Journal of Mathematical Research & Exposition
Jul., 2010, Vol. 30, No. 4, pp. 675–686
DOI:10.3770/j.issn:1000-341X.2010.04.012
Http://jmre.dlut.edu.cn

A New Multiplicity Formula for the Weyl Modules of Type B and C

Jia Chen YE*, **Ben Sheng YUN**, **Jia ZHAN**
Department of Mathematics, Tongji University, Shanghai 200092*, P. R. China*

Abstract A monomial basis and a filtration of subalgebras for the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\mathfrak{l})$ of a complex simple Lie algebra $\mathfrak{g}_\mathfrak{l}$ of type B_l and C_l are given, and the decomposition of the Weyl module $V(\lambda)$ as a $\mathfrak{U}(\mathfrak{g}_\mathfrak{l})$-module into a direct sum of Weyl modules $V(\mu)$'s as $\mathfrak{U}(\mathfrak{g}_{\mathfrak{l}-\mathfrak{1}})$-modules is described. In particular, a new multiplicity formula for the Weyl module $V(\lambda)$ is obtained in this note.

Keywords simple Lie algebra; multiplicity formula; weight; irreducible module.

Document code A
MR(2000) Subject Classification 17B10; 20G05
Chinese Library Classification O152.6

Let $\mathfrak{g}_\mathfrak{l}$ be a complex simple Lie algebra of type B_l or C_l, and $\mathfrak{U} = \mathfrak{U}(\mathfrak{g}_\mathfrak{l})$ its universal enveloping algebra. For any dominant integral weight $\lambda \in \Lambda^+$, $V(\lambda)$ denotes a finite dimensional irreducible $\mathfrak{U}(\mathfrak{g}_\mathfrak{l})$-module, the Weyl module. Following Littelmann [4], we define a monomial basis, and then construct a filtration of subalgebras for $\mathfrak{U}(\mathfrak{g}_\mathfrak{l})$. Furthermore, we describe a monomial basis for the Weyl module, and show how one can decompose the Weyl module $V(\lambda)$ as a $\mathfrak{U}(\mathfrak{g}_\mathfrak{l})$-module into a direct sum of Weyl modules as $\mathfrak{U}(\mathfrak{g}_{\mathfrak{l}-\mathfrak{1}})$-modules. Finally, we obtain a new multiplicity formula for the Weyl module $V(\lambda)$ of $\mathfrak{U}(\mathfrak{g}_\mathfrak{l})$.

The paper is organized as follows: In Section 1 we introduce some preliminaries; In Section 2 we construct a monomial basis and a filtration of subalgebras of $\mathfrak{U}(\mathfrak{g}_\mathfrak{l})$; In Section our main results concerning a $\mathbb{Z}$-basis and a new multiplicity formula for the Weyl module $V(\lambda)$ of $\mathfrak{U}(\mathfrak{g}_\mathfrak{l})$ is given; In Section 4 two examples for $\mathfrak{g}_\mathfrak{l}$ being of type B_3 and C_3 are given. We shall freely use the notations in Humphreys [1] without further comments.

1. Preliminaries

1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{U}$ the universal enveloping algebra of $\mathfrak{g}$. Let

$$\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$$

Received June 26, 2008; Accepted December 18, 2008
Supported by the National Natural Science Foundation of China (Grant No. 10671142).
* Corresponding author
E-mail: jcye@tongji.edu.cn (J. C. YE)

be the set of simple roots of $\mathfrak{g}$, Φ the set of roots, and Φ^+ the set of positive roots of $\mathfrak{g}$. Let Λ be the weight lattice of $\mathfrak{g}$, which is the $\mathbb{Z}$-span of fundamental weights, where we denote by ω_i ($1 \le i \le l$) the fundamental weights of $\mathfrak{g}$ such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, the Kronecker delta, and denote by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ the weight $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_l\omega_l$ with $\lambda_1, \lambda_2, \ldots, \lambda_l \in \mathbb{Z}$, the integer ring. Let $X(T)$ be the character group of T, which is also called the weight lattice of $\mathfrak{g}$. Then the set of dominant weights is

$$\Lambda^+ = \{(\lambda_1, \lambda_2, \ldots, \lambda_l) \in X(T) \mid \lambda_1, \lambda_2, \ldots, \lambda_l \ge 0\}.$$

Let W be the Weyl group of $\mathfrak{g}$. It is well-known that for $\lambda \in \Lambda^+$, the Weyl module $V(\lambda)$ is the finite dimensional irreducible $\mathfrak{g}$-module with the highest weight λ. We set $\mathrm{ch}\,(\lambda) = \mathrm{ch}\,(V(\lambda))$ for all $\lambda \in \Lambda^+$. Moreover, $\mathrm{ch}\,(\lambda)$ is given by the Weyl character formula, and for $\lambda \in \Lambda^+$, one has

$$\mathrm{ch}\,(\lambda) = \frac{\sum_{w\in W} \det(w) e(w(\lambda+\rho))}{\sum_{w\in W} \det(w) e(w\rho)}.$$

Let $e_\alpha, f_\alpha, h_i (\alpha \in \Phi^+, i = 1, 2, \ldots, l)$ be a Chevalley basis of $\mathfrak{g}$. The Kostant $\mathbb{Z}$-form $\mathfrak{U}_\mathbb{Z}$ of $\mathfrak{U}$ is the $\mathbb{Z}$-subalgebra of $\mathfrak{U}$ generated by the elements $e_\alpha^{(k)} := e_\alpha^k/k!, f_\alpha^{(k)} := f_\alpha^k/k!$ for $\alpha \in \Phi^+$ and $k \in \mathbb{N}$, the set of non-negative integers. Set

$$\binom{h_i + c}{k} := \frac{(h_i+c)(h_i+c-1)\cdots(h_i+c-k+1)}{k!}.$$

Then $\binom{h_i+c}{k} \in \mathfrak{U}_\mathbb{Z}$, for $i = 1, 2, \ldots, l, c \in \mathbb{Z}$, $k \in \mathbb{N}$. Moreover, $\mathfrak{U} := \mathfrak{U}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$. Let $\mathfrak{U}_\mathbb{Z}^+, \mathfrak{U}_\mathbb{Z}^-, \mathfrak{U}_\mathbb{Z}^\circ$ be the positive part, negative part and zero part of $\mathfrak{U}_\mathbb{Z}$, respectively. They are generated by $e_\alpha^{(k)}$, $f_\alpha^{(k)}$ and $\binom{h_i}{k}$, respectively. By abuse of notations, the images in $\mathfrak{U}$ of $e_\alpha^{(k)}, f_\alpha^{(k)}, \binom{h_i+c}{k}$, etc. will be denoted by the same notations, respectively. The algebra $\mathfrak{U}$ is a Hopf algebra, and $\mathfrak{U}$ has also a triangular decomposition $\mathfrak{U} = \mathfrak{U}^-\mathfrak{U}^\circ\mathfrak{U}^+$. Given an ordering in Φ^+, it is known that a $\mathbb{Z}$-basis for $\mathfrak{U}_\mathbb{Z}$ has the form of

$$\prod_{\alpha\in\Phi^+} f_\alpha^{(a_\alpha)} \prod_{i=1}^{l} \binom{h_i}{b_i} \prod_{\alpha\in\Phi^+} e_\alpha^{(c_\alpha)}$$

with $a_\alpha, b_i, c_\alpha \in \mathbb{N}$.

1.2. When $\mathfrak{g}$ is of type B_l with α_1 being the short simple root, we set

$$\alpha_{i\ j} = \alpha_i + \alpha_{i-1} + \cdots + \alpha_j, \alpha_{i\ 1\ j} = \alpha_{i\ 1} + \alpha_{j\ 1} = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_j) + \alpha_{j+1} + \cdots + \alpha_i,$$
$$1 \le i \le l, \quad 1 \le j < i \le l.$$

Then

$$\Phi^+ = \{\alpha_i, \alpha_{i\ j}, \alpha_{i\ 1\ j}; \quad 1 \le i \le l,\ 1 \le j < i \le l\}$$

is the set of positive roots which has l^2 elements. Fix an ordering of positive roots as follows:

$$\alpha_1, \alpha_{2\ 1\ 1}, \alpha_{2\ 1}, \alpha_2, \ldots, \alpha_{l\ 1\ l-1}, \alpha_{l\ 1\ l-2}, \ldots, \alpha_{l\ 1\ 2}, \alpha_{l\ 1\ 1}, \alpha_{l\ 1}, \alpha_{l\ 2}, \ldots, \alpha_{l\ l-1}, \alpha_l.$$

For example, when $l = 3$ the set of positive roots is $\{\alpha_1, \alpha_{2\ 1\ 1} = 2\alpha_1 + \alpha_2, \alpha_{2\ 1} = \alpha_1 + \alpha_2, \alpha_2, \alpha_{3\ 1\ 2} = 2(\alpha_1+\alpha_2)+\alpha_3, \alpha_{3\ 1\ 1} = 2\alpha_1+\alpha_2+\alpha_3, \alpha_{3\ 1} = \alpha_1+\alpha_2+\alpha_3, \alpha_{3\ 2} = \alpha_2+\alpha_3, \alpha_3\}$.

