Journal of Mathematical Research & Exposition
Jul., 2010, Vol. 30, No. 4, pp. 653–663
DOI:10.3770/j.issn:1000-341X.2010.04.010
Http://jmre.dlut.edu.cn

Modified Thiele-Werner Rational Interpolation

Chang Wen LI[1,2,*], **Xiao Lin ZHU**[2], **Le ZOU**[2]

1. *Department of Mathematics, Huaibei Normal University, Anhui* 235000*, P. R. China;*
2. *Department of Mathematics, Hefei University of Technology, Anhui* 230009*, P. R. China*

Abstract Through adjusting the order of interpolation nodes, we gave a kind of modified Thiele-Werner rational interpolation. This interpolation method not only avoids the infinite value of inverse differences in constructing the Thiele continued fraction interpolation, but also simplifies the interpolating polynomial coefficients with constant coefficients in the Thiele-Werner rational interpolation. Unattainable points and determinantal expression for this interpolation are considered. As an extension, some bivariate analogy is also discussed and numerical examples are given to show the validness of this method.

Keywords interpolation; modified Thiele-Werner algorithm; unattainable point.

Document code A
MR(2000) Subject Classification 41A20; 65D05
Chinese Library Classification O241.5

1. Introduction

As an approximation tool, Thiele continued fractions interpolation has many advantages. But in the construction of this interpolation, some inverse differences will probably be ∞, which may produce unattainable points or cause unreliability. To avoid these problems, Werner gave a generalized Thiele interpolation, namely, Thiele-Werner rational interpolation with the form

$$R^{(0)}(x) = P_0(x) + \frac{\omega_0(x)}{P_1(x)} {}_{+\cdots+} \frac{\omega_{t-1}(x)}{P_t(x)}, \tag{1}$$

where $\omega_s(x) = (x-x_{c_s})(x-x_{c_s+1})\cdots(x-x_{d_s})$, $s=0,1,\ldots,t-1$ and $\sum_{s=0}^{t}(d_s-c_s+1)=n+1$, and each $P_s(x)\ (s=0,1,\ldots,t-1)$ is a Newton interpolating polynomial that interpolates $f^{(s)}(x)$ on $X_n^s=\{x_i\,|_i=c_S,c_s+1,\ldots,d_s\}$, and

$$f^{(s+1)}(x_i) = \frac{\omega_s(x_i)}{f^{(s)}(x_i)-P_s(x_i)} \qquad \begin{array}{l} s=0,1,\ldots,t-1; \\ i=c_{s+1},c_{s+1}+1,\ldots,n. \end{array} \tag{2}$$

Graves-Morris [1] had proved that this interpolation is a reliable method which avoids the infinite value of inverse difference. It is obvious that Thiele continued fractions interpolation is a special case of (1). In fact, if Thiele continued fractions interpolation is called a point-based

Received April 16, 2008; Accepted June 30, 2009
Supported by the National Natural Science Foundation of China (Grant No. 60473114) and the Natural Science Foundation of Anhui Province (Grant No. 070416227).
* Corresponding author
E-mail address: cwli2008@163.com (C. W. LI)

interpolation, then Thiele-Werner rational interpolation can be called a block-based interpolation. Zhao [9, 10] generalized the point-based interpolation to the block-based one, and he gets many new interpolation schemes by dividing interpolation set of support points into many subsets. However [1, 9, 10] did not tell the way how to divide the set, and this is an important problem for using the method. In this paper, a modified Thiele-Werner rational interpolation (MTWRI) is presented, which not only gives a method of how to divide the interpolation set of support points into subsets to avoid the infinite value of inverse differences, but also has some interesting results such as uniqueness and determinantal expression.

2. Modified Thiele-Werner rational interpolation

2.1 Modified Thiele-Werner algorithm

Suppose n is an integer and values $\{f_i^0, i = 0, 1, \ldots, n\}$ are associated with distinct interpolation points in the set $X_0 = \{x_0, x_1, \ldots, x_n\}$, respectively.

Input: (x_i, f_i^0), $i = 0, 1, \ldots, n$;

Output: A modified Thiele-Werner rational interpolant (MTWRI).

Initialization Take an x_{c_0} from X_0 arbitrarily, say $x_{c_0} = x_0$, if

$$f_i^0 = f_0^0, \quad i = 0, 1, \ldots, n, \tag{3}$$

then the algorithm ends with $R^{(0)}(x) = f_0^0$.

