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Entire Functions Sharing One Value with Finite Weight

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Abstract This paper deals with some uniqueness problems of entire functions concerning differential polynomials that share one value with finite weight in a different form. We obtain some theorems which generalize some results given by Banerjee, Fang and Hua, Zhang and Lin, Zhang, etc.

Keywords entire functions; differential polynomial; weighted sharing.

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1. Introduction and main results

Suppose readers are familiar with the knowledge of value distribution and uniqueness theory of meromorphic functions. Some basic notions, for example, $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$, can be refered to [4, 5, 7, 8].

Some problems on the uniqueness of entire functions and their differential polynomials sharing the value 1 CM have been studied and some important results were obtained in [2, 3, 7, 9, 10]. For example, Zhang, Chen and Lin [10] obtained the following result:

Theorem A ([10]) *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n \geq 3m + 2k + 5$, and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ or $P(z) \equiv c_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0, c_0 \neq 0$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share 1 CM, then*

(i) when $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n), a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \cdots + a_1 \omega_1 + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \cdots + a_1 \omega_2 + a_0)$;

(ii) when $P(z) \equiv c_0$, either $f = c_1/\sqrt[n]{c_0} e^{cz}$, $g = c_2/\sqrt[n]{c_0} e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.

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Recently, many mathematicians (such as Yi, Lahiri, Fang, Banerjee, Lin, and others) are very interested in investigating the meromorphic functions sharing values with finite weight in the field of complex analysis. So one may ask: In Theorem A, can the nature of sharing 1 CM be further relaxed to finite weight?

In this paper the possible solutions of the above problem are investigated and the following theorems which are the main results of the paper are obtained.

Theorem 1.1 *Let $f(z)$, $g(z)$ be two transcendental entire functions and let n, k, m, l be four positive integers with $n > \frac{19}{3}m + \frac{14}{3}(k+1)$. If $\overline{E}_{l)}(1;[f^nP(f)]^{(k)}) = \overline{E}_{l)}(1;[g^nP(g)]^{(k)})$ and $E_{1)}(1;[f^nP(f))]^{(k)}) = E_{1)}(1;[g^nP(g)]^{(k)})$, where $l \geq 3$, then the conclusion of Theorem A still holds.*

Theorem 1.2 *Let $f(z)$, $g(z)$ be two transcendental entire functions and let n, k, m, l be four positive integers with $n > 5m+4k+4$. If $\overline{E}_{l)}(1;[f^nP(f)]^{(k)}) = \overline{E}_{l)}(1;[g^nP(g)]^{(k)})$ and $E_{2)}(1;[f^n P(f)]^{(k)}) = E_{2)}(1;[g^nP(g)]^{(k)})$, where $l \geq 4$, then the conclusion of Theorem A still holds.*

Though the standard definitions and notations of the value distribution theory are available in [4, 6], we explain some definitions and notations which are used in the paper.

Definition 1.1 ([1]) *Let k and r be two positive integers such that $1 \leq r < k-1$ and for $a \in \mathbf{C}$, $\overline{E}_{k)}(a;f) = \overline{E}_{k)}(a;g), E_{r)}(a;f) = E_{r)}(a;g)$. Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)(\overline{N}_L(r,a;g))$ the reduced counting function of those a-points of f and g for which $p > q \geq r+1 (q > p \geq r+1)$, by $\overline{N}_E^{(r+1}(r,a;f)$ the reduced counting function of those a-points of f and g for which $p = q \geq r+1$, by $\overline{N}_{f>s}(r,a;g)$ the reduced counting functions of those a-points of f and g for which $p > q = s$, and by $\overline{N}_{f\geq k+1}(r,a;f|g \neq a)(\overline{N}_{g\geq k+1}(r,a;g|f \neq a))$ the reduced counting functions of those a-points of f and g for which $p \geq k+1$ and $q = 0$ ($q \geq k+1$ and $p = 0$).*

If $r = 0$ in Definition 1.1, then we use the same notations as in Definition 1.1 except for that by $\overline{N}_E^{1)}(r,a;f)$ we mean the common simple a-points of f and g and by $\overline{N}_E^{(2}(r,a;f)$ we mean the reduced counting functions of those a-points of f and g for which $p = q \geq 2$.

