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Parameterization of Bivariate Nonseparable Orthogonal Symmetric Scaling Functions with Short Support

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Abstract Let I be the 2×2 identity matrix, and M a 2×2 dilation matrix with $M^2=2I$. First, we present the correlation of the scaling functions with dilation matrix M and $2I$. Then by relating the properties of scaling functions with dilation matrix $2I$ to the properties of scaling functions with dilation matrix M, we give a parameterization of a class of bivariate nonseparable orthogonal symmetric compactly supported scaling functions with dilation matrix M. Finally, a construction example of nonseparable orthogonal symmetric and compactly supported scaling functions is given.

Keywords nonseparable; bivariate; orthogonal; symmetric; compactly supported.

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1. Introduction

In recent years, nonseparable wavelets, especially bivariate nonseparable wavelets, have attracted the interest of many mathematicians. The details can be found in [1–6]. Although separable wavelet bases have a lot of advantages, they have a number of drawbacks. They are so special that they have very little design freedom, and separability imposes an unnecessary product structure on the plane which is artificial for natural images. One way to avoid this is through the construction of nonseparable wavelets.

Nonseparable wavelets have enough degrees of freedom to construct bases which have several properties simultaneously such as orthogonality, symmetry and compact support which is not possible for tensor-product scalar wavelets except for the Harr tensor. It is well-known that the construction of nonseparable scaling functions with dilation matrix $2I$ is mature [2–4, 7]. Lai and Roach [2] presented the complete solution of all bivariate nonseparable orthogonal symmetric and compactly supported scaling functions with dilation matrix $2I$ and filter size up to 6×6. Currently, it turns out that many researchers proceed to study the scaling function with dilation matrix M satisfying $M^2=2I$ (see [1, 5, 6]). Such dilation matrices make the MRA involve a unique wavelet which is easy to construct from the scaling function. Despite the success in

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constructing bivariate nonseparable orthogonal wavelets with arbitrarily high smoothness in [1], the nonseparable orthogonal symmetric and compactly supported wavelet is still a challenging problem. Obviously, if the dilation matrix M satisfies $M^2 = 2I$, then a nonseparable orthogonal symmetric scaling function with dilation matrix M must be a nonseparable orthogonal symmetric scaling function with dilation matrix $2I$. But the converse is not true.

In this paper, we first give the sufficient condition that a scaling function with dilation matrix $2I$ is also a scaling function with dilation matrix M. Then we study the nonseparable scaling functions with dilation matrix M by the nonseparable scaling functions with dilation matrix $2I$. Based on [2], we construct a complete parameterization of orthogonal symmetric scaling functions with dilation matrix M and filter size up to 6×6. The wavelet is easy to construct from the scaling function, so we deal mainly with the scaling functions.

2. Preliminaries and main results

Let M be an expanding 2×2 matrix, with integer entries such that $|\det M| = 2$, and the module of all eigenvalues are greater than 1. The key ingredients to an MRA with such a dilation matrix M are two functions: a scaling function ϕ and a wavelet ψ. The scaling function $\phi : R^2 \to R$ satisfies a dilation equation of the form

$$\phi(x) = 2\sum_{k\in Z^2} p(k)\phi(Mx-k), \tag{2.1}$$

where $\sum_{k\in Z^2} p(k) = 1$, $k = (k_1, k_2)$. The numbers $p(k)$ are called the scaling coefficients of $\phi(x)$. We assume that they are real and $p(k) \neq 0$ for only finitely many $k \in Z^2$. A convenient way to work with the scaling coefficients is to consider the coefficient mask (z-transform):

$$P(z_1, z_2) = \sum_{(k_1,k_2)\in Z^2} p(k_1, k_2) z_1^{k_1} z_2^{k_2},$$

where$(z_1, z_2) \in C^2$. By taking the Fourier transform on both sides of (2.1), we get