Otherwise adjusting the order of the elements in X_0, and renumbering the support points (x_i, f_i^0), $x_i \in X_0$, we have

$$\begin{cases} f_i^0 = f_0^0, \ i = c_0, c_0 + 1, \ldots, d_0, (c_0 = 0) \\ f_i^0 \neq f_0^0, \ i = d_0 + 1, d_0 + 2, \ldots, n, \end{cases} \tag{4}$$

where $d_0 \in \{0, 1, \ldots, n-1\}$. Define $b_0 = f_0^0$, $X_n^0 = \{x_{c_0}, x_{c_0+1}, \ldots, x_{d_0}\}$, $X_1 = X_0 \backslash X_n^0$ and $\omega_0(x) = (x - x_{c_0})(x - x_{c_0+1}) \cdots (x - x_{d_0})$, go to the next step.

Iteration For $j \in \{1, 2, \ldots, n\}$, define

$$f_i^j = \frac{\omega_{j-1}(x_i)}{f_i^{j-1} - b_{j-1}}, i = c_j, c_j + 1, \ldots, n, (c_j = d_{j-1} + 1), \tag{5}$$

where $\omega_{j-1}(x) = (x - x_{c_{j-1}})(x - x_{c_{j-1}+1}) \cdots (x - x_{d_{j-1}})$, if

$$f_i^j = f_{c_j}^j, \quad i = c_j, c_j + 1, \ldots, n, \tag{6}$$

then the algorithm ends with $t = j$ and $b_t = b_j$.

Otherwise adjusting the order of the elements in $X_j = X_{j-1} \backslash X_n^{j-1}$, and renumbering the support points $(x_i, f_i^j), x_i \in X_j$, we have

$$\begin{cases} f_i^j = f_{c_j}^j, \quad i = c_j, c_j + 1, \ldots, d_j, (c_j = d_{j-1} + 1) \\ f_i^j \neq f_{c_j}^j, \quad i = d_j + 1, d_j + 2, \ldots, n, \end{cases} \tag{7}$$

where $d_j \in \{c_j, c_j + 1, \ldots, n-1\}$. Define $b_j = f_i^j = f_{c_j}^j$ $(i = c_j, c_j + 1, \ldots, d_j)$, $X_n^j = \left\{x_{c_j}, x_{c_j+1}, \ldots, x_{d_j}\right\}, X_{j+1} = X_j \backslash X_n^j$, and the recurrence stops as $X_j = \emptyset$ for some j.

Termination For $j \in \{1, 2, \ldots, t-1\}$,

$$R^{(j)}(x) = b_j + \frac{\omega_j(x)}{b_{j+1}} \,+\cdots+\, \frac{\omega_{t-1}(x)}{b_t}. \tag{8}$$

The corresponding modified Thiele-Werner rational interpolation is obtained as follows

$$R^{(0)}(x) = b_0 + \frac{\omega_0(x)}{b_1} \,+\cdots+\, \frac{\omega_{t-1}(x)}{b_t}. \tag{9}$$

Obviously all the inverse differences f_i^j are not ∞, and $f_i^j \neq 0$ $(i = c_1, c_1+1, \ldots, n;\ j = 1, 2, \ldots, t)$, and therefore $b_j \neq 0$ $(j = 1, 2, \ldots, t)$. By the tail-to-head rationalization, we obtain a rational interpolant $R^{(0)}(x) = N(x)/D(x)$. We will clarify in Theorem 1 that the fraction $R^{(0)}(x) = N(x)/D(x)$ has the interpolation properties $R^{(0)}(x_i) = N(x_i)/D(x_i) = f_i^0$ $(i = 0, 1, \ldots, n)$.

2.2 Existence

Definition 1 *Let $R^{(0)}(x) = N(x)/D(x)$ be an MTWRI. A point (x_i, f_i^0) $(i \in \{0, 1, \ldots, n\})$ is called an unattainable point for $R^{(0)}(x)$ if*

$$N(x_i) - D(x_i)f_i^0 = 0, \quad \text{but} \quad R^{(0)}(x_i) = \frac{N(x_i)}{D(x_i)} \neq f_i^0.$$

Obviously, the MTWRI (x) satisfies the interpolation condition $R^{(0)}(x_i) = N(x_i)/D(x_i) = f_i^0$ $(i = 0, 1, \ldots, n)$ is equivalent to that there is no unattainable point for $R^{(0)}(x)$ in the set $\{(x_i, f_i^0) \,|\, i = 0, 1, \ldots, n\}$.

