q(x))$, we have

$$C(x)=C(q(x))\Longleftrightarrow B(x)=B(q(x)).\ \Box$$

Now we give some methods by which we can create the self-inverse sequences related to $p_n(x)$.

Theorem 3 *$\{a_n\}$ is a self-inverse sequence related to $p_n(x)$ if and only if there exists a sequence $\{\lambda_n\}$ such that*

$$a_n=\sum_{k=0}^{n}A(n,k)\lambda_k+\lambda_n,\quad n\ge0.$$

Proof Suppose that $\{a_n\}$ is a self-inverse sequence related to $p_n(x)$. Then $a_n=\sum_{k=0}^{n}A(n,k)a_k$. Let $\lambda_n=\frac{a_n}{2}$. We have

$$\sum_{k=0}^{n}A(n,k)\frac{a_k}{2}+\frac{a_n}{2}=\frac{a_n}{2}+\frac{a_n}{2}=a_n.$$

Conversely, let $\Lambda(x)=\sum_{k\ge0}\lambda_k\frac{x^k}{k!}$. Then

$$\begin{aligned}A(x)&=\sum_{n\ge0}\Big(\sum_{k=0}^{n}A(n,k)\lambda_k+\lambda_n\Big)\frac{x^n}{n!}=\sum_{k\ge0}\lambda_k\sum_{n\ge k}A(n,k)\frac{x^n}{n!}+\sum_{n\ge0}\lambda_n\frac{x^n}{n!}\\&=\sum_{k\ge0}\lambda_k\frac{(q(x))^k}{k!}+\sum_{n\ge0}\lambda_n\frac{x^n}{n!}=\Lambda(q(x))+\Lambda(x).\end{aligned}$$

Thus we have

$$A(q(x))=\Lambda(q(q(x)))+\Lambda(q(x))=\Lambda(x)+\Lambda(q(x))=A(x).\ \Box$$

Corollary 2 *Let $a_0=0$, $a_1=A(1,1)+1$, and $a_n=A(n,1)$ $(n\ge2)$. Then $\{a_n\}$ is a self-inverse sequence related to $p_n(x)$.*

Proof Let $\lambda_0=0$, $\lambda_1=1$ and $\lambda_n=0$ $(n\ge2)$ in Theorem 3. $\Box$

Corollary 3 *Let $a_n=p_n(a)+a^n$, where a is an arbitrary constant. Then $\{a_n\}$ is a self-inverse sequence related to $p_n(x)$.*

Proof Let $\lambda_n=a^n$ $(n\ge0)$ in Theorem 3. $\Box$

Theorem 4 *Suppose $\{f_n\}$ is an arbitrary sequence, and let*

$$a_n=\sum_{l=0}^{n}\binom{n}{l}f_{n-l}\sum_{k=0}^{l}A(l,k)f_k.$$

Then $\{a_n\}$ is a self-inverse sequence related to $p_n(x)$.

Proof Let $f(x)=\sum_{k\geq 0} f_k \frac{x^k}{k!}$. Then

$$A(x)=\sum_{n\geq 0}\sum_{l=0}^{n}\binom{n}{l}f_{n-l}\sum_{k=0}^{l}A(l,k)f_k\frac{x^n}{n!}=\sum_{m\geq 0}f_m\frac{x^m}{m!}\sum_{l\geq 0}\frac{x^l}{l!}\sum_{k=0}^{l}A(l,k)f_k$$

$$=\sum_{m\geq 0}f_m\frac{x^m}{m!}\sum_{k\geq 0}f_k\sum_{l\geq k}A(l,k)\frac{x^l}{l!}=f(x)\sum_{k\geq 0}f_k\frac{(q(x))^k}{k!}=f(x)f(q(x)).$$

Moreover, we have

$$A(q(x))=f(q(x))f(q(q(x)))=f(q(x))f(x)=A(x).\ \square$$

Corollary 4 *Let $a_0=0$, $a_1=0$ and $a_n=nA(n-1,1)$ $(n\geq 2)$. Then $\{a_n\}$ is a self-inverse sequence related to $p_n(x)$.*