$$\hat{\phi}(M^T\omega) = m(\omega)\hat{\phi}(\omega), \quad \omega = (\omega_1, \omega_2), \tag{2.2}$$

where $m(\omega) = \sum_{k\in Z^2} p(k)e^{ik\cdot\omega}$ is called the mask symbol of the scaling function ϕ, and $k\cdot\omega$ denotes the Euclidean inner product. Then we have $m(\omega_1, \omega_2) = P(e^{i\omega_1}, e^{i\omega_2})$. The condition that all eigenvalues of the matrix M are greater than 1 means that $\lim_{j\to\infty} M^{-j} = 0$. As usual, it is convenient to normalize ϕ such that $\hat{\phi}(0) = 1$. Then from (2.2) it follows that

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} m\Big((M^T)^{-j}\omega\Big). \tag{2.3}$$

Since $(M^T)^{-2} = \frac{1}{2}I$, the infinite product (2.3) breaks into two parts:

$$\prod_{j=1}^{\infty} m\Big((M^T)^{-j}\omega\Big) = \prod_{j=1}^{\infty} m(2^{-j}\omega)\prod_{j=1}^{\infty} m(2^{-j}M^T\omega). \tag{2.4}$$

An important task in wavelet theory is to relate the properties of the scaling function $\phi(x)$ to the properties of the coefficient mask $P(z_1, z_2)$. Our goal is to find coefficients in (2.1) that produce a scaling function with orthogonality, symmetry and compact support.

In what follows we need Lemma 2.1 formulated below [6].

Lemma 2.1 *A 2×2 integer matrix M is a dilation matrix with $M^2 = 2I$ if and only if $M = \pm \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ for some $a \in Z_+$, and some $b, c \in Z$ satisfying $bc = 2 - a^2$.*

It is esay to check that a scaling function with dilation matrix M must be a scaling function with dilation matrix $2I$. Next we give the sufficient condition that a scaling function with dilation matrix $2I$ is also a scaling function with dilation matrix M.

Theorem 2.2 *Let $M = \pm \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, for some $a \in Z_+$, and some $b, c \in Z$ satisfying $bc = 2 - a^2$. Let $\phi(x)$ be a scaling function satisfying the following dilation equation*

$$\phi(x) = 4 \sum_{k \in Z^2} q(k)\phi(2Ix - k). \tag{2.5}$$

If the scaling coefficients $q(k)$ satisfiy the following conditon:

$$q(k) = \sum_{\ell \in Z^2} p(k)p(k - M\ell),$$

where $p(k)$ are the scaling coefficients given in (2.1), then $\phi(x)$ also satisfies the equation (2.1).

Proof First we assume that $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where a, b and c are defined as above. According to Lemma 2.1, we get $M^2 = 2I$. Since $\phi(x)$ satisfies the dilation equation (2.5), by (2.3), we get $\hat{\phi}(\omega) = \prod_{j=1}^{\infty} Q\Big(e^{i((2I)^T)^{-j}\omega}\Big)$. Since $q(k) = \sum_{\ell \in Z^2} p(k)p(k - M\ell)$, we get

$$Q(z_1, z_2) = P(z_1, z_2)P({z_1}^a {z_2}^c, {z_1}^b {z_2}^{(-a)}), \tag{2.6}$$

where $P(z_1, z_2)$, $Q(z_1, z_2)$ are the z-transform of scaling coefficients $p(k)$ and $q(k)$, respectively. Applying (2.6) and (2.4), we get