Theorem 1 *Consider an MTWRI of the form*

$$R^{(0)}(x) = b_0 + \frac{\omega_0(x)}{b_1} \,+\cdots+\, \frac{\omega_{t-1}(x)}{b_t}. \tag{10}$$

For some $x_i \in X_n^{s-1} = \{x_{c_{s-1}}, x_{c_{s-1}+1}, \ldots, x_{d_{s-1}}\}$ and $s \in \{1, 2, \ldots, t\}$, the point (x_i, f_i^0) is an unattainable point for $R^{(0)}(x)$ if and only if $R^{(s)}(x_i) = 0$, where $R^{(s)}(x) = N^{(s)}(x)/D^{(s)}(x)$ defined by

$$R^{(t)}(x) = b_t, \quad R^{(j)}(x) = b_j + \omega_j(x)\left[R^{(j+1)}(x)\right]^{-1} (j = t-1, t-2, \ldots, s), \tag{11}$$

where $\omega_j(x) = (x - x_{c_j})(x - x_{c_j+1})\cdots(x - x_{d_j})$ $(j = t-1, t-2, \ldots, s)$.

Proof Suppose $R^{(s)}(x_i) = 0$ for some $x_i \in X_n^{s-1} = \{x_{c_{s-1}}, x_{c_{s-1}+1}, \ldots, x_{d_{s-1}}\}$, that is, $x - x_i$ is the factor of $N_s(x)$. From Eq.(10), we have

$$\begin{aligned} R^{(0)}(x) &= b_0 + \frac{\omega_0(x)}{b_1} + \frac{\omega_1(x)}{b_2} +\cdots+ \frac{\omega_{s-2}(x)}{b_{s-1}} + \frac{\omega_{s-1}(x)}{R_s(x)} \\ &= b_0 + \frac{\omega_0(x)}{b_1} + \frac{\omega_1(x)}{b_2} +\cdots+ \frac{\omega_{s-2}(x)}{b_{s-1}} + \frac{\omega_{s-1}(x)/(x - x_i)}{\bar{R}_s(x)}, \end{aligned}$$

where $\bar{R}^{(s)}(x) = \bar{N}^{(s)}(x)/D^{(s)}(x)$ and $N^{(s)}(x) = (x - x_i)\bar{N}^{(s)}(x)$, $\omega_j(x) = (x - x_{c_j})(x -$

$x_{c_j+1})\cdots(x-x_{d_j})$, $j=0,1,\ldots,s-1$. Since $[\omega_{s-1}(x)/(x-x_i)]|_{x=x_i}\neq 0$,

$$R^{(0)}(x_i)=b_0+\frac{\omega_0(x_i)}{b_1}+\frac{\omega_1(x_i)}{b_2}+\cdots+\frac{\omega_{s-2}(x_i)}{b_{s-1}}+\frac{[\omega_{s-1}(x)/(x-x_i)]|_{x=x_i}}{\bar{R}_s(x_i)}$$

$$\neq b_0+\frac{\omega_0(x_i)}{b_1}+\cdots+\frac{\omega_{s-2}(x_i)}{b_{s-1}}=f_i^0.$$

Hence $R^{(0)}(x_i)\neq f_i^0$, which contradicts the assumption, and the sufficiency is proved.

If $R^{(0)}(x_i)\neq 0$, then from the Eq.(6), we have $b_k=f_i^k=\frac{\omega_{k-1}(x_i)}{f_i^{k-1}-b_{k-1}}$ $(k=0,1,\ldots,t)$. So

$$R^{(0)}(x_i)=b_0+\frac{\omega_0(x_i)}{b_1}+\frac{\omega_1(x_i)}{b_2}+\cdots+\frac{\omega_{s-3}(x_i)}{b_{s-2}}+\frac{\omega_{s-2}(x_i)}{b_{s-1}}$$

$$=b_0+\frac{\omega_0(x_i)}{b_1}+\cdots+\frac{\omega_{s-3}(x_i)}{f_i^{s-2}}=\cdots=b_0+\frac{\omega_0(x_i)}{f_i^1}$$

$$=b_0+f_i^0-b_0=f_i^0,$$

and the necessity is proved.

Remark 1 If $t=n$, then the corresponding modified Thiele-Werner rational interpolation is Thiele rational interpolation.

Remark 2 From Theorem 1, if we know all points (x_i,f_i^0) for x_i in X_n^{t-1} and X_n^t are attainable, then, by theorem 1, the points (x_i,f_i^0) for x_i in the subset X_n^{t-2} can be directly tested to be attainable or not. In fact,

$$R^{(t-2)}(x_i)=b_{t-1}+(f_{c_t}^{t-1}-b_{t-1})\cdot\frac{(x_{c_{t-1}}-x_i)(x_{c_{t-1}+1}-x_i)\cdots(x_{d_{t-1}}-x_i)}{(x_{c_{t-1}}-x_{c_t})(x_{c_{t-1}+1}-x_{c_t})\cdots(x_{d_{t-1}}-x_{c_t})}$$

$$=b_{t-1}+(f_{c_t}^{t-1}-b_{t-1})\cdot k.$$

So the point (x_i,f_i^0) for x_i in X_n^{t-2} is unattainable if and only if $b_{t-1}+(f_{c_t}^{t-1}-b_{t-1})\cdot k=0$, namely $\frac{f_{c_t}^{t-1}}{b_{t-1}}=\frac{k-1}{k}$.