Definition 1.2 ([6]) *Let $a, b \in \mathbf{C} \cup \{\infty\}$. We denote by $N(r,a;f|g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g; by $N(r,a;f|g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.*

2. Some lemmas

For the proof of our results we need the following lemmas.

Lemma 2.1 ([7]) *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r,P(f)) = nT(r,f) + S(r,f).$$

Lemma 2.2 ([4]) *Let $f(z)$ be a transcendental entire function, k a positive integer, and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r,f) &\le N(r,0;f) + N(r,c;f^{(k)}) - N(r,0;f^{(k+1)}) + S(r,f) & (1)\\ &\le N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) - N_0(r,0;f^{(k+1)}) + S(r,f), & (2)\end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.3 ([4]) *Let $f(z)$ be a meromorphic function and $\alpha_1(z)$, $\alpha_2(z)$ be two meromorphic functions such that $T(r,\alpha_i) = S(r,f)$ $(i = 1,2)$. Then*

$$T(r,f) \le \overline{N}(r,\infty;f) + \overline{N}(r,\alpha_1(z);f) + \overline{N}(r,\alpha_2(z);f) + S(r,f).$$

Lemma 2.4 *Let f be a nonconstant entire function. Then*

$$\overline{N}(r,0;f^{(k)}) \le N_k(r,0;f) + \overline{N}(r,0;f) + S(r,f).$$

Proof Since

$$\begin{aligned} \overline{N}(r,0;f^{(k)}) &\le \overline{N}(r,\frac{f}{f^{(k)}}) + \overline{N}(r,0;f) \le T(r,\frac{f}{f^{(k)}}) + \overline{N}(r,0;f)\\ &\le T(r,\frac{f^{(k)}}{f}) + \overline{N}(r,0;f) + S(r,f)\\ &\le N(r,\frac{f^{(k)}}{f}) + m(r,\frac{f^{(k)}}{f}) + \overline{N}(r,0;f) + S(r,f),\end{aligned}$$

we estimate $N(r,\frac{f^{(k)}}{f})$ in the following.

Obviously, the poles of $\frac{f^{(k)}}{f}$ may only occur at the zeros of f. If z_0 is a $q(q \le k)$ order zero of f, then z_0 is a at most q order pole of $\frac{f^{(k)}}{f}$, and if z_0 is a $q(q > k)$ order zero of f, then z_0 is a k order pole of $\frac{f^{(k)}}{f}$. Hence we have

$$N(r,\frac{f^{(k)}}{f}) \le N_k(r,0;f) + S(r,f),$$

i.e.,

$$\overline{N}(r,0;f^{(k)}) \le N_k(r,0;f) + \overline{N}(r,0;f) + S(r,f). \quad \square$$

Lemma 2.5 ([1]) *Let F, G be two nonconstant entire functions such that $E_{1)}(1;F) = E_{1)}(1;G)$ and $H \not\equiv 0$. Then*

$$N_E^{1)}(r,1;F) \le N(r,\infty;H) + S(r,F) + S(r,G),$$

where $H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1})$.