1.3. When $\mathfrak{g}$ is of type C_l with α_1 being the long simple root, we set $\alpha_{i\ i} = \alpha_i$ and

$$\alpha_{i\ j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \alpha_{j\ i} = \alpha_{1\ j} + \alpha_{2\ i+1} = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{i+1}) + \alpha_{i+2} + \cdots + \alpha_j,$$

$$1 \le i \le l, 1 \le i < j \le l.$$

Then

$$\Phi^+ = \{\alpha_{i\ j}; \quad 1 \le i, j \le l\}$$

is the set of positive roots which has l^2 elements. Fix an ordering of positive roots as follows:

$$\alpha_1, \alpha_{1\ 2}, \alpha_{2\ 1}, \alpha_2, \ldots, \alpha_{l\ l-2}, \alpha_{l\ l-3}, \ldots, \alpha_{l\ 1}, \alpha_{1\ l}, \alpha_{l\ l-1}, \alpha_{2\ l}, \ldots, \alpha_{l-1\ l}, \alpha_l.$$

For example, when $l = 3$ the set of positive roots is $\{\alpha_1, \alpha_{1\ 2} = \alpha_1 + \alpha_2, \alpha_{2\ 1} = \alpha_1 + 2\alpha_2, \alpha_2, \alpha_{3\ 1} = \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_{1\ 3} = \alpha_1 + \alpha_2 + \alpha_3, \alpha_{3\ 2} = \alpha_1 + 2(\alpha_2 + \alpha_3), \alpha_{2\ 3} = \alpha_2 + \alpha_3, \alpha_3\}$.

1.4. Following Littelmann [3] and Littelmann [4], for $I = (i_1, i_2, \ldots, i_{l^2}) \in \mathbb{N}^{l^2}$, we define

$$f^I = f_1^{(i_1)} f_{2\ 1\ 1}^{(i_2)} f_{2\ 1}^{(i_3)} f_2^{(i_4)} \cdots f_{l\ 1\ l-1}^{(i_{l^2-2l+2})} \cdots f_{l\ 1\ 1}^{(i_{l^2-l})} f_{l\ 1}^{(i_{l^2-l+1})} \cdots f_{l\ l-1}^{(i_{l^2-1})} f_l^{(i_{l^2})}$$

for $\mathfrak{g}$ of type B_l, and

$$f^I = f_1^{(i_1)} f_{1\ 2}^{(i_2)} f_{2\ 1}^{(i_3)} f_2^{(i_4)} \cdots f_{l\ l-2}^{(i_{l^2-2l+2})} \cdots f_{l\ 1}^{(i_{l^2-l-1})} f_{1\ l}^{(i_{l^2-l})} f_{l\ l-1}^{(i_{l^2-l+1})} f_{2\ l}^{(i_{l^2-l+2})} \cdots f_{l-1\ l}^{(i_{l^2-1})} f_l^{(i_{l^2})}$$

for $\mathfrak{g}$ of type C_l.

Note that $\{f^I | I \in \mathbb{N}^{l^2}\}$ is a $\mathbb{Z}$-basis of $\mathfrak{U}_{\mathbb{Z}}^-$. Here we write $f_i, f_{i\ j}, f_{i\ j\ k}$ for f_α when $\alpha = \alpha_i, \alpha_{i\ j}, \alpha_{i\ j\ k}$, respectively. In particular, one has $f^{\mathbf{0}} = 1$ when $I = (0, 0, \ldots, 0) = \mathbf{0}$.

Moreover, we define an ordering "$\prec$"on $\mathbb{N}^{l^2}$ as follows: for any $I, I' \in \mathbb{N}^{l^2}$, $I = (i_1, i_2, \ldots, i_{l^2})$ and $I' = (i'_1, i'_2, \ldots, i'_{l^2})$, if there exists a k with $1 \le k \le l^2$ such that $i_k < i'_k$ and $i_j = i'_j$ for all $j > k$, then we say $I \prec I'$; otherwise, one has $I = I'$. Therefore, we can define an ordering on the basis of $\mathfrak{U}_{\mathbb{Z}}^-$ "$\prec$" in the same way: we say $f^I \prec f^{I'}$ if and only if $I \prec I'$. Any element in $\mathfrak{U}^-$ can be written uniquely in terms of $f = \sum_{I \in \mathbb{N}^{l^2}} a_I f^I$ with $a_I \in \mathbb{C}$.

2. Some commutator formulas and a class of special subalgebras in $\mathfrak{U}(\mathfrak{g}_l)$

2.1. For $1 \le i, j \le l$, one has the following commutator formulas [1].

(1) $$e_i^{(a)} f_i^{(b)} = \sum_{k=0}^{\min(a,b)} f_i^{(b-k)} \binom{h_i - a - b + 2k}{k} e_i^{(a-k)};$$

(2) $$h_i f_j^{(k)} = f_j^{(k)} h_i - k\alpha_j(h_i) f_j^{(k)};$$

(3) $$\binom{h_i + a}{b} f_j^{(k)} = f_j^{(k)} \binom{h_i - k\alpha_j(h_i) + a}{b};$$

(4) $$\begin{aligned} & e_i f_l^{(a_l)} \cdots f_i^{(a_i)} \cdots f_2^{(a_2)} f_1^{(a_1)} f_2^{(a'_2)} \cdots f_i^{(a'_i)} \cdots f_l^{(a'_l)} \\ &= f_l^{(a_l)} \cdots f_i^{(a_i)} \cdots f_2^{(a_2)} f_1^{(a_1)} f_2^{(a'_2)} \cdots f_i^{(a'_i)} \cdots f_l^{(a'_l)} e_i + \\ & f_l^{(a_l)} \cdots f_i^{(a_i-1)} (h_i - a_i + 1) f_{i-1}^{(a_{i-1})} \cdots f_2^{(a_2)} f_1^{(a_1)} f_2^{(a'_2)} \cdots f_i^{(a'_i)} \cdots f_l^{(a'_l)} + \\ & f_l^{(a_l)} \cdots f_i^{(a_i)} \cdots f_2^{(a_2)} f_1^{(a_1)} f_2^{(a'_2)} \cdots f_i^{(a'_i-1)} (h_i - a'_i + 1) f_{i+1}^{(a'_{i+1})} \cdots f_l^{(a'_l)} \end{aligned}$$

$$
\begin{aligned}
&= f_l^{(a_l)}\cdots f_i^{(a_i)}\cdots f_2^{(a_2)}f_1^{(a_1)}f_2^{(a_2')}\cdots f_i^{(a_i')}\cdots f_l^{(a_l')}e_i+\\
&\quad f_l^{(a_l)}\cdots f_i^{(a_i-1)}f_{i-1}^{(a_{i-1})}\cdots f_2^{(a_2)}f_1^{(a_1)}f_2^{(a_2')}\cdots f_i^{(a_i')}\cdots f_l^{(a_l')}\\
&\left(h_i-a_i+1-\sum_{k=1}^{i-1}a_k\alpha_k(h_i)-\sum_{k=2}^{l}a_k'\alpha_k(h_i)\right)+\\
&\quad f_l^{(a_l)}\cdots f_i^{(a_i)}\cdots f_2^{(a_2)}f_1^{(a_1)}f_2^{(a_2')}\cdots f_i^{(a_i'-1)}f_{i+1}^{(a_{i+1}')}\cdots f_l^{(a_l')}\\
&\left(h_i-a_i'+1-\sum_{k=i+1}^{l}a_k'\alpha_k(h_i)\right).
\end{aligned}
$$

2.2. Let us construct a class of special subalgebras $\mathfrak{U}_{\mathbb{Z},i}, 1\le i\le l$, of $\mathfrak{U}_{\mathbb{Z}}$ as follows. Set

$$\mathfrak{U}_{\mathbb{Z},i} := \langle e_j^{(a_j)}, f_j^{(b_j)}, \tbinom{h_j+c}{k} \mid a_j, b_j, c, k\in\mathbb{N},\ 1\le j\le i\rangle.$$

Then one has

$$0\subseteq \mathfrak{U}_{\mathbb{Z},1}\subseteq\mathfrak{U}_{\mathbb{Z},2}\subseteq\cdots\subseteq\mathfrak{U}_{\mathbb{Z},l}=\mathfrak{U}_{\mathbb{Z}}.$$

The set of positive roots in $\mathfrak{U}(\mathfrak{g_i})$ is just that of the first i^2 roots according to the ordering of Φ^+.

