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Existence and Nonexistence of Positive Solutions to a Singular p-Laplacian Differential Equation

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Abstract In this paper we give a necessary and sufficient condition for the existence of positive solutions for the one-dimensional singular p-Laplacian differential equation. The methods used to show existence rely on upper-lower solutions method and compactness techniques, while the methods used to prove nonexistence are based on monotone techniques and scaling arguments.

Keywords p-Laplacian; singularity; positive solution; existence; nonexistence.

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1. Introduction and main results

In the paper we consider the differential equation:

$$\big(\Phi_p(u')\big)' + \frac{\lambda}{t}\Phi_p(u') - \frac{\gamma}{u}|u'|^p + f(t) = 0, \quad 0 < t < 1, \tag{1}$$

subject to the Dirichlet boundary condition:

$$u(1) = u(0) = 0, \tag{2}$$

where $\Phi_p(s) = |s|^{p-2}s$ with $p > 1$, $\lambda \geqslant 0, \gamma > 0$ and $f \in C[0,1]$. We say that $u \in C^1[0,1]$ with $|u'|^{p-2}u' \in C^1(0,1)$ is a solution to BVP (1) and (2) if it is positive in $(0,1)$ and satisfies (1) and (2).

The interesting feature of (1) is that the nonlinear term both is singular at $t = 0$ and $u = 0$ and depends on the derivative explicitly.

This kind of equation arises in the study of a class of degenerate parabolic filtration-absorption equations and in the theory of non-Newton fluids [1, 4–6, 19]. When $p = 2$, (1) with the boundary value condition:

$$u(1) = u'(0) = 0$$

has been studied in [5] under the assumptions: $\lambda = N - 1$, $\gamma > 0$ and $f \equiv 1$ and in [21] under the assumptions: $\lambda > 0$, $\gamma > \frac{1}{2}(1 + \lambda)$ and $f \in C[0,1]$ with $f > 0$ on $[0,1]$, respectively. By different methods, a decreasing solution was obtained in [5], while a solution u obtained in [21] is not decreasing and satisfies (see Theorem 1 and (19) in [21])

$$C_1t^2 \leqslant u(t) \leqslant C_2t^2, \quad 0 \leqslant t \leqslant 1/4,$$

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where C_i $(i=1,2)$ are positive constants, so it also satisfies (2). When $p>2$ and $\lambda=0$, by regularization method, it was shown in [20] that if $\gamma>\frac{p-1}{p}$ and $f\in C[0,1]$ with $f>0$ on $[0,1]$, then BVP (1) and (2) has a solution u satisfying (see Theorem 2 and (18) in [20])

$$C_1t^2\leqslant u(t)\leqslant C_2t^{\frac{p}{p-1}},\quad 0\leqslant t\leqslant 1/4, \tag{3}$$

where C_i $(i=1,2)$ are positive constants. It is easy to see that in the case where $1<p<2$, any solution to BVP (1) and (2) does not satisfy (3) since $p/(p-1)>2$.

Motivated by the above papers, we investigate the existence and nonexistence of solutions to BVP (1) and (2) in the present paper. By considering an approximate problem and using the upper and lower solutions method and compactness techniques, we first establish the following existence theorem which in particular covers the case where $1<p<2$.

Theorem 1 *Let $p>1$, $\lambda\geqslant 0$, $\gamma>0$, and let $f\in C[0,1]$ with $f>0$ on $[0,1]$. If $\gamma>\frac{p-1}{p}(1+\lambda)$, then BVP (1) and (2) has at least one solution in $\mathcal{C}$, which is defined as*

$$\mathcal{C}=\{v\in C[0,1]; C^{-1}t^{\frac{p}{p-1}}\leqslant v(t)\leqslant Ct^{\frac{p}{p-1}}, 0\leqslant t\leqslant\tau,$$

$$\textit{for some constants }\ \tau\in(0,1/4],\ C\geqslant 1\}\ \textit{near } t=0 \textit{ for } p>2.$$

Clearly, Theorem 1 is an extension to the existence results in [20] and [21] and an improvement to (3).