$$\begin{aligned}
\hat{\phi}(\omega) &= \prod_{j=1}^{\infty} Q\Big(e^{i((2I)^T)^{-j}\omega}\Big) = \prod_{j=1}^{\infty} Q(e^{i2^{-j}\omega_1}, e^{i2^{-j}\omega_2}) \\
&= \prod_{j=1}^{\infty} P(e^{i2^{-j}\omega_1}, e^{i2^{-j}\omega_2}) \prod_{j=1}^{\infty} P\Big(e^{i2^{-j}(a\omega_1 + c\omega_2)}, e^{i2^{-j}(b\omega_1 - a\omega_2)}\Big) \\
&= \prod_{j=1}^{\infty} P(e^{i2^{-j}\omega})P(e^{i2^{-j}M^T\omega}) = \prod_{j=1}^{\infty} P\Big(e^{i(M^T)^{-j}\omega}\Big),
\end{aligned}$$

where $\omega = (\omega_1, \omega_2)$. Applying (2.3) again, then $\phi(x)$ also satisfies the dilation equation (2.1). Similarly, for $M = -\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where a, b, c are defined as above, Theorem 2.2 remains valid.

In the following, for simplicity, we assume that $M = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

Let $Q(x, y) = \sum_{0 \le i, j \le 5} q(i, j)x^i y^j$ be a trigonometric polynomial with $x = e^{i\omega_1}, y = e^{i\omega_2}$, and $q(i,j)$ satisfying (2.5). And $Q(x,y)$ satisfies the following properties:

(i) Existence: $Q(1,1)=1$;

(ii) Orthogonality: $\mid Q(x,y)\mid^2+\mid Q(-x,y)\mid^2\mid Q(x,-y)\mid^2\mid Q(-x,-y)\mid^2=1$;

(iii) Symmetry: $Q(1/x,1/y)=x^{-5}y^{-5}Q(x,y)$;

(iv) Vanishing moments: $Q(x,y)=\frac{(x+1)}{2}\frac{(y+1)}{2}\widetilde{Q}(x,y)$, where $\widetilde{Q}(x,y)$ is another trigonometric polynomial.

According to the symmetric condition (iii), we can write the $q(i,j)$ in their polyphase form as shown below.

$$[q(i,j)]_{0\le i,j\le 5}=\begin{pmatrix} a_0 & b_0 & a_1 & b_1 & a_2 & b_2\\ b_8 & a_8 & b_7 & a_7 & b_6 & a_6\\ a_3 & b_3 & a_4 & b_4 & a_5 & b_5\\ b_5 & a_5 & b_4 & a_4 & b_3 & a_3\\ a_6 & b_6 & a_7 & b_7 & a_8 & b_8\\ b_2 & a_2 & b_1 & a_1 & b_0 & a_0 \end{pmatrix}. \tag{2.7}$$

In the following, the coefficients of trigonometric polynomials $Q(x,y)$ which satisfy properties (i)–(iv) are parameterized [2].

Lemma 2.3 *Let $Q(x,y)=\sum_{0\le i,j\le 5}q(i,j)x^iy^j$, where $q(i,j)$ is defined in (2.7). For any β, $\gamma\in[0,2\pi]$, let $\alpha=2(\beta-\gamma)+\pi/4$ or $\alpha=\pi/4$. Denote $p=\frac{1}{16}-\frac{1}{8\sqrt{2}}\cos\alpha$ and $q=\frac{1}{16}-\frac{1}{8\sqrt{2}}\sin\alpha$. If*