2.3. Let $K=(k_l^l, k_{l-1}^{l-1}, k_l^{l-1}, k_{l+1}^{l-1}, \dots, k_{l-i+1}^{l-i+1}, k_{l-i+2}^{l-i+1}, \dots, k_{l-1}^{l-i+1}, k_l^{l-i+1}, k_{l+1}^{l-i+1}, \dots, k_{l+i-2}^{l-i+1}, k_{l+i-1}^{l-i+1}, \dots, k_1^1, k_2^1, \dots, k_{l-1}^1, k_l^1, k_{l+1}^1, \dots, k_{2l-2}^1, k_{2l-1}^1)\in\mathbb{N}^{l^2}$. Define an index set

$$
\begin{aligned}
\Pi := \{K\in\mathbb{N}^{l^2} | 2k_{l-i+1}^{l-i+1}\ge 2k_{l-i+2}^{l-i+1}\ge\cdots\ge 2k_{l-1}^{l-i+1}\ge k_l^{l-i+1}\ge 2k_{l+1}^{l-i+1}\ge\cdots\\
\ge 2k_{l+i-2}^{l-i+1}\ge 2k_{l+i-1}^{l-i+1},\ \ 1\le i\le l\}
\end{aligned}
$$

for $\mathfrak{g}$ of type B_l, and

$$
\begin{aligned}
\Pi := \{K\in\mathbb{N}^{l^2} \mid k_{l-i+1}^{l-i+1}\ge k_{l-i+2}^{l-i+1}\ge\cdots\ge k_{l-1}^{l-i+1}\ge k_l^{l-i+1}\ge k_{l+1}^{l-i+1}\ge\cdots\\
\ge k_{l+i-2}^{l-i+1}\ge k_{l+i-1}^{l-i+1},\ \ 1\le i\le l\}
\end{aligned}
$$

for $\mathfrak{g}$ of type C_l.

For any $K\in\Pi$, one has such a monomial

$$
\begin{aligned}
\theta^K =& f_1^{(k_l^l)}f_2^{(k_{l-1}^{l-1})}f_1^{(k_l^{l-1})}f_2^{(k_{l+1}^{l-1})}\cdots f_i^{(k_{l-i+1}^{l-i+1})}f_{i-1}^{(k_{l-i+2}^{l-i+1})}\cdots f_1^{(k_l^{l-i+1})}\cdots f_{i-1}^{(k_{l+i-2}^{l-i+1})}f_i^{(k_{l+i-1}^{l-i+1})}\cdots\\
&f_l^{(k_1^1)}f_{l-1}^{(k_2^1)}\cdots f_1^{(k_l^1)}\cdots f_l^{(k_{2l-1}^1)}\in\mathfrak{U}_{\mathbb{Z}}^-.
\end{aligned}
$$

The following theorem was first proved in Littelmann [4, Theorem 4.2].

Theorem 2.4 *The set $\{\theta^K | K\in\Pi\}$ forms a $\mathbb{Z}$-basis of $\mathfrak{U}_{\mathbb{Z}}^-$.*

2.5. Moreover, we define

$$
\begin{aligned}
\Pi_{l-1} :&= \{K\in\Pi \mid k_j^1=0, 1\le j\le 2l-1\}\subseteq\Pi,\\
\Pi' &= \{K\in\Pi \mid k_j^i=0, 1<i\le l, i\le j\le 2l-i\}.
\end{aligned}
$$

Then the set $\{\theta^K \mid K\in\Pi_{l-1}\}$ forms a $\mathbb{Z}$-basis of $\mathfrak{U}_{\mathbb{Z},l-1}{}^-$. If we define the ordinary vector addition in Π, one has the following claims:

(1) $\Pi=\Pi_{l-1}\oplus\Pi'$;

(2) If $K_2\in\Pi_{l-1}$ and $K_1\in\Pi'$, then $\theta^{K_2}\theta^{K_1}=\theta^{K_2+K_1}$;

(3) If $K_1,\ K_1' \in \Pi'$ with $K_1 \prec K_1'$, then $K_2 + K_1 \prec K_1'$ for any $K_2 \in \Pi_{l-1}$.

3. A new multiplicity formula of the Weyl module $V(\lambda)$

3.1. It is known that the irreducible $\mathfrak{g}$-module $V(\lambda)$ has a $\mathbb{Z}$-lattice $V(\lambda)_{\mathbb{Z}}$. Let $\mathfrak{U}_{\mathfrak{i}} = \mathfrak{U}_{\mathbb{Z},i}\otimes_{\mathbb{Z}}\mathbb{C}$ for $1 \le i \le l$.

Let E be the real vector space spanned by $\alpha_1, \alpha_2, \ldots, \alpha_l$. It is well known that $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_l^\vee$ again form a basis of E, and $\omega_1, \omega_2, \ldots, \omega_l$ form the dual basis relative to the inner product on E: $(\omega_i, \alpha_j^\vee) = \frac{2(\omega_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$. If we restrict ourselves to consider the $(l-1)$-dimensional subspaces E' of E spanned by $\alpha_1, \alpha_2, \ldots, \alpha_{l-1}$, then $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_{l-1}^\vee$ and $\omega_1, \omega_2, \ldots, \omega_{l-1}$ remain the dual bases of E' relative to the inner product on E. Therefore, we can consider the restriction of $\mathfrak{U}_{\mathfrak{l}}$ to $\mathfrak{U}_{\mathfrak{l-1}}$, and the restriction of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ as a weight of $\mathfrak{U}_{\mathfrak{l}}$ to $\lambda_{\mathfrak{U}_{\mathfrak{l-1}}} = (\lambda_1, \lambda_2, \ldots, \lambda_{l-1})$ as a weight of $\mathfrak{U}_{\mathfrak{l-1}}$. Moreover, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \Lambda^+$ be a dominant weight, and v a maximal vector of weight λ of the Weyl module $V(\lambda)$ of $\mathfrak{U}_{\mathfrak{l}}$. Then $V(\lambda)|_{\mathfrak{U}_{\mathfrak{l-1}}}$ denotes the restriction of $V(\lambda)$ to a $\mathfrak{U}_{\mathfrak{l-1}}$-module.

3.2. Following Littelmann [4, §6], we define λ_i^j for $\mathfrak{g}$ as follows. Set $\lambda_{2l-1}^1 = \lambda_l$ and $k_j^i = 0$, if $i + j > 2l$. For all $1 \le i \le l$, $i \le j < l$, λ_j^i is defined to be $h_{l-j+1}w = \lambda_j^i w$, where

$$w = f_{l-j}^{(k_{j+1}^i)} \cdots f_2^{(k_{l-1}^i)} f_1^{(k_l^i)} f_2^{(k_{l+1}^i)} \cdots f_{l-i+1}^{(k_{2l-i}^i)} \cdots f_l^{(k_1^1)} f_{l-1}^{(k_2^1)} \cdots f_1^{(k_l^1)} \cdots f_{l-1}^{(k_{2l-2}^1)} f_l^{(k_{2l-1}^1)} v$$

and

$$\lambda_j^i = \lambda_{l-j+1} + k_{2l-j+1}^i + \sum_{n=1}^{i-1} (k_{j-1}^n + k_{2l-j+1}^n) - 2k_{2l-j}^i - 2\sum_{n=1}^{i-1} (k_j^n + k_{2l-j}^n) + \sum_{n=1}^{i} (k_{j+1}^n + ck_{2l-j-1}^n).$$