Proof Let $f_0=0$, $f_1=1$ and $f_n=0$ $(n\geq 2)$ in Theorem 4. $\square$

Corollary 5 *Let $a_n=\sum l=0^n\binom{n}{l}p_l(a)a^{n-l}$, where a is an arbitrary constant. Then $\{a_n\}$ is a self-inverse sequence related to $p_n(x)$.*

Proof Let $f_n=a^n$ $(n\geq 0)$ in Theorem 4. $\square$

Theorem 3 has an equivalent form:

Theorem 5 *$\{a_n\}$ is a self inverse sequence related to $p_n(x)$ if and only if there exists a function $f:\mathbb{Z}\to\mathbb{C}$ such that*

$$a_n=\sum_{k=0}^{n}\binom{n}{k}p_{n-k}(1)p_k(\Delta)f(0)+f(n)\ .$$

Proof Following the string of identities:

$$\sum_{k=0}^{n}\binom{n}{k}p_{n-k}(1)p_k(\Delta)f(0)=\Big[\Big(\sum_{k=0}^{n}\binom{n}{k}p_{n-k}(1)p_k(\Delta)\Big)f(x)\Big]_{x=0}$$

$$=[p_n(\Delta+I)f(x)]_{x=0}=[p_n(E)f(x)]_{x=0}=[\sum_{k=0}^{n}A(n,k)E^kf(x)]_{x=0}=\sum_{k=0}^{n}A(n,k)f(k),$$

we get the result. $\square$

The following theorem is shown by operator method.

Theorem 6 *Suppose that $r_n(x)=\sum_{k=0}^{n}r_{n,\,k}x^k$ is a basic sequence for delta operator $R=r(D)$. And let $q_n(x)=r_n(\mathbf{p}(x))=\sum_{k=0}^{n}q_{n,\,k}x^k$. If $\{a_n\}$ is a self-inverse sequence related to $p_n(x)$, then we have the following identity:*

$$\sum_{k=0}^{n}r_{n,\,k}a_k=\sum_{k=0}^{n}q_{n,\,k}a_k.$$

Proof By Lemma 5, we know that $q_n(x)$ is a basic sequence for delta operator $r(q(D))$. Let T be a linear operator such that $Tx^n=a_n$. Then

$$Tp_n(x)=T\sum_{k=0}^{n}A(n,k)x^k=\sum_{k=0}^{n}A(n,k)Tx^k=\sum_{k=0}^{n}A(n,k)a_k=a_n.$$

Therefore, we have $Tx^n = Tp_n(x)$. If $f(x)$ is a polynomial, then we get $Tf(x) = Tf(\mathbf{p}(x))$ by linearity. Let $f(x) = r_n(x)$. Immediately, we have

$$Tr_n(x) = Tr_n(\mathbf{p}(x)) = Tq_n(x),$$

i.e.,

$$\sum_{k=0}^{n} r_{n,\,k} Tx^k = \sum_{k=0}^{n} q_{n,\,k} Tx^k.$$

The result follows $Tx^k = a_k$. □

4. Self-inverse sequences related to the Laguerre

The Laguerre polynomials

$$L_n(x) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(-1)^k x^k$$

is a basic sequence for the delta operator $Q = q(D) = \frac{D}{D-I}$. If $a_n = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(-1)^k a_k + a_0$ $(n \geq 1)$, we say that $\{a_n\}$ is a self-inverse sequence related to $L_n(x)$. From Theorem 1, we have:

Proposition 1 *$\{a_n\}$ is a self-inverse sequence related to $L_n(x)$ if and only if the exponential generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ satisfies $A(x) = A(\frac{x}{x-1})$.*