$$\begin{aligned}
a_0&=[-p(1+\cos(\beta-\gamma))-q\sin(\beta-\gamma)-\sqrt{p^2+q^2}(\cos\beta+\cos\gamma)]/4,\\
a_2&=[-p(1-\cos(\beta-\gamma))+q\sin(\beta-\gamma)-\sqrt{p^2+q^2}(\cos\beta-\cos\gamma)]/4,\\
a_6&=[-p(1-\cos(\beta-\gamma))+q\sin(\beta-\gamma)+\sqrt{p^2+q^2}(\cos\beta-\cos\gamma)]/4,\\
a_8&=[-p(1+\cos(\beta-\gamma))-q\sin(\beta-\gamma)+\sqrt{p^2+q^2}(\cos\beta+\cos\gamma)]/4,\\
b_0&=[-q(1+\cos(\beta-\gamma))+p\sin(\beta-\gamma)-\sqrt{p^2+q^2}(\sin\beta+\sin\gamma)]/4,\\
b_2&=[-q(1-\cos(\beta-\gamma))-p\sin(\beta-\gamma)-\sqrt{p^2+q^2}(\sin\beta-\sin\gamma)]/4,\\
b_6&=[-q(1-\cos(\beta-\gamma))-p\sin(\beta-\gamma)+\sqrt{p^2+q^2}(\sin\beta-\sin\gamma)]/4,\\
b_8&=[-q(1+\cos(\beta-\gamma))+p\sin(\beta-\gamma)+\sqrt{p^2+q^2}(\sin\beta+\sin\gamma)]/4,\\
a_1&=\frac{p}{2}+\frac{1}{2}\sqrt{p^2+q^2}\cos\beta,\quad b_1=\frac{q}{2}+\frac{1}{2}\sqrt{p^2+q^2}\sin\beta,\\
a_3&=\frac{p}{2}+\frac{1}{2}\sqrt{p^2+q^2}\cos\gamma,\quad b_3=\frac{q}{2}+\frac{1}{2}\sqrt{p^2+q^2}\sin\gamma,\\
a_5&=\frac{p}{2}-\frac{1}{2}\sqrt{p^2+q^2}\cos\gamma,\quad b_5=\frac{q}{2}-\frac{1}{2}\sqrt{p^2+q^2}\sin\gamma,\\
a_7&=\frac{p}{2}-\frac{1}{2}\sqrt{p^2+q^2}\cos\beta,\quad b_7=\frac{q}{2}-\frac{1}{2}\sqrt{p^2+q^2}\sin\beta,\\
a_4&=\frac{1}{4}-p,\qquad\qquad\qquad\quad b_4=\frac{1}{4}-q,
\end{aligned}$$

then $Q(x,y)$ satisfies properties (i)–(iv). On the other hand, if $Q(x,y)$ satisfies properties (i)–(iv), *and the conditions*

$$a_1 + a_4 + a_7 = 1/4, \ a_3 + a_4 + a_5 = 1/4 \tag{2.8}$$

hold, where a_v, $v = 0, \ldots, 5$, *is defined in (2.7), then the coefficients* a_v, b_v, $v = 0, \ldots, 5$, *of* $Q(x, y)$ *can be expressed in the above format.*

According to [2], the solutions of a_v, b_v, $v = 0, \ldots, 5$, in Lemma 2.3 can generate a nonseparable orthogonal symmetric scaling function.

Theorem 2.4 *Let* $M = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. *Let* ϕ *be a scaling function with dilation matrix* $2I$ *satisfying the dilation equation (2.5). If the mask symbol of* ϕ *has the form*

$$Q(x, y) = \sum_{0 \le i,j \le 5} \sum_{0 \le m,n \le 5} p(m, n) p(i - 2n, j - m) x^i y^j, \tag{2.9}$$

then $P(x, y) = \sum_{0 \le i,j \le 5} p(i, j) x^i y^j$ *can generate the scaling function* ϕ *with dilation matrix* M. *If* $Q(x, y)$ *satisfies properties (i)–(iv), and (2.8) holds, then* ϕ *defined in (2.1) is a nonseparable orthogonal symmetric and compactly supported scaling function with dilation matrix* M. *Furthermore, the scaling coefficients* $p(i, j)$, $i, j = 0, 1, \ldots, 5$, *can be expressed with two parameters.*