λ_{2l-j}^i is defined to be $h_{l-j+1}w = \lambda_{2l-j}^i w$, where

$$w = f_{l-j+2}^{(k_{2l-j+1}^i)} f_{l-j+3}^{(k_{2l-j+2}^i)} \cdots f_{l-i+1}^{(k_{2l-i}^i)} \cdots f_l^{(k_1^1)} f_{l-1}^{(k_2^1)} \cdots f_1^{(k_l^1)} \cdots f_{l-1}^{(k_{2l-2}^1)} f_l^{(k_{2l-1}^1)} v$$

and

$$\lambda_{2l-j}^i = \lambda_{l-j+1} + k_{2l-j+1}^i + \sum_{n=1}^{i-1} (k_{j-1}^n + k_{2l-j+1}^n) - 2\sum_{n=1}^{i-1} (k_j^n + k_{2l-j}^n) + \sum_{n=1}^{i-1} (k_{j+1}^n + ck_{2l-j-1}^n).$$

λ_l^i is defined to be $h_1 w = \lambda_l^i w$, where

$$w = f_2^{(k_{l+1}^i)} f_3^{(k_{l+2}^i)} \cdots f_{l-i+1}^{(k_{2l-i}^i)} \cdots f_l^{(k_1^1)} \cdots f_2^{(k_{l-1}^1)} f_1^{(k_l^1)} f_2^{(k_{l+1}^1)} \cdots f_l^{(k_{2l-1}^1)} v$$

and

$$\lambda_l^i = \lambda_1 + dk_{l+1}^i + d\sum_{n=1}^{i-1} (k_{l-1}^n + k_{l+1}^n) - 2\sum_{n=1}^{i-1} k_l^n.$$

Where d and c are defined as follows: when $\mathfrak{g}$ is of type B_l, $d = 2$ and $c = 0$ if $j = l-1$ or $c = 1$ otherwise; and when $\mathfrak{g}$ is of type C_l, $d = 1$ and $c = 1$.

Moreover, we define two index sets Π_λ and Π'_λ, which are related to λ, as follows:

$$\Pi_\lambda = \Pi_{l,\lambda} := \{K \in \Pi | 0 \le k_j^i \le \lambda_j^i,\ 1 \le i \le l, i \le j \le 2l-i\}.$$

Let $P = (0, \ldots, 0, p_l, p_{l-1}, \ldots, p_2, p_1, \overline{p}_2, \ldots, \overline{p}_l) \in \Pi'$. For $\mathfrak{g}$ of type B_l, we set $\overline{p}_1 = \overline{p}_{l+1} = 0$, and define

$$\Pi'_\lambda := \{P \in \Pi' | p_1 - 2\overline{p}_2 \leq \lambda_1, \overline{p}_i - \overline{p}_{i+1} \leq \lambda_i,\ p_i + 2\overline{p}_i - (p_{i-1} + \overline{p}_{i-1}) - \overline{p}_{i+1} \leq \lambda_i,\ 2 \leq i \leq l\},$$

and for $\mathfrak{g}$ of type C_l, we set $\overline{p}_{l+1} = 0, \overline{p}_1 = p_1$, and define

$$\Pi'_\lambda := \{P \in \Pi' | \overline{p}_i - \overline{p}_{i+1} \leq \lambda_i, 1 \leq i \leq l,\ p_j + 2\overline{p}_j - p_{j-1} - \overline{p}_{j-1} - \overline{p}_{j+1} \leq \lambda_j,\ 2 \leq j \leq l\}.$$

Let $\lambda - \sum_{i=2}^{l} (p_i + \overline{p}_i)\alpha_i - p_1\alpha_1 = \lambda - P\alpha$.

3.3. It is easy to see that Π_λ is a finite set. We shall show that the set $\{\theta^K v | K \in \Pi_\lambda\}$ forms a $\mathbb{Z}$-basis of $V(\lambda)_\mathbb{Z}$. Also, we shall see that Π'_λ is also a finite set, and it becomes an index set of highest weights of irreducible components of $V(\lambda)$ to be viewed as a $\mathfrak{U}_{\mathfrak{l}-1}$-module.

Denote by $\Pi(\lambda)$ the set of weights of the Weyl module $V(\lambda)$. For $P = (0, \ldots, 0, p_l, p_{l-1}, \ldots, p_2, p_1, \overline{p}_2, \ldots, \overline{p}_l) \in \Pi'_\lambda$, we say $P\alpha = p_1\alpha_1 + \sum_{i=2}^{l} (p_i + \overline{p}_i)\alpha_i \ll \sum_{i=1}^{l} a_i\alpha_i$ if and only if $p_l + \overline{p}_l = a_l$, $p_i + \overline{p}_i \leq a_i$, $i = 2, \ldots, l-1$, and $p_1 \leq a_1$.

3.4. Let V be a $\mathfrak{U}_\mathfrak{l}$-module. We say a vector $v \in V$ to be a primitive vector of V, if there are two submodules V_1, V_2 with $V_2 \subset V_1 \subseteq V$ such that $v \in V_1$, $v \notin V_2$, and all e_i with $1 \leq i \leq l$ kill the canonical image of v in V_1/V_2.

The following Lemmas can be proved as in Ye and Zhou [5, Lemmas 4.3 and 4.5].

Lemma 3.5 *Let w be a primitive vector of weight λ in V. Then V has a composition factor isomorphic to $V(\lambda)$.*

Lemma 3.6 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \Lambda^+$ be a dominant weight. Let V be a finite dimensional $\mathfrak{U}_\mathfrak{l}$-module generated by a maximal vector v of weight λ of V. Then one has $V \simeq V(\lambda)$.*

Moreover, one has the following lemma (Humphreys [1,§21.4]).

Lemma 3.7 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \Lambda^+$ be a dominant weight, and v a maximal vector of weight λ of $V(\lambda)$. Then one has*

$$f_i^{(\lambda_i+1)} v = 0, \quad 1 \leq i \leq l.$$

Then one has the following theorems.

Theorem 3.8 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \Lambda^+$ be a dominant weight. As a $\mathfrak{U}_{\mathfrak{l}-1}$-module, the irreducible $\mathfrak{U}_\mathfrak{l}$-module $V(\lambda)$ has the following direct sum decomposition*

$$V(\lambda)|_{\mathfrak{U}_{\mathfrak{l}-1}} = \bigoplus_{P \in \Pi'_\lambda} V\left((\lambda - P\alpha)_{\mathfrak{U}_{\mathfrak{l}-1}}\right).$$

Proof By definition, Π'_λ is a finite set. Let $|\Pi'_\lambda| = t$. We can arrange elements of Π'_λ according to the ordering of Π'_λ defined in §1.4. Then one has

$$\Pi'_\lambda = \{\mathbf{0} = P_1 \prec P_2 \prec \cdots \prec P_t\}.$$

Set
$$M_{P_s} = \sum\nolimits_{K \in \Pi, K \prec P_{s+1}} \mathbb{C}\theta^K v, \quad 1 \leq s \leq t-1,$$

where v is a maximal vector of $V(\lambda)$ and $M_{P_t} = V(\lambda)$. Then one has

$$0 \subseteq M_{P_1} \subseteq M_{P_2} \subseteq \cdots \subseteq M_{P_t} = V(\lambda).$$

First of all, we can show that M_{P_s}, $1 \leq s \leq t$, is a $\mathfrak{U}_{\mathfrak{l-1}}$-submodule of $V(\lambda)$. In order do so, we need only to show that M_{P_s} is stable under actions of e_i, h_i and f_i with $1 \leq i \leq l-1$, and then M_{P_s} is a $\mathfrak{U}_{\mathfrak{l-1}}$-module.