Obviously, using Corollary 2, we have the sequence $\{a_n\}$, which satisfies $a_0 = a_1 = 0$ and $a_n = -n!$ $(n \geq 2)$, is a self-inverse sequence related to $L_n(x)$. Moreover, by Corollary 3, the sequence $\{a_n\}$, which satisfies $a_0 = 2$ and

$$a_n = L_n(-1) + (-1)^n = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1} + (-1)^n = \sum_{k=1}^{n} L_{n,\,k} + (-1)^n$$

where $L_{n,\,k}$ are known as the signless Lah numbers, is a self-inverse sequence related to $L_n(x)$. From Corollary 5, we have the sequence $\{a_n\}$, which satisfies $a_0 = 1$ and

$$a_n = \sum_{l=0}^{n} \binom{n}{l} L_l(-1)(-1)^{n-l} = \sum_{l=1}^{n} \binom{n}{l}(-1)^{n-l} \sum_{k=1}^{l} L_{l,\,k} + (-1)^n,$$

is also a self-inverse sequence related to $L_n(x)$.

Applying Theorem 2, we have the following proposition.

Proposition 2 *Let $c_0 = c_1 = 0$ and $c_n = -\sum_{k=2}^{n} \frac{n!}{(n-k)!} b_{n-k}$. Then $\{c_n\}$ is a self-inverse sequence related to $L_n(x)$ if and only if $\{b_n\}$ is a self-inverse sequence related to $L_n(x)$.*

Proof Let $a_0 = a_1 = 0$ and $a_n = -n!$ $(n \geq 2)$ in Theorem 2. □

For the Laguerre polynomials $L_n(x)$, Theorems 3, 4 and 5 can be restated as follows.

Proposition 3 *$\{a_n\}$ is a self-inverse sequence related to $L_n(x)$ if and only if there exists a* *sequence* $\{\lambda_n\}$ *such that* $a_0 = 2\lambda_0$ *and*

$$a_n = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(-1)^k\lambda_k + \lambda_n, \quad n \geq 1.$$

Proposition 4 *Suppose* $\{f_n\}$ *is an arbitrary sequence, and let* $a_0 = f_0^2$ *and*

$$a_n = \sum_{l=1}^{n}\binom{n}{l} f_{n-l}\sum_{k=1}^{l}\frac{l!}{k!}\binom{l-1}{k-1}(-1)^k f_k + f_n f_0, \quad n \geq 1.$$

Then $\{a_n\}$ *is a self-inverse sequence related to* $L_n(x)$*.*

Proposition 5 $\{a_n\}$ *is a self-inverse sequence related to* $L_n(x)$ *if and only if there exists a function* $f : \mathbb{Z} \to \mathbb{C}$ *such that*

$$a_n = \sum_{k=0}^{n}\binom{n}{k} L_{n-k}(1)L_k(\Delta)f(0) + f(n).$$

By Corollary 1 and Theorem 6, we get:

Proposition 6 *If* $\{a_n\}$ *is a self-inverse sequence related to* $L_n(x)$*, then we have the following identities:*

$$a_{m+1} = -\sum_{l=1}^{m}\binom{m}{l}(m-l+1)!\sum_{n=1}^{l} a_{n+1}\frac{l!}{n!}\binom{l-1}{n-1}(-1)^n - (m+1)!a_1, \quad m \geq 1, \tag{6}$$

and

$$\sum_{k=0}^{n-1}\binom{2n}{k}(-1)^{n-k}(n-1)_k a_{n-k} = \sum_{k=0}^{n-1}\binom{-n}{k}(-1)^k(n-1)_k a_{n-k}, \quad n \geq 1. \tag{7}$$

Proof (6) holds from Corollary 1.

Using Theorem 6, we can get (7). By the closed forms for basic polynomials [3], we have that the delta operator $r(D) = D - D^2$ has a unique sequence of basic polynomials $r_0(x) = 1$ and

$$r_n(x) = \sum_{k=0}^{n-1}\binom{-n}{k}(-1)^k(n-1)_k x^{n-k}, \quad n \geq 1.$$

Denote by $q_n(x)$ the basic sequence for delta operator $r(q(D)) = -\frac{D}{(D-I)^2}$. Then we have

$$q_n(x) = \sum_{k=0}^{n-1}\binom{2n}{k}(-1)^{n-k}(n-1)_k x^{n-k}.$$

(7) is immediate from Theorem 6. $\square$

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