Next we study the nonexistence of solutions in $\mathcal{C}$. By monotone techniques and scaling arguments, we obtain

Theorem 2 *Assume that the hypotheses of Theorem 1 hold. If $\gamma\leqslant\frac{p-1}{p}(1+\lambda)$, then BVP (1) and (2) has no solution in $\mathcal{C}$.*

As an immediate consequence of the above results, we have

Theorem 3 *Under the hypotheses of Theorem 1, BVP (1) and (2) has one solution in $\mathcal{C}$ if and only if $\gamma>\frac{p-1}{p}(1+\lambda)$.*

Finally, let us remark that when $F(t,u,v)$ does not depend on the derivative, some existence results for boundary value problems to the following singular p-Laplacian have been obtained in a large number of papers (see [2, 3, 8, 11, 14, 18] and references therein):

$$\big(\Phi_p(u')\big)'+F(t,u,u')=0,\quad 0<t<1. \tag{4}$$

However, up to now, only a few papers deal with the case when $F(t,u,v)$ has a derivative dependence, see for example [9, 10, 12, 13, 15–17]. We point out that the case considered here, namely, $F(t,u,v)=\frac{\lambda}{t}\Phi_p(v)-\frac{\gamma}{u}|v|^p+f(t)$, is not contained in those papers mentioned above since it does not satisfy some sufficient conditions imposed on F.

We will prove Theorems 1 and 2 in Sections 2 and 3, respectively. In addition, an example is also given to illustrate our main result.

2. Proof of Theorem 1

Let $\epsilon \in (0,1)$, and define $H_\epsilon(r,v,\xi) : (0,1)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$H_\epsilon(r,v,\xi) = -\frac{\lambda}{t+\epsilon^{1/\alpha}}|\xi|^{p-2}\xi + \gamma\frac{|\xi|^p}{I_\epsilon(v)} - f(t),$$

where $\alpha = \frac{p}{p-1}$, and $I_\epsilon(v) = v+\epsilon^2$ if $v \geqslant 0$, $I_\epsilon(v) = \epsilon^2$ if $v < 0$. By the inequality: $a^{p-1} \leqslant a^p + 1$ $(a \geqslant 0)$, we have

$$\begin{aligned}
|H_\epsilon(t,v,\xi)| &\leqslant \frac{\lambda}{\epsilon^{1/\alpha}}|\xi|^{p-1} + \frac{\gamma}{\epsilon^2}|\xi|^p + \max_{[0,1]} f \\
&\leqslant \frac{\lambda}{\epsilon^{1/\alpha}}(1+|\xi|^p) + \frac{\gamma}{\epsilon^2}|\xi|^p + \max_{[0,1]} f \\
&\leqslant \Big(\frac{\lambda}{\epsilon^{1/\alpha}} + \frac{\gamma}{\epsilon^2} + \max_{[0,1]} f\Big)\mathcal{H}(|\xi|) \qquad (5)
\end{aligned}$$

for all $(t,v,\xi) \in (0,1)\times\mathbb{R}\times\mathbb{R}$, where $\mathcal{H}(s) = 1+s^p$ for $s \geqslant 0$. Denote $\mathcal{M} = \{u \in C^1(0,1); |u'|^{p-2}u' \in C^1(0,1)\}$ and define operator $\mathcal{L}_\epsilon : \mathcal{M} \to C(0,1)$ by

$$(\mathcal{L}_\epsilon u)(t) = -\big(|u'|^{p-2}u'\big)' + H_\epsilon(t,u,u'), \ \ 0<t<1.$$

Consider the problem:

$$\begin{cases} (\mathcal{L}_\epsilon u)(t) = 0, \quad 0<t<1, \\ u(1) = u(0) = 0. \end{cases} \qquad (6)$$

We say that u is an upper solution (lower solution) to BVP (6) if $\mathcal{L}_\epsilon u \geqslant (\leqslant) 0$ in $(0,1)$ and $u(t) \geqslant (\leqslant) 0$ at $t = 0,1$.

To show the existence of a solution to BVP (1), we will use the upper and lower solutions method (see Theorem 1 and Remark 2.4 in [9]). Note that $\int_0^{+\infty}\frac{s^{p-1}}{\mathcal{H}(t)}\mathrm{d}t = +\infty$. Then it suffices to find a lower solution and an upper solution to obtain a solution to BVP (6).

Lemma 1 *Let $W = C\psi^\alpha$ with $\alpha = \frac{p}{p-1}$, where $\psi(t)$ is defined by*

$$\psi(t) = \frac{p-1}{p}\Big[(\frac{1}{2})^{p/(p-1)} - |\frac{1}{2} - t|^{p/(p-1)}\Big], \qquad (7)$$

and the constant $C \in (0,1)$ such that $[(1+\lambda)\alpha^{p-1} + \gamma\alpha^p]C^{p-1} \leqslant \min_{[0,1]} f$. Then for any $\epsilon \in (0,1)$, W is a lower solution to BVP (6).

Proof It is easy to check that ψ possesses the following properties:

(a) $\psi > 0$ in $(0,1)$, $\psi \in C^1[0,1]$.