For any $\theta^K v \in M_{P_s}$ with $K \prec P_{s+1}$, it is still a weight vector, and for any h_i with $1 \leq i \leq l$, one has by § 2.1 (2)

$$h_i \theta^K v = a_{i_K} \theta^K v \in M_{P_s} \ \text{ with } \ a_{i_K} \in \mathbb{Z}.$$

By § 2.5 (1), $K = K_1 + K_2$ with $K_1 \in \Pi'$ and $K_2 \in \Pi_{l-1}$. Therefore, one has for any $f_i \in \mathfrak{U}(\mathfrak{g}_{\mathfrak{l-1}})$ with $1 \leq i \leq l-1$,

$$\begin{aligned} f_i \theta^K v &= f_i \theta^{K_1+K_2} v = f_i \left(\theta^{K_2} \theta^{K_1}\right) v \ \text{ by § 2.5 (2)} \\ &= \left(f_i \theta^{K_2}\right) \theta^{K_1} v = \Big(\sum_{K' \in \Pi_{l-1}} a_{_{K'}} \theta^{K'} \Big) \theta^{K_1} v \\ &= \sum_{K' \in \Pi_{l-1}} a_{_{K'}} \theta^{K'+K_1} v \ \text{ with } \ a_{K'} \in \mathbb{Z}. \end{aligned}$$

Note the fact that $K = K_1 + K_2 \prec P_{s+1}$, one has $K_1 \prec P_{s+1}$, and $K' + K_1 \prec P_{s+1}$ for any $K' \in \Pi_{l-1}$. Therefore,

$$f_i \theta^K v = \sum_{K' \in \Pi_{l-1}} a_{_{K'}} \theta^{K'+K_1} v \in M_{P_s}.$$

Furthermore, one has for any e_i with $1 \leq i \leq l$,

$$\begin{aligned} e_i \theta^K v =& e_i f_1^{(k_l^l)} f_2^{(k_{l-1}^{l-1})} f_1^{(k_l^{l-1})} f_2^{(k_{l+1}^{l-1})} \cdots f_i^{(k_{l-i+1}^{l-i+1})} f_{i-1}^{(k_{l-i+2}^{l-i+1})} \cdots f_1^{(k_l^{l-i+1})} \cdots f_{i-1}^{(k_{l+i-2}^{l-i+1})} \\ & f_i^{(k_{l+i-1}^{l-i+1})} \cdots f_l^{(k_1^1)} f_{l-1}^{(k_2^1)} \cdots f_1^{(k_l^1)} \cdots f_{l-1}^{(k_{2l-2}^1)} f_l^{(k_{2l-1}^1)} v \\ =& \theta^k e_i v + \sum_{n=1}^{l-i+1} f_1^{(k_l^l)} f_2^{(k_{l-1}^{l-1})} f_1^{(k_l^{l-1})} f_2^{(k_{l+1}^{l-1})} \cdots f_i^{(k_{l-i+1}^n - 1)} (h_i - k_{l-i+1}^n + 1) \\ & f_{i-1}^{(k_{l-i+2}^n)} \cdots f_1^{(k_l^n)} \cdots f_{i-1}^{(k_{l+i-2}^n)} f_i^{(k_{l+i-1}^n)} \cdots f_l^{(k_1^1)} f_{l-1}^{(k_2^1)} \cdots f_1^{(k_l^1)} \cdots f_{l-1}^{(k_{2l-2}^1)} f_l^{(k_{2l-1}^1)} v + \\ & \sum_{n=1}^{l-i+1} f_1^{(k_l^l)} f_2^{(k_{l-1}^{l-1})} f_1^{(k_l^{l-1})} f_2^{(k_{l+1}^{l-1})} \cdots f_i^{(k_{l-i+1}^n)} f_{i-1}^{(k_{l-i+2}^n)} \cdots f_1^{(k_l^n)} \cdots f_{i-1}^{(k_{l+i-2}^n)} \\ & f_i^{(k_{l+i-1}^n - 1)} (h_i - k_{l+i-1}^n + 1) \cdots f_l^{(k_1^1)} f_{l-1}^{(k_2^1)} \cdots f_1^{(k_l^1)} \cdots f_{l-1}^{(k_{2l-2}^1)} f_l^{(k_{2l-1}^1)} v \text{ by § 2.1(4)} \\ =& \sum_{n=1}^{l-i+1} (a_n \theta^{K-K_n} v + \overline{a}_n \theta^{K-\overline{K}_n} v), \end{aligned}$$

where

$$\begin{aligned} a_n =& \lambda_i - k_{l-i+1}^n + 1 - b \sum_{r=1}^{n-1} (k_{l-i+1}^r + k_{l+i-1}^r) - 2k_{l+i-1}^n + \\ & \sum_{r=1}^{n} (k_{l-i+2}^r + c k_{l+i-2}^r) + d \sum_{r=1}^{n-1} (k_{l-i}^r + k_{l+i}^r) \in \mathbb{Z}, \end{aligned}$$

$$\overline{a}_n = \lambda_i - k^n_{l+i-1} + 1 - b\sum_{r=1}^{n-1}(k^r_{l-i+1} + k^r_{l+i-1}) + \sum_{r=1}^{n-1}(k^r_{l-i+2} + ck^r_{l+i-2}) + d\sum_{r=1}^{n-1}(k^r_{l-i} + k^r_{l+i}) \in \mathbb{Z},$$

and $b=1, d=2$ if $i=1$ or $b=2, d=1$ otherwise, and $c=0$ if $i=2$ or $c=1$ otherwise for $\mathfrak{g}$ of type B_l; and $b=1$ if $i=1$ or $b=2$ otherwise and $c=1$, $d=1$ for $\mathfrak{g}$ of type C_l.

$K_n = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{N}^{l^2}$ with 1 occurring in the place where k^n_{l-i+1} lies in the corresponding K, $\overline{K}_n = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{N}^{l^2}$ with 1 occurring in the place where k^n_{l+i-1} lies in the corresponding K. Since $K - K_n \prec K \prec P_{s+1}$ and $K - \overline{K}_n \prec K \prec P_{s+1}$, one has

$$e_i\theta^K v = \sum_{n=1}^{l-i+1} a_n \theta^{K-K_n} v + \sum_{n=1}^{l-i+1} \overline{a}_n \theta^{K-\overline{K}_n} v \in M_{P_s}.$$

It shows that M_{P_s} is stable under actions of e_i, h_i with $1 \le i \le l$ and f_i with $1 \le i \le l-1$, and M_{P_s} is a $\mathfrak{U}_{\mathfrak{l}-1}$-module.

Secondly, we show that $\theta^{P_s}v$, $1 \le s \le t$, is a primitive vector in $V(\lambda)$ when $V(\lambda)$ is viewed as a $\mathfrak{U}_{\mathfrak{l}-1}$-module. We show first that $\theta^{P_s}v \neq 0$. Let $P_s = (0,\ldots 0, p_l, p_{l-1}, \ldots, p_2, p_1, \overline{p}_2, \ldots, \overline{p}_l) \in \Pi'_\lambda$.

One has

$$\begin{aligned}
& e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} e_l^{(p_l)} \theta^{P_s} v \\
&= e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} e_l^{(p_l)} f_l^{(p_l)} f_{l-1}^{(p_{l-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_l^{(\overline{p}_l)} v \\
&= e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} \Big(\sum_{k=0}^{p_l} f_l^{(p_l-k)} \binom{h_l - 2p_l + 2k}{k} e_l^{(p_l-k)} \Big) \\
& f_{l-1}^{(p_{l-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_l^{(\overline{p}_l)} v \quad \text{by § 2.1(1)} \\
&= e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} \binom{h_l}{p_l} f_{l-1}^{(p_{l-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_l^{(\overline{p}_l)} v \\
&= e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} f_{l-1}^{(p_{l-1})} \binom{h_l - p_{l-1}\alpha_{l-1}(h_l)}{p_l} \\
& f_{l-2}^{(p_{l-2})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_l^{(\overline{p}_l)} v \\
&= e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} f_{l-1}^{(p_{l-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_l^{(\overline{p}_l)} \\
& \binom{h_l - \sum_{k=1}^{l-1} p_k\alpha_k(h_l) - \sum_{k=2}^{l} \overline{p}_k\alpha_k(h_l)}{p_l} v \\
&= e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} f_{l-1}^{(p_{l-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_l^{(\overline{p}_l)} \\
& \binom{\lambda_l + p_{l-1} - 2\overline{p}_l + \overline{p}_{l-1}}{p_l} v = \cdots \\
&= \prod_{k=2}^{l} \binom{\lambda_k + \overline{p}_{k+1}}{\overline{p}_k} \binom{\lambda_1 + c\overline{p}_2}{p_1} \prod_{k=2}^{l} \binom{\lambda_k + p_{k-1} - 2\overline{p}_k + \overline{p}_{k-1} + \overline{p}_{k+1}}{p_k} v,
\end{aligned}$$

where for $\mathfrak{g}$ of type B_l, $c=2$, $\overline{p}_1 = \overline{p}_{l+1} = 0$, the last third equality is because $\alpha_j(h_i) \neq 0$ if and only if $|i-j| \le 1$, and $\alpha_2(h_1) = -2$, $\alpha_k(h_{k\pm1}) = -1, (2 \le k \le l), \alpha_j(h_j) = 2, (1 \le j \le l)$; and for $\mathfrak{g}$ of type C_l, $c=1$, $\overline{p}_1 = p_1, \overline{p}_{l+1} = 0$, the last third equality is because $\alpha_j(h_i) \neq 0$ if and only if

$|i-j| \le 1$, $\alpha_1(h_2) = -2$, $\alpha_k(h_{k\pm1}) = -1$, $(2 \le k \le l)$, $\alpha_j(h_j) = 2$, $(1 \le j \le l)$.