2.3 Uniqueness and determinant expression

Next we will give the determinantal expression of the modified Thiele-Werner rational interpolant introduced above.

Corollary 1 *Let $R^{(t)}(x)=N_t(x)/D_t(x)$ be a modified Thiele-Werner rational interpolation of the form*

$$R^{(t)}(x)=b_0+\frac{\omega_0(x)}{b_1}+\cdots+\frac{\omega_{t-1}(x)}{b_t},\qquad(12)$$

then we have

$$N_t(x) = \begin{vmatrix} b_0 & \omega_0(x) & & & & & \\ -1 & b_1 & \omega_1(x) & & & & \\ & -1 & b_2 & \omega_2(x) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & b_{t-2} & \omega_{t-2}(x) & \\ & & & & -1 & b_{t-1} & \omega_{t-1}(x) \\ & & & & & -1 & b_t \end{vmatrix}, \tag{13}$$

$$D_t(x) = \begin{vmatrix} 1 & 0 & & & & & \\ -1 & b_1 & \omega_1(x) & & & & \\ & -1 & b_2 & \omega_2(x) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & b_{t-2} & \omega_{t-2}(x) & \\ & & & & -1 & b_{t-1} & \omega_{t-1}(x) \\ & & & & & -1 & b_t \end{vmatrix}. \tag{14}$$

Proof From Eqs (13) and (14), we get

$$N_j(x) = b_j N_{j-1}(x) + \omega_{j-1}(x) N_{j-2}(x), \quad j = t, t-1, \ldots, 2, \tag{15}$$

$$D_j(x) = b_j D_{j-1}(x) + \omega_{j-1}(x) D_{j-2}(x), \quad j = t, t-1, \ldots, 2, \tag{16}$$

where for $j = t, t-1, \ldots, 2$,

$$N_j(x) = \begin{vmatrix} b_0 & \omega_0(x) & & & & & \\ -1 & b_1 & \omega_1(x) & & & & \\ & -1 & b_2 & \omega_2(x) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & b_{j-2} & \omega_{j-2}(x) & \\ & & & & -1 & b_{j-1} & \omega_{j-1}(x) \\ & & & & & -1 & b_j \end{vmatrix}, \tag{17}$$

$$D_j(x) = \begin{vmatrix} 1 & 0 & & & & & \\ -1 & b_1 & \omega_1(x) & & & & \\ & -1 & b_2 & \omega_2(x) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & b_{j-2} & \omega_{j-2}(x) & \\ & & & & -1 & b_{j-1} & \omega_{j-1}(x) \\ & & & & & -1 & b_j \end{vmatrix} \tag{18}$$

and

$$N_0(x) = b_0, \quad D_0(x) = 1, \quad N_1(x) = b_0 b_1 + \omega_0(x), \quad D_1(x) = b_1.$$

If we define $N_{-1}(x) = 1$ and $D_{-1}(x) = 0$, then $N_t(x)$ and $D_t(x)$ can be obtained by the calculation through the recurrence relation for continued fractions of the form

$$b_0 + \frac{\omega_0(x)}{b_1} + \cdots + \frac{\omega_{t-1}(x)}{b_t}. \tag{19}$$

So we have

$$\frac{N_t(x)}{D_t(x)} = b_0 + \frac{\omega_0(x)}{b_1} + \cdots + \frac{\omega_{t-1}(x)}{b_t}. \tag{20}$$

□

2.4 Error estimation

Now we turn to discuss the error estimation of the modified Thiele-Werner rational interpolation.

Theorem 2 *Let $[a, b]$ be the smallest interval containing $X_n = \{x_0, x_1, \ldots, x_n\}$, $f(x)$ be differentiable on $[a, b]$ up to $n+1$ times and $N(x)$ be a modified Thiele-Werner rational interpolation of the form*

$$b_0 + \frac{\omega_0(x)}{b_1} + \cdots + \frac{\omega_{t-1}(x)}{b_t} \triangleq \frac{N(x)}{D(x)}. \tag{21}$$

Then there exists a point $\xi \in [a, b]$ for $\forall x \in [a, b]$, such that

$$f(x) - R(x) = \frac{\omega(x)}{D(x)} \cdot \frac{[f(x)D(x) - N(x)]^{(n+1)}_{x=\xi}}{(n+1)!}, \tag{22}$$

where $\omega(x) = \prod_{i=0}^{n}(x - x_i)$.