(b) $(|\psi'|^{p-2}\psi')' = -1$ in $(0,1)$, $\psi(1) = \psi(0) = 0$.

(c) $|\psi(t)| \leqslant t$, $|\psi'(t)| \leqslant 1$, $\forall t \in [0,1]$.

Using the properties of ψ, one arrives at

$$\begin{aligned}
\mathcal{L}_\epsilon W =& - \big(|W'|^{p-2}W'\big)' - \frac{\lambda}{t+\epsilon^{1/\alpha}}|W'|^{p-2}W' + \frac{\gamma}{W+\epsilon^2}|W'|^p - f(t) \\
\leqslant& - \big(|W'|^{p-2}W'\big)' - \frac{\lambda}{t+\epsilon^{1/\alpha}}|W'|^{p-2}W' + \frac{\gamma}{W}|W'|^p - \min_{[0,1]} f(t) \\
=& - (C\alpha)^{p-1}\psi(|\psi'|^{p-2}\psi')' - (C\alpha)^{p-1}|\psi'|^p -
\end{aligned}$$

$$
\begin{aligned}
&\frac{\lambda(C\alpha)^{p-1}}{t+\epsilon^{1/\alpha}}\psi|\psi'|^{p-2}\psi'+\gamma C^{p-1}\alpha^p|\psi'|^p-\min_{[0,1]}f(t)\\
=&(C\alpha)^{p-1}\psi-(C\alpha)^{p-1}|\psi'|^p-\\
&\frac{\lambda(C\alpha)^{p-1}}{t+\epsilon^{1/\alpha}}\psi|\psi'|^{p-2}\psi'+\gamma C^{p-1}\alpha^p|\psi'|^p-\min_{[0,1]}f(t)\\
\leqslant&(C\alpha)^{p-1}\psi+\frac{\lambda(C\alpha)^{p-1}}{t+\epsilon^{1/\alpha}}\psi|\psi'|^{p-1}+\gamma C^{p-1}\alpha^p|\psi'|^p-\min_{[0,1]}f(t)\\
\leqslant&(C\alpha)^{p-1}+\lambda(C\alpha)^{p-1}+\gamma C^{p-1}\alpha^p-\min_{[0,1]}f(t)\\
&\leqslant 0,\quad 0<t<1.
\end{aligned}
$$

The proof of the lemma is completed. □

Remark 1 Since $\psi'(t)=(1/2-t)^{1/(p-1)}$ on $[0,1/2]$, $\psi'(t)\geqslant 4^{-1/(p-1)}$ on $[0,1/4]$, and hence, by noticing $\psi(0)=0$,

$$\psi(t)\geqslant 4^{-1/(p-1)}t,\quad 0\leqslant t\leqslant 1/4. \tag{8}$$

Lemma 2 *There exist constants $\epsilon_0\in(0,1)$ and $C>0$ independent of ϵ, such that for any $\epsilon\in(0,\epsilon_0)$, $V_\epsilon=C(t+\epsilon^{1/\alpha})^\alpha$ with $\alpha=\frac{p}{p-1}$ is an upper solution to BVP (6).*

Proof Since $V_\epsilon\geqslant\epsilon$, we have

$$
\begin{aligned}
\mathcal{L}_\epsilon V_\epsilon&=-\big(|V_\epsilon'|^{p-2}V_\epsilon'\big)'-\frac{\lambda}{t+\epsilon^{1/\alpha}}|V_\epsilon'|^{p-2}V_\epsilon'+\frac{\gamma}{V_\epsilon+\epsilon^2}|V_\epsilon'|^p-f\\
&=-(C\alpha)^{p-1}-\lambda(C\alpha)^{p-1}+\frac{\gamma C^{p-1}\alpha^p}{1+C^{-1}(t+\epsilon^{1/\alpha})^{-\alpha}\epsilon^2}-f\\
&\geqslant-(C\alpha)^{p-1}-\lambda(C\alpha)^{p-1}+\frac{\gamma C^{p-1}\alpha^p}{1+C^{-1}\epsilon}-\max_{[0,1]}f\\
&=C^{p-1}\alpha^p\Big[\gamma-\frac{p-1}{p}(1+\lambda)\Big]-\max_{[0,1]}f+R_C(\epsilon),\quad 0<t<1,
\end{aligned}
$$

where

$$R_C(\epsilon)=\gamma\alpha^pC^{p-1}[(1+C^{-1}\epsilon)^{-1}-1].$$

Since $\gamma>\frac{p-1}{p}(1+\lambda)$, one can choose a positive constant C_* such that

$$C_*^{p-1}\alpha^p\Big[\gamma-\frac{p-1}{p}(1+\lambda)\Big]-\max_{[0,1]}f\geqslant 1.$$