Note that $\overline{p}_k - \overline{p}_{k+1} \le \lambda_k, p_1 - 2\overline{p}_2 \le \lambda_1, p_k + 2\overline{p}_k - (p_{k-1} + \overline{p}_{k-1}) - \overline{p}_{k+1} \le \lambda_k$, one has $0 \le \overline{p}_k \le \lambda_k + \overline{p}_{k+1}$, $0 \le p_1 \le \lambda_1 + 2\overline{p}_2$, $0 \le p_k \le \lambda_k + p_{k-1} + \overline{p}_{k-1} + \overline{p}_{k+1} - 2\overline{p}_k$, and then $\binom{\lambda_1+2\overline{p}_2}{p_1} \ne 0, \binom{\lambda_k+\overline{p}_{k+1}}{\overline{p}_k} \ne 0, \binom{\lambda_k+p_{k-1}-2\overline{p}_k+\overline{p}_{k-1}+\overline{p}_{k+1}}{p_k} \ne 0, 2 \le k \le l$, i.e.,

$$e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} e_l^{(p_l)} \theta^{P_s} v \ne 0$$

for $\mathfrak{g}$ of type B_l; and $\overline{p}_k - \overline{p}_{k+1} \le \lambda_k$, $1 \le k \le l$, $p_k - p_{k-1} - \overline{p}_{k-1} + 2\overline{p}_k - \overline{p}_{k+1} \le \lambda_k$, $2 \le k \le l$, one has $\overline{p}_k \le \overline{p}_{k+1} + \lambda_k$, $1 \le k \le l$, $p_k \le p_{k-1} + \overline{p}_{k-1} - 2\overline{p}_k + \overline{p}_{k+1} + \lambda_k$, $2 \le k \le l$, and then $\prod_{n=1}^{l} \binom{\lambda_n\ +\overline{p}_{n+1}}{p_n} \ne 0$, $\prod_{k=2}^{l} \binom{\lambda_k+p_{k-1}+\overline{p}_{k-1}-2\overline{p}_k+\overline{p}_{k+1}}{p_k} \ne 0$, i.e.

$$e_l^{(\overline{p}_l)} \cdots e_2^{(\overline{p}_2)} e_1^{(p_1)} e_2^{(p_2)} \cdots e_{l-1}^{(p_{l-1})} e_l^{(p_l)} \theta^{P_s} v \ne 0$$

for $\mathfrak{g}$ of type C_l.

This shows that $\theta^{P_s} v \ne 0$. By our construction, it is easy to see that $\theta^{P_s} v \in M_{P_s}$ but $\theta^{P_s} v \notin M_{P_{s-1}}$. So we need only to prove that $e_i \theta^{P_s} v \in M_{P_{s-1}}$ for $1 \le i \le l-1$, and then we can conclude that $\theta^{P_s} v$ is a primitive vector in $V(\lambda)$. In fact

$$\begin{aligned}
e_i\theta^{P_s}v &= e_i f_l^{(p_l)} f_{l-1}^{(p_{l-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_l^{(\overline{p}_l)} v \\
&= \theta^{P_s} e_i v + f_l^{(p_l)} \cdots f_i^{(p_i-1)} (h_i - p_i + 1) f_{i-1}^{(p_{i-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_l^{(\overline{p}_l)} v + \\
&\quad f_l^{(p_l)} f_{l-1}^{(p_{l-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_i^{(\overline{p}_i-1)} (h_i - \overline{p}_i + 1) f_{i+1}^{(\overline{p}_{i+1})} \cdots f_l^{(\overline{p}_l)} v \quad \text{by § 2.1 (4)} \\
&= (\lambda_i - p_i + 1 + p_{i-1} + \overline{p}_{i-1} - 2\overline{p}_i + \overline{p}_{i+1}) f_l^{(p_l)} \cdots f_i^{(p_i-1)} f_{i-1}^{(p_{i-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots \\
&\quad f_l^{(\overline{p}_l)} v + (\lambda_i - \overline{p}_i + 1 + \overline{p}_{i+1}) f_l^{(p_l)} f_{l-1}^{(p_{l-1})} \cdots f_2^{(p_2)} f_1^{(p_1)} f_2^{(\overline{p}_2)} \cdots f_i^{(\overline{p}_i-1)} \\
&\quad f_{i+1}^{(\overline{p}_{i+1})} \cdots f_l^{(\overline{p}_l)} v \quad \text{by } §2.1(2)
\end{aligned}$$

Note that $(0, \ldots, 0, p_l, \ldots, p_{i+1}, p_i - 1, p_{i-1}, \ldots, p_1, \overline{p}_2, \ldots, \overline{p}_l) \prec P_s$, $(0, \ldots, 0, p_l, \ldots, p_2,\ p_1,\ \overline{p}_2, \ldots, \overline{p}_{i-1}, \overline{p}_i - 1, \overline{p}_{i+1}, \ldots, \overline{p}_l) \prec P_s$, one has $e_i\theta^{P_s} v \in M_{P_{s-1}}$.

Thirdly, we show that $M_{P_s} = M_{P_{s-1}} + \mathfrak{U}_{\mathfrak{l}-1}\theta^{P_s} v$. "$\supseteq$"is easy to be proved by definition of M_{P_s} and §2.5. Here we only prove "$\subseteq$". For any $K \in \Pi$ with $K \prec P_{s+1}$, one has a unique decomposition $K = K_2 + K_1$ with $K_2 \in \Pi'_\lambda$ and $K_1 \in \Pi_{l-1}$. If $K \prec P_s$, then $\theta^K v \in M_{P_{s-1}}$. Otherwise, when $P_s \preceq K \prec P_{s+1}$, we must have $K_2 = P_s$. Then

$$\theta^K v = \theta^{K_1+K_2} v = \theta^{K_1} \theta^{P_s} v \in \mathfrak{U}_{l-1} \theta^{P_s} v$$

as required.

Finally, we show that $M_{P_s}/M_{P_{s-1}} \simeq V\left((\lambda - P_s\alpha)_{\mathfrak{U}_{\mathfrak{l}-1}}\right)$. Let w be the canonical image of $\theta^{P_s} v$ in $M_{P_s}/M_{P_{s-1}}$. Then one has $M_{P_s}/M_{P_{s-1}} \simeq \mathfrak{U}_{\mathfrak{l}-1}\mathfrak{w}$. Since $\theta^{P_s} v$ is a primitive vector in $V(\lambda)$, w becomes a maximal vector of weight $(\lambda - P_s\alpha)_{\mathfrak{U}_{\mathfrak{l}-1}}$. Note the fact that $V(\lambda)$ is a finite dimensional module, and $M_{P_s}/M_{P_{s-1}}$ is also finite dimensional and generated by a maximal vector w, we must have $M_{P_s}/M_{P_{s-1}} \simeq V\left((\lambda - P_s\alpha)_{\mathfrak{U}_{\mathfrak{l}-1}}\right)$ by Lemma 3.6.

Using the complete reducibility, we prove Theorem 3.8. □

The following theorem was first proved in Littlemann [4] Corollary 6 of Theorem 6.1. We use induction on the rank of $\mathfrak{g}$ to give a different proof.

Theorem 3.9 *Let v be a maximal vector of $V(\lambda)$. Then $\{\theta^K v \mid K \in \Pi_\lambda\}$ forms a $\mathbb{Z}$-basis of $V(\lambda)_\mathbb{Z}$.*

Proof We use induction on l. When $l = 2$, one has for any $\lambda = (\lambda_1, \lambda_2) \in \Lambda^+$, by the Weyl character formula,

$$\dim V(\lambda) = \frac{1}{6}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)(\lambda_1 + 2\lambda_2 + 3),$$

for $\mathfrak{g}$ of type B_2, and

$$\dim V(\lambda) = \frac{1}{6}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)(2\lambda_1 + \lambda_2 + 3),$$

for $\mathfrak{g}$ of type C_2. Let

$$\Pi_\lambda = \{K = (k_2^2, k_1^1, k_2^1, k_3^1) | k_3^1 \le \lambda_2, 2k_3^1 \le k_2^1 \le \lambda_1 + 2k_3^1, \frac{1}{2}k_2^1 \le k_1^1 \le \lambda_2 + k_2^1 - 2k_3^1,$$
$$k_2^2 \le \lambda_1 + 2k_1^1 - 2k_2^1 + 2k_3^1\}$$

for $\mathfrak{g}$ of type B_2, and

$$\Pi_\lambda = \{K = (k_2^2, k_1^1, k_2^1, k_3^1) | k_3^1 \le \lambda_2, k_3^1 \le k_2^1 \le \lambda_1 + k_3^1, k_2^1 \le k_1^1 \le \lambda_2 + 2k_2^1 - 2k_3^1,$$
$$k_2^2 \le \lambda_1 + k_1^1 - 2k_2^1 + k_3^1\}$$

for $\mathfrak{g}$ of type C_2. Now we need only to show that the number of Π_λ is equal to $\dim V(\lambda)$, i.e., $\{\theta^K v | K \in \Pi_\lambda\}$ forms a $\mathbb{Z}$-basis of the $\mathbb{Z}$-form of $V(\lambda)$ and spans $V(\lambda)$ over $\mathbb{C}$. However, it could be done easily by an elementary but prolix calculation. We omit the detail here.