Proof Let $E(x) = f(x)D(x) - N(x)$. Then $E(x_i) = 0$ $(i = 0, 1, \ldots, n)$. Making use of the Newton interpolation formula, we have

$$E(x) = \sum_{i=0}^{n} E[x_0, x_1, \ldots, x_i]\overline{\omega_i}(x) + \omega(x)\frac{E^{(n+1)}(\xi)}{(n+1)!} = \omega(x)\frac{E^{(n+1)}(\xi)}{(n+1)!},$$

where $\overline{\omega_i}(x) = (x - x_0)(x - x_1)\cdots(x - x_{i-1})$, $\xi \in (a, b)$. It is easy to verify that

$$f(x) - R(x) = \frac{E(x)}{D(x)} = \frac{\omega(x)}{D(x)} \cdot \frac{E^{(n+1)}(\xi)}{(n+1)!} = \frac{\omega(x)}{D(x)} \cdot \frac{[f(x)D(x) - N(x)]^{(n+1)}_{x=\xi}}{(n+1)!},$$

and the theorem is proved. □

2.5 Numerical examples

Let $X_0 = \{0, 1, 2, 3, 4, 5\}$ and $\{f_1^0, f_2^0, f_3^0, f_4^0, f_5^0\} = \{2, 4, 8, 20, 10, 8\}$. Try to find a Thiele rational interpolant and a modified Thiele-Werner rational interpolant to satisfy the interpolating condition.

According to the Thiele algorithm, we have the table of inverse differences for Thiele rational interpolation

2					
4	1/2				
8	1/3	− 6			
20	1/6	− 6	∞		
10	1/2	∞	0	0	
8	5/6	12	1/6	0	∞

So we get the Thiele rational interpolation

$$R(x) = 2 + \frac{x}{0.5} + \frac{x-1}{-6} + \frac{x-2}{\infty} + \frac{x-3}{0} + \frac{x-4}{\infty}$$

$$= 2 + \frac{x}{0.5} + \frac{x-1}{-6} = \frac{-4x-8}{x-4}.$$

Obviously, the points (4,10) and (5,8) are unattainable for $R(x)$.

According to the modified Thiele-Werner rational interpolation algorithm, we can choose $x_{c_0} = 0$. Then dividing X_0 into $X_5^0 = \{0\}$, $X_5^1 = \{1,4\}$, $X_5^2 = \{2,5\}$, $X_5^2 = \{3\}$, we have the table of inverse differences for a modified Thiele-Werner rational interpolant

2			
4	1/2		
10	1/2		
8	1/3	12	
8	5/6	12	
20	1/6	6	1/3

So we have a modified Thiele-Werner rational interpolant

$$R^{(0)}(x) = 2 + \frac{x}{1/2} + \frac{(x-1)(x-4)}{12} + \frac{(x-2)(x-5)}{1/3}$$

$$= \frac{6x^3 - 10x^2 - 62x + 100}{5x^2 - 31x + 50}.$$

After verification, the modified Thiele-Werner rational interpolant $R^{(0)}(x)$ satisfies all the given interpolating conditions.

3. Multivariate modified Newton-Thiele-Werner blending interpolation

The modified Thiele-Werner rational interpolation method can be generalized to the multivariate case. Here we consider the case of Newton-Thiele-Werner blending interpolation with the form

$$R_{m,n}(x,y) = A_0(y) + (x-x_0)A_1(y) + \cdots + (x-x_0)\cdots(x-x_{m-1})A_m(y) \tag{23}$$

where $A_i(y) = P_i^0(y) + \frac{\omega_i^0(y)}{P_i^1(y)} + \cdots + \frac{\omega_i^{t_i-1}(y)}{P_i^{t_i}(y)}$, and $\omega_i^s(y) = (y - y_{c_i^s})(y - y_{c_i^s+1}) \cdots (y - y_{d_i^s})$ $(i = 0, 1, \ldots, m; \quad s = 0, 1, \ldots, t_i - 1)$.

Let n, m be nonnegative integers and values $\{f_{ij}^{00} | i = 0, 1, \ldots, m;\ j = 0, 1, \ldots, n\}$ be associated with the distinct interpolation points in $\prod_{mn}$, where $\prod_{mn} = \prod_m \times \prod_n$, $\prod_m = \{x_i \,| i = 0, 1, \ldots, m\}$ and $\prod_n = \{y_j \,| j = 0, 1, \ldots, n\}$. To avoid the infinite value inverse difference, we give a method of dividing $\prod_{in} = \{(x_i, y_j) \,| j = 0, 1, \ldots, n\}$ into subsets $\prod_{in}^{is} = \{(x_i, y_j) | j = c_s, c_s + 1, \ldots, d_s\}$ with $s = 0, 1, \ldots, t_i$ for $i \in \{0, 1, \ldots, m\}$.