Note that $R_{C_*}(\epsilon)\to 0$ $(\epsilon\to 0^+)$. Then there exists a constant $\epsilon_0\in(0,1)$, such that for all $\epsilon\in(0,\epsilon_0)$,

$$C_*^{p-1}\alpha^p\Big[\gamma-\frac{p-1}{p}(1+\lambda)\Big]-\max_{[0,1]}f+R_{C_*}(\epsilon)\geqslant 0.$$

So that for all $\epsilon\in(0,\epsilon_0)$,

$$\mathcal{L}_\epsilon V_\epsilon\geqslant 0,\quad 0<t<1.$$

The proof is completed. □

From Theorem 1 and Remark 2.4 in [9], it follows that for any fixed $\epsilon \in (0, \epsilon_0)$ BVP (6) has a solution $u_\epsilon \in C^1[0,1] \cap \mathcal{M}$ satisfying

$$C(t+\epsilon^{1/\alpha})^\alpha \geqslant u_\epsilon \geqslant C\psi^\alpha, \quad t \in [0,1]. \tag{9}$$

Hence, $u_\epsilon > 0$ in $(0,1)$ and satisfies

$$-\big(|u_\epsilon'|^{p-2}u_\epsilon'\big)' - \frac{\lambda}{t+\epsilon^{1/\alpha}}|u_\epsilon'|^{p-2}u_\epsilon' + \gamma\frac{|u_\epsilon'|^p}{u_\epsilon+\epsilon^2} - f(t) = 0, \quad t \in (0,1). \tag{10}$$

Next we shall establish some (locally) uniform estimates for u_ϵ'.

Lemma 3 *There exists a positive constant C independent of ϵ, such that for all $\epsilon \in (0, \epsilon_0)$,*

$$|u_\epsilon'(t)| \leqslant C, \quad \forall t \in [0,1]. \tag{11}$$

Proof Noticing $u_\epsilon(1) = u_\epsilon(0) = 0$ and $u_\epsilon \geqslant 0$ on $[0,1]$, we have

$$u_\epsilon'(0) \geqslant 0 \geqslant u_\epsilon'(1). \tag{12}$$

Integrating (10) over $(0,1)$ yields

$$-\big(|u_\epsilon'|^{p-2}u_\epsilon'\big)\Big|_0^1 + \gamma\int_0^1 \frac{|u_\epsilon'|^p}{u_\epsilon+\epsilon^2}\mathrm{d}t = \lambda\int_0^1 \frac{|u_\epsilon'|^{p-2}u_\epsilon'}{t+\epsilon^{1/\alpha}}\mathrm{d}t + \int_0^1 f(t)\mathrm{d}t.$$

By (12), one derives that

$$\gamma\int_0^1 \frac{|u_\epsilon'|^p}{u_\epsilon+\epsilon^2}\mathrm{d}t \leqslant \lambda\int_0^1 \frac{|u_\epsilon'|^{p-1}}{t+\epsilon^{1/\alpha}}\mathrm{d}t + \int_0^1 f(t)\mathrm{d}t. \tag{13}$$

Using Young's inequality: $ab \leqslant \sigma a^l + \sigma^{-q/l}b^q (a, b \geqslant 0, \sigma > 0, q, l > 1, \frac{1}{l} + \frac{1}{q} = 1)$, one deduces by taking $l = p/(p-1)$,

$$\frac{|u_\epsilon'|^{p-1}}{t+\epsilon^{1/\alpha}} \leqslant \sigma\frac{|u_\epsilon'|^p}{(t+\epsilon^{1/\alpha})^\alpha} + \sigma^{1-p}. \tag{14}$$

On the other hand, by the first estimate in (9) we obtain

$$u_\epsilon(t) + \epsilon^2 \leqslant C(t+\epsilon^{1/\alpha})^\alpha + \epsilon^2 \leqslant 2C(t+\epsilon^{1/\alpha})^\alpha, \quad t \in [0,1]. \tag{15}$$

Therefore

$$\frac{|u_\epsilon'|^p}{(t+\epsilon^{1/\alpha})^\alpha} \leqslant C\frac{|u_\epsilon'|^p}{u_\epsilon+\epsilon^2}, \quad t \in [0,1],$$

where C is a positive constant independent of ϵ. Combining this and (14), one obtains

$$\frac{|u_\epsilon'|^{p-1}}{t+\epsilon^{1/\alpha}} \leqslant \sigma C\frac{|u_\epsilon'|^p}{u_\epsilon+\epsilon^2} + \sigma^{1-p}, \quad t \in [0,1]. \tag{16}$$