Assume that our theorem holds for $l-1$, and then we have to show that the theorem holds for l. Let us use the same notations as in the proof of Theorem 3.9, and construct the bases of M_{P_s} for $1 \le s \le t$. For $s = 1$, one has $M_{P_1} \simeq V(\lambda_{\mathfrak{U}_{l-1}})$ as $\mathfrak{U}_{l-1}$-module, and $\{\theta^K v \mid K \in \Pi_{l-1, \lambda_{\mathfrak{U}_{l-1}}}\}$ is a $\mathbb{Z}$-basis of M_{P_1} by the induction hypothesis. When $s = 2$, note the following facts:

i) $\theta^{K+P_2} v \in M_{P_2}$ if $K \in \Pi_{l-1, (\lambda - P_2\alpha)_{\mathfrak{U}_{l-1}}}$ by § 2;

ii) The number of $\{\theta^K \mid K \in \Pi_{l-1, (\lambda - P_2\alpha)_{\mathfrak{g}_{l-1}}}\}$ is equal to $\dim V\left((\lambda - P_2\alpha)_{\mathfrak{g}_{l-1}}\right)$ by the induction hypothesis;

iii) $M_{P_2}/M_{P_1} \simeq V\left((\lambda - P_2\alpha)_{\mathfrak{g}_{l-1}}\right)$.

Therefore, we see that

$$\{\theta^K v | K \in \Pi_{l-1, \lambda_{\mathfrak{g}_{l-1}}}\} \bigcup \{\theta^K \theta^{P_2} v = \theta^{K+P_2} v | K \in \Pi_{l-1, (\lambda - P_2\alpha)_{\mathfrak{g}_{l-1}}}\}$$

forms a $\mathbb{Z}$-basis of M_{P_2}.

In this way, the set of

$$\{\theta^K v | K \in \Pi_{l-1, \lambda_{\mathfrak{g}_{l-1}}}\} \bigcup \{\theta^{K+P_2} v | K \in \Pi_{l-1, (\lambda - P_2\alpha)_{\mathfrak{g}_{l-1}}}\} \bigcup \cdots \bigcup$$
$$\{\theta^{K+P_t} v | K \in \Pi_{l-1, (\lambda - P_t\alpha)_{\mathfrak{g}_{l-1}}}\}$$

forms a $\mathbb{Z}$-basis of $M_{P_t} = V(\lambda)$. Note that elements in both the above set and the set of $\{\theta^K v | K \in \Pi_\lambda\}$ are the same, this proves our theorem. □

Theorem 3.10 *Let $\mu \in \Pi(\lambda)$ be a weight of $V(\lambda)$. Then the multiplicity $m_\lambda(\mu)$ of μ in $V(\lambda)$*

is equal to

$$m_\lambda(\mu) = \dim V(\lambda)_\mu = \sum_{P\in\Pi'_\lambda,\ P\alpha\ll\lambda-\mu} \dim V\left((\lambda - P\alpha)_{\mathfrak{U}_{l-1}}\right)_{\mu_{\mathfrak{U}_{l-1}}}$$
$$= \sum_{P\in\Pi'_\lambda,\ P\alpha\ll\lambda-\mu} m_{(\lambda-P\alpha)_{\mathfrak{U}_{l-1}}}(\mu_{\mathfrak{U}_{l-1}}).$$

Proof Let us use the same notations as in the proof of Theorem 3.8, and let $\lambda - \mu = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_l\alpha_l$ with all $a_i \geq 0$, $i = 1, 2, \ldots, l$. Let $\mathcal{M}$ be the set of the basis elements in the weight space of μ of $V(\lambda)$. They satisfy the following conditions:

$$\begin{aligned}\mathcal{M} = \{\theta^K v \mid K = (&k_l^l, k_{l-1}^{l-1}, k_l^{l-1}, k_{l+1}^{l-1}, \ldots, k_{l-i+1}^{l-i+1}, \ldots, k_{l-1}^{l-i+1}, k_l^{l-i+1}, k_{l+1}^{l-i+1}, \ldots,\\ &k_{l+i-1}^{l-i+1}, \ldots, k_1^1, k_2^1, \ldots, k_{l-1}^1, k_l^1, k_{l+1}^1, \ldots, k_{2l-1}^1) \in \Pi_\lambda, \ \text{ with } \ k_1^1 + k_{2l-1}^1 = a_l,\\ &k_2^1 + k_{2l-2}^1 + k_2^2 + k_{2l-2}^2 = a_{l-1}, \ldots, k_{l-1}^1 + k_{l+1}^1 + \cdots + k_{l-1}^{l-1} + k_{l+1}^{l-1} = a_2,\\ &k_l^1 + k_l^2 + \cdots + k_l^l = a_1\}.\end{aligned}$$

Then the number of $\mathcal{M}$ is equal to $m_\lambda(\mu)$. If we divide $\mathcal{M}$ into a disjoint union of $\mathcal{M}_i$, where $\mathcal{M}_i = \{\theta^K v \mid K \in \mathcal{M} \text{ with } P_i \prec K \prec P_{i+1}\}$. From Theorem 3.9, we see that $\mathcal{M}_i \subseteq M_{P_i}$, and the number of $\mathcal{M}_i$ is equal to $m_{(\lambda-P_i\alpha)}(\mu_{\mathfrak{U}_{l-1}})$. Theorem 3.10 follows from Theorem 3.8. □

4. Examples

When $\mathfrak{g}_l$ is of type C_3, for any $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3 = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda^+$, one has the following index sets:

$$\begin{aligned}\Pi =& \{(k_3^3, k_2^2, k_3^2, k_4^2, k_1^1, k_2^1, k_3^1, k_4^1, k_5^1) | k_2^2 \geq k_3^2 \geq k_4^2, k_1^1 \geq k_2^1 \geq k_3^1 \geq k_4^1 \geq k_5^1\} \subseteq \mathbb{N}^9;\\ \Pi' =& \{(0, \ldots, 0, k_1^1, k_2^1, k_3^1, k_4^1, k_5^1) | k_1^1 \geq k_2^1 \geq k_3^1 \geq k_4^1 \geq k_5^1\} \subseteq \mathbb{N}^9;\\ \Pi_\lambda =& \{(k_3^3, k_2^2, k_3^2, k_4^2, k_1^1, k_2^1, k_3^1, k_4^1, k_5^1) \in \Pi | k_5^1 \leq \lambda_3, k_4^1 \leq \lambda_2 + k_5^1, k_3^1 \leq \lambda_1 + k_4^1,\\ & k_2^1 \leq \lambda_2 + 2k_3^1 - 2k_4^1 + k_5^1, k_1^1 \leq \lambda_3 + k_2^1 + k_4^1 - 2k_5^1, k_4^2 \leq \lambda_2 + k_1^1 - 2k_2^1 + 2k_3^1 -\\ & 2k_4^1 + k_5^1, \ k_3^2 \leq \lambda_1 + k_4^2 + k_2^1 - 2k_3^1 + k_4^1, k_2^2 \leq \lambda_2 + 2k_3^2 - 2k_4^2 + k_1^1 - 2k_2^1 +\\ & 2k_3^1 - 2k_4^1 + k_5^1, k_3^3 \leq \lambda_1 + k_2^2 - 2k_3^2 + k_4^2 + k_2^1 - 2k_3^1 + k_4^1\};\\ \Pi'_\lambda =& \{(0, \ldots, 0, p_3, p_2, p_1, \overline{p}_2, \overline{p}_3) \in \Pi' | \overline{p}_3 \leq \lambda_3, \overline{p}_2 \leq \lambda_2 + \overline{p}_3, p_1 \leq \lambda_1 + \overline{p}_2,\\ & p_2 \leq \lambda_2 + 2p_1 - 2\overline{p}_2 + \overline{p}_3, p_3 \leq \lambda_3 + p_2 + \overline{p}_2 - 2\overline{p}_3\}.\end{aligned}$$