3.1 Modified Newton-Thiele-Werner algorithm and property

Step 1. Define

$$M = \begin{bmatrix} f_{0,0}^{(0,0)} & f_{1,0}^{(0,0)} & \cdots & f_{m,0}^{(0,0)} \\ f_{0,1}^{(0,0)} & f_{1,1}^{(0,0)} & \cdots & f_{m,1}^{(0,0)} \\ \vdots & \vdots & \vdots & \vdots \\ f_{0,n}^{(0,0)} & f_{1,n}^{(0,0)} & \cdots & f_{m,n}^{(0,0)} \end{bmatrix}. \tag{24}$$

For $j = 0, 1, \ldots, n;\ p = 1, 2, \ldots, m;\ i = p, p+1, \ldots, m$, define

$$f_{i,j}^{(p,0)} = \frac{f_{i,j}^{(p-1,0)} - f_{p-1,j}^{(p-1,0)}}{x_i - x_{p-1}}. \tag{25}$$

By Eq.(25), we change M into

$$M_1 = \begin{bmatrix} f_{0,0}^{(0,0)} & f_{1,0}^{(1,0)} & \cdots & f_{m,0}^{(m,0)} \\ f_{0,1}^{(0,0)} & f_{1,1}^{(1,0)} & \cdots & f_{m,1}^{(m,0)} \\ \vdots & \vdots & & \vdots \\ f_{0,n}^{(0,0)} & f_{1,n}^{(1,0)} & \cdots & f_{m,n}^{(m,0)} \end{bmatrix} = [M_{01}, M_{11}, \ldots, M_{m1}]^{\mathrm{T}}. \tag{26}$$

Step 2. We change the elements of each row M_{i1} of M_1 into M_{i2} by Modified Thiele-Werner algorithm, and denote

$$M_2 = \begin{bmatrix} b_0^0 & b_1^0 & \cdots & b_m^0 \\ b_0^1 & b_1^1 & \cdots & b_m^1 \\ \vdots & \vdots & & \vdots \\ b_0^{t_0} & b_1^{t_1} & \cdots & b_m^{t_m} \end{bmatrix} = [M_{02}, M_{12}, \ldots, M_{m2}]^{\mathrm{T}}. \tag{27}$$

Therefore, the set $\prod_{in} = \{(x_i, y_j) \,| j = 0, 1, \ldots, n\}$ is divided into

$$\prod_{in}^{is} = \{(x_i, y_j) \,| j = c_s, c_s + 1, \ldots, d_s\} \text{ for every } i \in \{0, 1, \ldots, m\}, \text{ where } s = 0, 1, \ldots, t_i.$$

Remark 3 $t_0, t_1, \ldots, t_m$ are not always the same, so M_2 may not be a matrix. For convenience, we still note it with a matrix form.

Step 3. Using the elements of $M_{i2} = [b_i^0, b_i^0, \ldots, b_i^{t_i}]^{\mathrm{T}}$ $(i = 0, 1, \ldots, m)$, we can construct the modified Thiele-Werner rational interpolation

$$A_i(y) = b_i^0 + \frac{\omega_i^0(y)}{b_i^1} + \cdots + \frac{\omega_i^{t_i-1}(y)}{b_i^{t_i}} \quad (i = 0, 1, \ldots, m), \tag{28}$$

where $\omega_i^s(y) = (y - y_{c_i^s})(y - y_{c_i^s+1}) \cdots (y - y_{d_i^s})$ with $s = 0, 1, \ldots, t_i - 1$ and $\sum_{s=0}^{t_i} (d_i^s - c_i^s + 1) = n + 1$.

Step 4. Let

$$R_{m,n}(x, y) = A_0(y) + (x - x_0)A_1(y) + \cdots + (x - x_0)\cdots(x - x_{m-1})A_m(y). \tag{29}$$

Finally we get a blending fraction $R_{m,n}(x, y)$ and if

$$R_{m,n}(x_i, y_j) = f_{ij}^{00} \ , \quad i = 0, 1, \ldots, m; \quad j = 0, 1, \ldots, n, \tag{30}$$

then $R_{m,n}(x, y)$ is called a modified Newton-Thiele-Werner blending interpolant (MNTWBI).