Taking $\sigma = \frac{\gamma}{2\lambda C}$ in (16), one derives from (13) that

$$\frac{\gamma}{2}\int_0^1 \frac{|u_\epsilon'|^p}{u_\epsilon+\epsilon^2}\mathrm{d}t \leqslant \lambda(\frac{\gamma}{2\lambda C})^{1-p} + \int_0^1 f(t)\mathrm{d}t. \tag{17}$$

This and (16) imply that

$$\int_0^1 \frac{|u_\epsilon'|^{p-1}}{t+\epsilon^{1/\alpha}}\mathrm{d}t \leqslant C. \tag{18}$$

Again integrating (10) over (t_1, t_2), one gets

$$\left(|u'_\epsilon|^{p-2}u'_\epsilon\right)\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\Big(\frac{\gamma}{u_\epsilon+\epsilon^2}|u'_\epsilon|^p - \frac{\lambda}{t+\epsilon^{1/\alpha}}|u'_\epsilon|^{p-2}u'_\epsilon - f(t)\Big)\mathrm{d}t.$$

Combining this with (17) and (18), we find that there exists a positive constant C independent of ϵ, such that

$$\left||u'_\epsilon(t_2)|^{p-2}u'_\epsilon(t_2) - |u'_\epsilon(t_1)|^{p-2}u'_\epsilon(t_1)\right| \leqslant C, \quad \forall t_2, t_1 \in [0,1]. \tag{19}$$

Since $u_\epsilon(0) = u_\epsilon(1) = 0$ and $u_\epsilon \in C^1[0,1]$, by the mean value theorem, there exists $t_\epsilon^* \in (0,1)$, such that $u'_\epsilon(t_\epsilon^*) = 0$. Then taking $t_1 = t_\epsilon^*$ in (19) gives

$$\left||u'_\epsilon(t)|^{p-2}u'_\epsilon(t)\right| \leqslant C, \quad \forall t \in [0,1].$$

This completes the proof. □

Lemma 4 *For any $\delta \in (0, 1/2)$, there exists a positive constant C_δ independent of ϵ, such that for all $\epsilon \in (0, \epsilon_0)$,*

$$|u'_\epsilon(t_2) - u'_\epsilon(t_1)| \leqslant C_\delta |t_2 - t_1|^\beta, \quad \forall t_2, t_1 \in [\delta, 1-\delta], \tag{20}$$

where $\beta = 1/(p-1)$ if $p \geqslant 2$, $\beta = 1$ if $1 < p < 2$.

Proof By (9) and (11), it is easy to derive from (10) that for any $\delta \in (0, 1/2)$ there exists a positive constant C_δ independent of ϵ, such that for all $\epsilon \in (0, \epsilon_0)$,

$$\left|\left(|u'_\epsilon|^{p-2}u'_\epsilon\right)'\right| \leqslant C_\delta, \quad \delta \leqslant t \leqslant 1-\delta. \tag{21}$$

Recalling the inequality [7]:

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta')(\eta - \eta') \geqslant \begin{cases} C_1|\eta - \eta'|^p, & p \geqslant 2, \\ C_2|\eta-\eta'|^2(|\eta| + |\eta'|)^{p-2}, & 1 < p < 2 \end{cases}$$

for any $\eta, \eta' \in \mathbb{R}$, where C_i $(i = 1, 2)$ are positive constants depending only on p, one derives that if $p \geqslant 2$, then using (21) yields

$$\begin{aligned} |u'_\epsilon(t_2) - u'_\epsilon(t_1)|^p &\leqslant C_1^{-1}[u'_\epsilon(t_2) - u'_\epsilon(t_1)] \cdot [|u'_\epsilon(t_2)|^{p-2}u'_\epsilon(t_2) - |u'_\epsilon(t_1)|^{p-2}u'_\epsilon(t_1)] \\ &\leqslant C_\delta |u'_\epsilon(t_2) - u'_\epsilon(t_1)||t_2 - t_1|, \quad \forall t_2, t_1 \in [\delta, 1-\delta], \end{aligned}$$

hence

$$|u'_\epsilon(t_2) - u'_\epsilon(t_1)| \leqslant C_\delta |t_2 - t_1|^{1/(p-1)}, \quad \forall t_2, t_1 \in [\delta, 1-\delta],$$