Take $\lambda = \omega_1 + \omega_2 + \omega_3 = (1, 1, 1)$, then one has

$$\begin{aligned}\Pi_\lambda =& \{(k_3^3, k_2^2, k_3^2, k_4^2, k_1^1, k_2^1, k_3^1, k_4^1, k_5^1) \mid k_5^1 \leq 1, \ k_4^1 \leq 1 + k_5^1, k_3^1 \leq 1 + k_4^1,\\ & k_2^1 \leq 1 + 2k_3^1 - 2\ k_4^1 + k_5^1, \ k_1^1 \leq 1 + k_2^1 + k_4^1 - 2k_5^1, \ k_4^2 \leq 1 + k_1^1 - 2k_2^1 +\\ & 2k_3^1 - 2k_4^1 + k_5^1, \ k_3^2 \leq 1 + k_4^2 + k_2^1 - 2k_3^1 + k_4^1, \ k_2^2 \leq 1 + 2k_3^2 - 2k_4^2 +\\ & k_1^1 - 2k_2^1 + 2k_3^1 - 2k_4^1 + k_5^1, \ k_3^3 \leq 1 + k_2^2 - 2k_3^2 + k_4^2 + k_2^1 - 2k_3^1 + k_4^1\} \subseteq \Pi,\\ \Pi'_\lambda =& \{P_1 = (0, \ldots, 0, 0, 0, 0, 0, 0) \prec P_2 = (0, \ldots, 0, 1, 0, 0, 0, 0) \prec P_3 = (0, \ldots, 0, 1, 1, 0, 0, 0)\\ & \prec P_4 = (0, \ldots, 0, 2, 1, 0, 0, 0) \prec P_5 = (0, \ldots, 0, 1, 1, 1, 0, 0) \prec P_6 = (0, \ldots, 0, 2, 1, 1, 0, 0)\end{aligned}$$

$$
\begin{aligned}
&\prec P_7=(0,\dots,0,2,2,1,0,0)\prec P_8=(0,\dots,0,3,2,1,0,0)\prec P_9=(0,\dots,0,3,3,1,0,0)\\
&\prec P_{10}=(0,\dots,0,4,3,1,0,0)\prec P_{11}=(0,\dots,0,1,1,1,1,0)\prec P_{12}=(0,\dots,0,2,1,1,1,0)\\
&\prec P_{13}=(0,\dots,0,3,1,1,1,0)\prec P_{14}=(0,\dots,0,2,2,2,1,0)\prec P_{15}=(0,\dots,0,3,2,2,1,0)\\
&\prec P_{16}=(0,\dots,0,4,2,2,1,0)\prec P_{17}=(0,\dots,0,3,3,2,1,0)\prec P_{18}=(0,\dots,0,4,3,2,1,0)\\
&\prec P_{19}=(0,\dots,0,5,3,2,1,0)\prec P_{20}=(0,\dots,0,1,1,1,1,1)\prec P_{21}=(0,\dots,0,2,2,1,1,1)\\
&\prec P_{22}=(0,\dots,0,2,2,2,1,1)\prec P_{23}=(0,\dots,0,3,3,2,1,1)\prec P_{24}=(0,\dots,0,4,4,2,1,1)\\
&\prec P_{25}=(0,\dots,0,2,2,2,2,1)\prec P_{26}=(0,\dots,0,3,2,2,2,1)\prec P_{27}=(0,\dots,0,3,3,3,2,1)\\
&\prec P_{28}=(0,\dots,0,4,3,3,2,1)\prec P_{29}=(0,\dots,0,4,4,3,2,1)\prec P_{30}=(0,\dots,0,5,4,3,2,1)\}.
\end{aligned}
$$

Therefore, one has the following $\mathfrak{U}_2$-module isomorphisms:

$$
\begin{aligned}
&M_{P_1}\simeq V(1,1), && M_{P_2}/M_{P_1}\simeq V(1,2), && M_{P_3}/M_{P_2}\simeq V(2,0),\\
&M_{P_4}/M_{P_3}\simeq V(2,1), && M_{P_5}/M_{P_4}\simeq V(0,2), && M_{P_6}/M_{P_5}\simeq V(0,3),\\
&M_{P_7}/M_{P_6}\simeq V(1,1), && M_{P_8}/M_{P_7}\simeq V(1,2), && M_{P_9}/M_{P_8}\simeq V(2,0),\\
&M_{P_{10}}/M_{P_9}\simeq V(2,1), && M_{P_{11}}/M_{P_{10}}\simeq V(1,0), && M_{P_{12}}/M_{P_{11}}\simeq V(1,1),\\
&M_{P_{13}}/M_{P_{12}}\simeq V(1,2), && M_{P_{14}}/M_{P_{13}}\simeq V(0,1), && M_{P_{15}}/M_{P_{14}}\simeq V(0,2),\\
&M_{P_{16}}/M_{P_{15}}\simeq V(0,3), && M_{P_{17}}/M_{P_{16}}\simeq V(1,0), && M_{P_{18}}/M_{P_{17}}\simeq V(1,1),\\
&M_{P_{19}}/M_{P_{18}}\simeq V(1,2), && M_{P_{20}}/M_{P_{19}}\simeq V(1,1), && M_{P_{21}}/M_{P_{20}}\simeq V(2,0),\\
&M_{P_{22}}/M_{P_{21}}\simeq V(0,2), && M_{P_{23}}/M_{P_{22}}\simeq V(1,1), && M_{P_{24}}/M_{P_{23}}\simeq V(2,0),\\
&M_{P_{25}}/M_{P_{24}}\simeq V(1,0), && M_{P_{26}}/M_{P_{25}}\simeq V(1,1), && M_{P_{27}}/M_{P_{26}}\simeq V(0,1),\\
&M_{P_{28}}/M_{P_{27}}\simeq V(0,2), && M_{P_{29}}/M_{P_{28}}\simeq V(1,0), && M_{P_{30}}/M_{P_{29}}\simeq V(1,1).
\end{aligned}
$$

Moreover, $V(\omega_1+\omega_2+\omega_3)|_{\mathfrak{U}_2}\simeq\bigoplus_{i=1}^{30}V(\lambda-P_i\alpha)|_{\mathfrak{U}_2}$. Take $\mu=\omega_1+\omega_3=(1,0,1)$, one has $m_\lambda(\mu)=6$, $\lambda-\mu=\alpha_1+2\alpha_2+\alpha_3$. Using Theorem 3.11, one has

$$
\begin{aligned}
m_\lambda(\mu) &= m_{(\lambda-P_2\alpha)_{\mathfrak{U}_2}}(\mu_{\mathfrak{U}_2})+m_{(\lambda-P_3\alpha)_{\mathfrak{U}_2}}(\mu_{\mathfrak{U}_2})+m_{(\lambda-P_5\alpha)_{\mathfrak{U}_2}}(\mu_{\mathfrak{U}_2})+m_{(\lambda-P_{11}\alpha)_{\mathfrak{U}_2}}(\mu_{\mathfrak{U}_2})\\
&= m_{(1,2)}(1,0)+m_{(2,0)}(1,0)+m_{(0,2)}(1,0)+m_{(1,0)}(1,0)=3+1+1+1=6.
\end{aligned}
$$

References

[1] HUMPHREYS J E. *Introduction to Lie Algebras and Representation Theory* [M]. Springer-Verlag, New York-Berlin, 1972.

[2] JANTZEN J C. *Representations of Algebraic Groups* [M]. Second edition. American Mathematical Society, Providence, RI, 2003.

[3] LITTELMANN P. *An algorithm to compute bases and representation matrices for SL_{n+1}-representations* [J]. J. Pure Appl. Algebra, 1997, **117/118**: 447–468.

[4] LITTELMANN P. *Cone, crystal and pattern* [J]. Transform. Groups, 1998, **3**(2): 145–179.

[5] YE Jiachen, ZHOU Zhongguo. *A new multiplicity formula for the Weyl modules of type A* [J]. Comm. Algebra, 2005, **33**(12): 4361–4373.