Definition 2 *Let $R_{m,n}(x, y) = N_{m,n}(x, y)/D_{m,n}(x, y)$ be an MNTWBI with the form (23). A point (x_i, y_j, f_{ij}^{00}) ($i \in \{0, 1, \ldots, m\}$, $j \in \{0, 1, \ldots, n\}$) is called an unattainable point for $R_{m,n}(x, y)$ if*

$$R_{m,n}(x_i, y_j) - f_{ij}^{00} \cdot D_{m,n}(x_i, y_j) = 0, \quad \text{but} \quad R_{m,n}(x_i, y_j) = \frac{N_{m,n}(x_i, y_j)}{D_{m,n}(x_i, y_j)} \neq f_{ij}^{00}.$$

Let $R_{m,n}(x, y) = N_{m,n}(x, y)/D_{m,n}(x, y)$ be an MNTWBI with the form (23). A point (x_i, y_j, f_{ij}^{00}) ($i \in \{0, 1, \ldots, m\}$, $j \in \{0, 1, \ldots, n\}$) is called an unattainable point for $R_{m,n}(x, y)$ if

$$R_{m,n}(x_i, y_j) - f_{ij}^{00} \cdot D_{m,n}(x_i, y_j) = 0, \quad \text{but} \quad R_{m,n}(x_i, y_j) = \frac{N_{m,n}(x_i, y_j)}{D_{m,n}(x_i, y_j)} \neq f_{ij}^{00}.$$

Theorem 3 *Consider an MNTWBI of the form (23). The point (x_i, y_j, f_{ij}^{00}) is an unattainable point for $R_{m,n}(x, y)$ if and only if $A_i^{(s)}(y_j) = 0$ for some $s \in \{1, 2, \ldots, t_i\}$, where $j = t_i - 1, t_i - 2, \ldots, s$, and $\omega_i^j(y) = (y - y_{c_i^j})(y - y_{c_i^j+1}) \cdots (y - y_{d_i^j})$.*

The proof is analogous to that of Theorem 1.

We now turn to discuss the error estimation of the modified Newton-Thiele-Werner blending interpolation. It is easy to verify the following theorem based on bivariate Newton interpolation formula.

Theorem 4 *Suppose $D = [a, b] \times [c, d]$ is a rectangular domain containing $\prod_{nm}$ and $f(x, y) \in C^{(n+m+2)}(D)$. Let*

$$R_{m,n}(x, y) = A_0(y) + (x - x_0)A_1(y) + \cdots + (x - x_0)\cdots(x - x_{m-1})A_m(y) = \frac{P(x, y)}{Q(x, y)} \tag{31}$$

be a modified Newton-Thiele-Werner blending interpolant on $\prod_{nm}$. Then for $\forall (x, y) \in D$, we have

$$\begin{aligned} &f(x, y) - R_{m,n}(x, y) \\ &= \frac{1}{(n+1)!} \cdot \frac{\omega(x)}{Q(x, y)} \cdot \frac{\partial^{n+1}[fQ - P]}{\partial x^{n+1}}\bigg|_{x=\xi} A\# + \frac{1}{(m+1)!} \cdot \frac{\omega^*(y)}{Q(x, y)} \cdot \frac{\partial^{m+1}[fQ - P]}{\partial y^{m+1}}\bigg|_{y=\eta} - \\ &\quad \frac{1}{(n+1)!(m+1)!} \cdot \frac{\omega(x)\omega^*(y)}{Q(x, y)} \cdot \frac{\partial^{n+m+2}[fQ - P]}{\partial x^{n+1} \partial y^{m+1}}\bigg|_{x=\overline{\xi}, y=\overline{\eta}} \end{aligned}$$

with $\xi,\ \overline{\xi} \in (a,b)$ *and* $\eta,\ \overline{\eta} \in (c,d)$, *where*

$$\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_{n-1}),$$

$$\omega^*(y) = (y-y_0)(y-y_1)\cdots(y-y_{m-1}).$$

3.2 Numerical examples

Given the data

	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$
$x_0 = 1$	1	2/3	1	7/4
$x_1 = 2$	3	3/11	5	19/4
$x_2 = 3$	7	20/3	37/3	231/20

Table 1 Interpolation data

By the Newton-Thiele blending interpolation algorithm, we have the corresponding inverse difference table

1	-3	0	7/2
2	1	∞	0
1	2	1	2

Table 2 Inverse difference

So we get the classical Newton-Thiele interpolation function

$$T(x,y) = 1 + \frac{y}{-3}{}_{+}\frac{y-1}{0}{}_{+}\frac{y-2}{7/2} +$$

$$(x-1)\left[2+\frac{y}{1}{}_{+}\frac{y-1}{\infty}{}_{+}\frac{y-2}{0}\right] + (x-1)(x-2)\left[1+\frac{y}{2}{}_{+}\frac{y-1}{1}{}_{+}\frac{y-2}{2}\right]$$

$$= \frac{2y^2-3y+5}{y+5} + (x-1)(2+y) + (x-1)(x-2)\frac{y^2+4y-2}{4y-2}.$$

Obviously the point $(1,3,7/4)$ is unattainable for $T(x,y)$.