Lemma 2.6 ([1]) *Let F, G be two nonconstant entire functions such that $\overline{E}_{l)}(1;F) = \overline{E}_{l)}(1;G)$, $E_{1)}(1;F) = E_{1)}(1;G)$ and $H \not\equiv 0$, where $l \ge 3$. Then*

$$\begin{aligned} N(r,\infty;H) \le &\overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) +\\ &\overline{N}_{F\ge l+1}(r,1;F|G \ne 1) + \overline{N}_{G\ge l+1}(r,1;G|F \ne 1) + \overline{N}_0(r,0;F') +\end{aligned}$$

$$\overline{N}_0(r,0;G'),$$

where $\overline{N}_0(r,0;F')$ *is the reduced counting function of those zeros of* F' *which are not the zeros of* $F(F-1)$ *and* $\overline{N}_0(r,0;G')$ *is similarly defined.*

Lemma 2.7 ([1]) *Let* F, G *be two nonconstant entire functions such that* $\overline{E}_{l)}(1;F)=\overline{E}_{l)}(1;G)$, $E_{1)}(1;F)=E_{1)}(1;G)$ *and* $H\not\equiv 0$, *where* $l\geq 3$. *Then*

$$\begin{aligned}&2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+l\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)-\overline{N}_{F>2}(r,1;G)\\ &\quad\leq N(r,1;G)-\overline{N}(r,1;G).\end{aligned}$$

Lemma 2.8 *Let* F, G *be two nonconstant entire functions such that* $\overline{E}_{l)}(1;F)=\overline{E}_{l)}(1;G)$, $E_{1)}(1;F)=E_{1)}(1;G)$, *where* $l\geq 3$. *Then*

$$\overline{N}_{F>2}(r,1;G)+2\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)\leq\frac{2}{3}\overline{N}(r,0;F)-\frac{2}{3}N_0(r,0;F')+S(r,F).$$

Proof We note that any 1-point of F with multiplicity ≥ 3 is counted at most twice. Hence we see that

$$\begin{aligned}&\overline{N}_{F>2}(r,1;G)+2\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)\leq\frac{2}{3}N(r,0;F'|F=1)\\ &\quad\leq\frac{2}{3}N(r,0;F'|F\neq 0)-\frac{2}{3}N_0(r,0;F')\leq\frac{2}{3}\overline{N}(r,0;F)-\frac{2}{3}N_0(r,0;F')+S(r,F).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.9 *Let* F^*, G^* *be two nonconstant entire functions and* $\overline{E}_{l)}(1;(F^*)^{(k)})=\overline{E}_{l)}(1;(G^*)^{(k)})$, $E_{1)}(1;(F^*)^{(k)})=E_{1)}(1;(G^*)^{(k)})$ *and* $H^*\not\equiv 0$, *where* $l\geq 3$. *Then*

$$\begin{aligned}T(r,F^*)\leq&\frac{5}{3}\overline{N}(r,0;F^*)+\frac{5}{3}N_k(r,0;F^*)+N_{k+1}(r,0;F^*)+\overline{N}(r,0;G^*)+\\ &N_k(r,0;G^*)+N_{k+1}(r,0;G^*)+S(r,F^*)+S(r,G^*),\end{aligned}$$

where

$$H^*\equiv\Big[\frac{(F^*)^{(k+2)}}{(F^*)^{(k+1)}}-\frac{2(F^*)^{(k+1)}}{(F^*)^{(k)}-1}\Big]-\Big[\frac{(G^*)^{(k+2)}}{(G^*)^{(k+1)}}-\frac{2(G^*)^{(k+1)}}{(G^*)^{(k)}-1}\Big].$$

Proof Let $F=(F^*)^{(k)}$ and $G=(G^*)^{(k)}$. Then the condition of this lemma is $\overline{E}_{l)}(1;F)=\overline{E}_{l)}(1;G), E_{1)}(1;F)=E_{1)}(1;G)$ and $H^*=H\not\equiv 0$. Using Lemmas 2.5 and 2.7, we get

$$\begin{aligned}&\overline{N}(r,1;F)+\overline{N}(r,1;G)\leq N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\\ &\quad\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+\overline{N}(r,1;G)\\ &\quad\leq\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\\ &\quad\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\\ &\quad\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\\ &\quad\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+T(r,G)-m(r,1;G)+\\ &\quad O(1)-2\overline{N}_L(r,1;F)-2\overline{N}_L(r,1;G)-\overline{N}_E^{(2}(r,1;F)-\\ &\quad l\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}_{F>2}(r,1;G)+\overline{N}_0(r,0;F')+\end{aligned}$$