and if $p \in (1, 2)$, then

$$\begin{aligned} &|u'_\epsilon(t_2) - u'_\epsilon(t_1)|^2[|u'_\epsilon(t_2)| + |u'_\epsilon(t_1)|]^{p-2} \\ &\quad\leqslant C_2^{-1}[u'_\epsilon(t_2) - u'_\epsilon(t_1)] \cdot [|u'_\epsilon(t_2)|^{p-2}u'_\epsilon(t_2) - |u'_\epsilon(t_1)|^{p-2}u'_\epsilon(t_1)] \\ &\quad\leqslant C_\delta |u'_\epsilon(t_2) - u'_\epsilon(t_1)||t_2 - t_1|, \quad \forall t_2, t_1 \in [\delta, 1-\delta], \end{aligned}$$

using (11) yields

$$|u'_\epsilon(t_2) - u'_\epsilon(t_1)| \leqslant C_\delta |t_2 - t_1| [|u'_\epsilon(t_2)| + |u'_\epsilon(t_1)|]^{2-p} \leqslant C_\delta |t_2 - t_1|$$

for all $t_2, t_1 \in [\delta, 1-\delta]$. This ends our proof. □

By (11) and (20) and using Arzelá-Ascoli theorem, there exist a subsequence of $\{u_\epsilon\}$, still denoted by $\{u_\epsilon\}$, and a function $u \in C^1(0,1) \cap C[0,1]$ such that, as $\epsilon \to 0^+$,

$$\begin{aligned} &u_\epsilon \to u \ \text{ uniformly in } \ C[0,1], \\ &u_\epsilon \to u \ \text{ uniformly in } \ C^1[\delta, 1-\delta], \end{aligned} \tag{22}$$

where $\delta \in (0, 1/2)$. Hence, it follows from $u_\epsilon(1) = u_\epsilon(0) = 0$ and (9) that $u(1) = u(0) = 0$ and

$$Ct^{p/(p-1)} \geqslant u(t) \geqslant C[\psi(t)]^{p/(p-1)}, \quad t \in [0,1],$$

therefore $u > 0$ in $(0,1)$. Noticing (8), we have $u \in \mathcal{C}$.

We now show that u satisfies (1). Integrating (10) over (t_0, r) $(0 < t_0, r < 1)$ gives

$$\begin{aligned} |u_\epsilon'(t)|^{p-2}u_\epsilon'(t) = \int_{t_0}^t \Big(\gamma \frac{|u_\epsilon'|^p}{u_\epsilon + \epsilon^2} - \frac{\lambda}{s + \epsilon^{1/\alpha}} |u_\epsilon'|^{p-2}u_\epsilon' - f(s)\Big) \mathrm{d}s + \\ |u_\epsilon'(t_0)|^{p-2}u_\epsilon'(t_0). \end{aligned}$$

Letting $\epsilon \to 0^+$ and using Lebesgue's dominated convergence theorem yield

$$|u'(t)|^{p-2}u'(t) = \int_{t_0}^t \Big(\gamma \frac{|u'|^p}{u} - \frac{\lambda}{s}|u'|^{p-2}u' - f\Big) ds + |u'(t_0)|^{p-2}u'(t_0). \tag{23}$$

This shows that $|u'(t)|^{p-2}u'(t) \in C^1(0,1)$ and (1) is satisfied.

It remains to show that $u \in C^1[0,1]$. In (17) and (18), letting $\epsilon \to 0^+$ and using Fatou's lemma yield

$$\int_0^1 \frac{|u'|^p}{u} \mathrm{d}t \leqslant C, \quad \int_0^1 \frac{|u'|^{p-1}}{t} \mathrm{d}t \leqslant C. \tag{24}$$

So, $\frac{|u'|^p}{u}, \frac{|u'|^{p-2}u'}{t} \in L^1[0,1]$. By (23), the function $\omega(t) = |u'(t)|^{p-2}u'(t) = \Phi_p(u'(t))$ is absolutely continuous on $[0,1]$. Since $u'(t) = \Phi_q(\omega(t))$, where $1/p + 1/q = 1$, we see that $u' \in C[0,1]$.