Next, we give the modified Newton-Thiele-Werner blending interpolation. For convenience, we choose $y_0 = 0$ for every x_i $(i = 0,1,2)$ by the modified Newton-Thiele-Werner blending interpolation algorithm,

$\prod_{03} = \{(x_0,y_j)\,|\,j=0,1,2,3\}$ is divided into $\prod_{03}^{00} = \{(1,0),(1,2)\}$, $\prod_{03}^{01} = (1,1)$, $\prod_{03}^{02} = (1,3)$, $\prod_{13} = \{(x_1,y_j)\,|\,j=0,1,2,3\}$ is divided into $\prod_{13}^{10} = (2,0)$, $\prod_{13}^{11} = \{(2,1),(2,2)\}$, $\prod_{13}^{12} = (2,3)$, and $\prod_{23} = \{(x_2,y_j)\,|\,j=0,1,2,3\}$ is divided into $\prod_{23}^{20} = (3,0)$, $\prod_{23}^{21} = (3,1)$, $\prod_{23}^{22} = (3,2)$, $\prod_{23}^{23} = (3,3)$.

From Eq.(28), we have

$$A_0(y) = 1 + \frac{y(y-2)}{3}{}_{+}\frac{(y-1)}{2},$$

$$A_1(y) = 2 + \frac{y}{1} \underset{+}{} \frac{(y-1)(y-2)}{1},$$

$$A_3(y) = 1 + \frac{y}{2} \underset{+}{} \frac{y-1}{1} \underset{+}{} \frac{(y-2)}{2}.$$

From Eq.(29), we finally obtain

$$R_{m,n}(x,y) = 1 + \frac{y(y-2)}{3} \underset{+}{} \frac{y-1}{2} +$$

$$(x{-}1)\left[2 + \frac{y}{1} \underset{+}{} \frac{(y-1)(y-2)}{1}\right] + (x{-}1(x{-}2)\left[1 + \frac{y}{2} \underset{+}{} \frac{y-1}{1} \underset{+}{} \frac{y-2}{2}\right]$$

$$= \frac{2y^2-3y+5}{y+5} + (x-1)\frac{2y^2-5y+6}{y^2-3y+3} + (x-1)(x-2)\frac{y^2+4y-2}{4y-2},$$

which has been verified to satisfy the interpolation condition.

4. Conclusion

This paper presents a kind of univariate and bivariate modified Thiele-Werner rational interpolation, which can be obtained by modified Newton-Thiele-Werner algorithm, and their existence and uniqueness are discussed. Our future work will be focused on the following aspects:

- Study the type of this interpolation.
- Find a method to solve the unattainable points of this interpolation.
- Study the application in image process.

We conclude this paper by pointing out that it is not difficult to generalize the Modified Thiele-Werner rational interpolation to vector-valued or matrix-valued case.

References

[1] GRAVES-MORRIS P R. *Efficient reliable rational interpolation* [C]. van Rossum H, deBruin M, Pade approximation and itsapplications, 1984: 29–63.

[2] GRAVES-MORRIS P R. *Vector valued rational interpolants (I)* [J]. Numer. Math., 1983, **42**(3): 331–348.

[3] TAN Jieqing. *Bivariate blending rational interpolants* [J]. Approx. Theory Appl. (N.S.), 1999, **15**(2): 74–83.

[4] WANG Jinbo, GU Chuanqing. *Vector valued Thiele-Werner-type osculatory rational interpolants* [J]. J. Comput. Appl. Math., 2004, **163**(1): 241–252.

[5] ZHU Xiaolin, ZHU Gongqin. *A method for directly finding the denominator value of rational interpolants* [J]. J. Comput. Appl. Math., 2002, **148**(2): 341–348.

[6] ZHU Xiaolin, ZHU Gongqin. *A study of the existence of vector valued rational interpolation* [J]. J. Infor. Comput. Science, 2005, **2**: 631–640.

[7] ZHU Xiaolin, ZHU Gongqin. *The note on matrix-valued rational interpolation* [J]. Numer. Math. J. Chinese Univ. (English Ser.), 2005, **14**(4): 305–314.

[8] WANG Renhong, ZHU Gongqin. *The Approximation of Rational Functions and Applications* [M]. Beijing: Science Publishers, 2004. (in Chinese)

[9] ZHAO Qianjin, TAN Jieqing. *Block based Newton-like blending interpolation* [J]. J. Comput. Math., 2006, **24**(4): 515–526.

[10] WANG Renhong. *Numerical Approximation* [M]. Beijing: Higher Education Press, 1999. (in Chinese)