Proof Since ϕ satisfies the dilation equation (2.5) and $Q(x, y)$ satisfies (2.9), according to Theorem 2.2, ϕ also satisfies the dilation equation (2.1) with the coefficient mask $P(x, y)$. Since $Q(x, y)$ satisfies properties (i)–(iv) and (2.8) holds, by Lemma 2.3, we know that ϕ defined in (2.5) is nonseparable orthogonal symmetric and compactly supported. So ϕ defined in (2.1) is also a nonseparable orthogonal symmetric scaling function. By comparing the coefficients of $Q(x, y)$ in Lemma 2.3 and in (2.9), we can obtain a group of nonlinear equations about the scaling coefficients $p(i, j)$, $i, j = 0, 1, \ldots, 5$, and the equations must have solutions which can be expressed with two parameters β and γ in Lemma 2.3. We can select proper $\beta, \gamma \in [0, 2\pi]$ such that $a_0 > 0$. Then we get $p(0, 0) = \pm\sqrt{a_0}$. Consequently, we get $p(1, 0) = b_8/p(0, 0) = \pm b_8/\sqrt{a_0}$, and $p(0, 1) = \Big(b_0 - p(1, 0)p(0, 0)\Big)/p(0, 0) = \pm(b_0 - b_8)/\sqrt{a_0}$. Step by step, we can get the complete solutions of $p(i, j), 0 \le i, j \le 5$ which are expressed with two parameters β and γ.

For $M = \pm \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ as defined in Lemma 2.1, we have the similar conclusion.

Remark Since the scaling function $\phi(x)$ with dilation matrix M satisfies $|\det(M)| = 2$, we know that there exists a unique wavelet $\psi(x)$ associated with $\phi(x)$. It is known that if the wavelet coefficients $d(k)$ are given by

$$d(k) = (-1)^{k_1} p(e - k),$$

where $k = (k_1, k_2)$, and $e = (1, 0)$, then the wavelet $\psi(x) = 2\sum_{k \in Z^2} d(k)\phi(Mx - k)$ is orthogonal to $\{\phi(x - k), k \in Z^2\}$, and its translations and dilations form an orthogonal basis of $L^2(R^2)$ (see [8, 9]).

3. Example

Example According to Theorem 2.4, we choose $\beta = \pi/2$, $\gamma = 2\pi/3$ and get a group solution

$$[p(i,j)]_{0\leq i,j\leq 5} = \begin{pmatrix} 0.1800 & -0.4582 & -0.2751 & 1.1263 & -0.6067 & 0.7347 \\ 0.0237 & 0.0541 & -0.0513 & -0.1497 & -0.0718 & -0.0534 \\ 0.2719 & -0.8066 & 1.4555 & 2.5312 & -3.2966 & -0.4522 \\ 0.0849 & 0.1786 & 0.7224 & 0.9768 & -1.3491 & -2.7910 \\ 0.9736 & -2.9277 & 2.0109 & 10.4391 & -4.2024 & -9.3260 \\ 0.2040 & 0.3754 & 2.0362 & 1.2152 & -4.4585 & -4.1597 \end{pmatrix}.$$

If choosing $\beta = \pi/2$, $\gamma = 5\pi/6$, we get another group solution

$$[p(i,j)]_{0\leq i,j\leq 5} = \begin{pmatrix} 0.2243 & -0.5119 & -0.2838 & 1.2375 & -0.5173 & 0.3912 \\ -0.0295 & -0.0752 & 0.0186 & 0.2050 & 0.2003 & 0.0513 \\ 0.3047 & -0.5529 & 0.1849 & 2.3593 & 0.0445 & -0.6655 \\ 0.0916 & 0.2145 & 0.3035 & 0.9500 & 0.1575 & -2.4067 \\ 0.9622 & -1.6585 & -0.9509 & 7.2236 & 3.9714 & -6.0578 \\ 0.0423 & 0.1206 & 1.6741 & 3.1986 & -0.5290 & -7.4962 \end{pmatrix}.$$

Then according to Theorem 2.4, $P(x,y) = \sum_{0\leq i,j\leq 5} p(i,j)x^i y^j$ can generate a nonseparable orthogonal symmetric and compactly supported scaling function.

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