The proof of Theorem 1 is completed. □

3. Proof of Theorem 2

Assume that $u \in \mathcal{C}$ is a solution to BVP (1) and (2). Denote $w(t) = t^{-p/(p-1)}u(t)$. Since $|u'|^{p-2}u' \in C^1(0,1)$, $u \in C^2$ in some neighborhood of the point t where $u'(t) \neq 0$, and hence $w \in C^2$ in some neighborhood of the point t where

$$(t^{p/(p-1)}w)' \equiv t^{p/(p-1)}w' + \frac{p}{p-1}t^{1/(p-1)}w \neq 0.$$

Substituting $u = t^{p/(p-1)}w(t)$ into (1) yields

$$\begin{aligned} &\Big|t^{p/(p-1)}w' + \frac{p}{p-1}t^{1/(p-1)}w\Big|^{p-2} \cdot \\ &\Big((p-1)t^{p/(p-1)}w'' + 2pt^{1/(p-1)}w' + \frac{p}{p-1}t^{(2-p)/(p-1)}w\Big) + \\ &\lambda\Big|tw' + \frac{p}{p-1}w\Big|^{p-2}\Big(tw' + \frac{p}{p-1}w\Big) - \gamma\frac{|tw' + \frac{p}{p-1}w|^p}{w} + f(t) = 0 \end{aligned} \tag{25}$$

in $(0,1)$ at the points where

$$(t^{p/(p-1)}w)' \equiv t^{p/(p-1)}w' + \frac{p}{p-1}t^{1/(p-1)}w \neq 0.$$

Since $u \in \mathcal{C}$, there exist positive constants $\tau \in [0,1/4]$ and $C \geqslant 1$, such that

$$C^{-1} \leqslant w(t) \leqslant C, \quad 0 < t \leqslant \tau. \tag{26}$$

We claim that

$$w \text{ is monotone in some interval } (0,t_0) \subseteq (0,\tau). \tag{27}$$

Assume that this is not true. Then there exist two points $0 < t_2, t_1 < \tau$ such that $w'(t_2) < 0 < w'(t_1)$. Without loss of generality, we assume that $t_2 > t_1$. Then there exists some $t_3 \in (0,t_1)$ such that $w'(t_3) < 0$. Therefore, w reaches a minimum at some $t_* \in (t_3,t_1)$ such that

$$w'(t_*) = 0. \tag{28}$$

Clearly, $(t^{p/(p-1)}w)'(t_*) \neq 0$, so $w \in C^2$ in some neighborhood of t_*. Then

$$w''(t_*) \geqslant 0. \tag{29}$$

Using (28) and (29), one derives from (25) that

$$(\frac{p}{p-1})^p\Big[\frac{p-1}{p}(1+\lambda)-\gamma\Big][w(t_*)]^{p-1} + f(t_*) \leqslant 0.$$

Since $\gamma \leqslant (p-1)(1+\lambda)/p$, $f(t_*) \leqslant 0$. This contradiction proves the claim.

It follows from (26) and (27) that $\lim_{t\to 0^+} w(t)$ exists and is positive. Denote $\lim_{t\to 0^+} w(t) = M > 0$ and $w(0) = M$. Then $w \in C[0,1]$.

Let $w_\varepsilon(t) = w(\varepsilon t)$, $0 \leqslant t \leqslant \varepsilon^{-1}$, $\varepsilon \in (0,1)$, and $v_\varepsilon(t) = \Psi(t^{p/(p-1)}w_\varepsilon(t))$, $0 < t < 1$, where $\Psi(s) : (0,\infty) \to \mathbb{R}$ is defined as follows

$$\Psi(s) = \begin{cases} \frac{s^{1-\gamma/(p-1)}}{1-\gamma/(p-1)}, & \gamma \neq p-1, \\ \ln(s), & \gamma = p-1. \end{cases} \tag{30}$$

Clearly, as $\varepsilon \to 0^+$,

$$w_\varepsilon(t) \to M \quad \text{uniformly on} \quad [0,1], \tag{31}$$

$$v_\varepsilon(t) \to \Psi(Mt^{p/(p-1)}) \quad \text{uniformly on} \quad [1/2,1]. \tag{32}$$

By (31), there exists $\varepsilon_0 \in (0,1)$, such that for all $\varepsilon \in (0,\varepsilon_0)$,

$$M/2 \leqslant w_\varepsilon(t) \leqslant 3M/2 \quad \text{on} \quad [0,1]. \tag{33}$$

Moreover, it is easy to derive from (1) that

$$\big(\Phi_p([\Psi(u)]')\big)' + \frac{\lambda}{t}\Phi_p([(\Psi(u)]') + \frac{f(t)}{u^\gamma} = 0, \quad 0 < t < 1.$$

From this and some elementary calculations it follows that

$$\big(\Phi_p(v_\varepsilon')\big)' + \frac{\lambda}{t}\Phi_p(v_\varepsilon') + \frac{f(\varepsilon t)}{(t^{p/(p-1)}w_\varepsilon)^\gamma} = 0, \quad 0 < t < 1, \tag{34}$$

which is equivalent to

$$\big(t^{\lambda}\Phi_p(v_\varepsilon')\big)' + \frac{t^{\lambda} f(\varepsilon t)}{(t^{p/(p-1)} w_\varepsilon)^{\gamma}} = 0, \quad 0 < t < 1,$$

i.e.,

$$\big(\Phi_p(t^{\lambda/(p-1)} v_\varepsilon')\big)' + \frac{t^{\lambda} f(\varepsilon t)}{(t^{p/(p-1)} w_\varepsilon)^{\gamma}} = 0, \quad 0 < t < 1.$$

By (33), there exists a constant $C > 0$ independent of ε, such that for all $\epsilon \in (0, \epsilon_0)$,

$$\big|\big(\Phi_p(t^{\lambda/(p-1)} v_\varepsilon')\big)'\big| \leqslant C, \quad 1/2 \leqslant t < 1.$$

Like in the proof to (20), one can show that for all $\epsilon \in (0, \epsilon_0)$,

$$\big|t_2^{\lambda/(p-1)} v_\varepsilon'(t_2) - t_1^{\lambda/(p-1)} v_\varepsilon'(t_1)\big| \leqslant C|t_2 - t_1|^{\beta}, \quad 1/2 \leqslant t_2, t_1 < 1,$$

where β is the same as that in Lemma 2.4. By Arzelá-Ascoli theorem and noticing (32), there exists a subsequence of $\{v_\varepsilon\}$, still denoted by $\{v_\varepsilon\}$, such that

$$v_\varepsilon'(t) \to [\Psi(Mt^{p/(p-1)})]' \quad \text{in} \quad (1/2, 1), \quad \text{as } \varepsilon \to 0^{+}. \tag{35}$$

Multiplying (34) by $\phi \in C_0^1(1/2, 1)$ and integrating over $(1/2, 1)$, one obtains

$$\int_{1/2}^{1} \Big(-\Phi_p(v_\varepsilon')\phi' + \frac{\lambda}{t}\Phi_p(v_\varepsilon')\phi + \frac{f(\varepsilon t)\phi}{(t^{p/(p-1)} w_\varepsilon)^{\gamma}} \Big) \mathrm{d}t = 0.$$

Letting $\varepsilon \to 0^{+}$ and using (31) and (35), one gets

$$\int_{1/2}^{1} \Big(-\Phi_p([\Psi(Mt^{p/(p-1)})]')\phi' + \frac{\lambda}{t}\Phi_p([\Psi(Mt^{p/(p-1)})]')\phi + \frac{f(0)\phi}{(Mt^{p/(p-1)})^{\gamma}} \Big) \mathrm{d}t = 0.$$

Therefore, noticing (30), one has

$$-\int_{1/2}^{1} M^{p-1-\gamma} (\frac{p}{p-1})^{p-1} t^{1-p\gamma/(p-1)} \phi' \mathrm{d}t +$$
$$\int_{1/2}^{1} \Big(\lambda M^{p-1-\gamma} (\frac{p}{p-1})^{p-1} \frac{\phi}{t^{p\gamma/(p-1)}} + \frac{f(0)\phi}{M^{\gamma} t^{p\gamma/(p-1)}} \Big) \mathrm{d}t = 0.$$

And integrating by parts to the first integral, one obtains

$$\Big[M^{p-1} (\frac{p}{p-1})^{p-1} (1 + \lambda - \frac{p\gamma}{p-1}) + f(0) \Big] \int_{1/2}^{1} \frac{\phi}{t^{p\gamma/(p-1)}} \mathrm{d}t = 0,$$

which implies that

$$M^{p-1} (\frac{p}{p-1})^{p-1} (1 + \lambda - \frac{p\gamma}{p-1}) + f(0) = 0,$$

so $\gamma > \frac{p-1}{p}(1+\lambda)$ by noticing $f(0) > 0$, which leads to a contradiction.

The proof of Theorem 2 is completed. □

Example Let $\lambda \geqslant 0$. Consider the problem

$$\begin{cases} \big(|u'|^4 u'\big)' + \frac{\lambda}{t}|u'|^4 u' - \frac{11|u'|^6}{6u} + \frac{\sin(\pi t)}{100t+1} + 1 = 0, \quad 0 < t < 1, \\ u(1) = u(0) = 0. \end{cases} \tag{36}$$

Let $p = 6$, $\gamma = 11/6$, and $f(t) = \frac{\sin(\pi t)}{100t+1} + 1$. Clearly, $f \in C[0,1]$ with $f \geqslant 1$ on $[0,1]$. According to Theorem 3, problem (36) has one solution in $\mathcal{C}$ if and only if $0 \leqslant \lambda < 6/5$.

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