$$\begin{aligned}
&\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\le&\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+T(r,G)-m(r,1;G)+\\
&2\overline{N}_{F\ge l+1}(r,1;F|G\ne 1)+\overline{N}_{F>2}(r,1;G)-\\
&(l-1)\overline{N}_{G\ge l+1}(r,1;G|F\ne 1)+\overline{N}_0(r,0;F')+\\
&\overline{N}_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned}$$

From Lemma 2.8, we can get

$$\begin{aligned}
\overline{N}(r,1;F)+\overline{N}(r,1;G)\le&\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+T(r,G)-m(r,1;G)+\\
&\frac{2}{3}\overline{N}(r,0;F)-(l-1)\overline{N}_{G\ge l+1}(r,1;G|F\ne 1)+\\
&\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \qquad (3)
\end{aligned}$$

Using Lemma 2.2 for F^* and G^*, we get

$$T(r,F^*)\le N_{k+1}(r,0;F^*)+\overline{N}(r,1;F)-N_0(r,0;F')+S(r,F^*), \qquad (4)$$

$$T(r,G^*)\le N_{k+1}(r,0;G^*)+\overline{N}(r,1;G)-N_0(r,0;G')+S(r,G^*). \qquad (5)$$

Adding (4) and (5) gives

$$\begin{aligned}
T(r,F^*)+T(r,G^*)\le&N_{k+1}(r,0;F^*)+N_{k+1}(r,0;G^*)+\overline{N}(r,1;F)+\overline{N}(r,1;G)-\\
&N_0(r,0;F')-N_0(r,0;G')+S(r,F^*)+S(r,G^*). \qquad (6)
\end{aligned}$$

Since

$$T(r,G)=T(r,(G^*)^{(k)})\le T(r,G^*)+S(r,G^*), \qquad (7)$$

from (3), (6), (7) and $S(r,F)=S(r,F^*)$, $S(r,G)=S(r,G^*)$, it follows

$$\begin{aligned}
T(r,F^*)\le&N_{k+1}(r,0;F^*)+N_{k+1}(r,0;G^*)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)-\\
&m(r,1;G)+\frac{2}{3}\overline{N}(r,0;F)+S(r,F^*)+S(r,G^*). \qquad (8)
\end{aligned}$$

Since $F=(F^*)^{(k)}$ and $G=(G^*)^{(k)}$, from Lemma 2.4, (8) becomes

$$\begin{aligned}
T(r,F^*)\le&N_{k+1}(r,0;F^*)+\frac{5}{3}\overline{N}(r,0;F^*)+\frac{5}{3}N_k(r,0;F^*)+N_{k+1}(r,0;G^*)+\\
&\overline{N}(r,0;G^*)+N_k(r,0;G^*)+S(r,F^*)+S(r,G^*). \qquad (9)
\end{aligned}$$

Lemma 2.10 *Let F^*, G^* be two transcendental entire functions and $\overline{E}_{l)}(1;(F^*)^{(k)})=\overline{E}_{l)}(1;(G^*)^{(k)})$, $E_{1)}(1;(F^*)^{(k)})=E_{1)}(1;(G^*)^{(k)})$ where $l\ge 3$. If*

$$\Delta_{1l}=\frac{5}{3}\Theta(0,F^*)+\frac{5}{3}\delta_k(0;F^*)+\delta_{k+1}(0;F^*)+\Theta(0,G^*)+\delta_k(0;G^*)+\delta_{k+1}(0;G^*)>\frac{19}{3},$$

Then $(F^*)^{(k)}(G^*)^{(k)}\equiv 1$ or $F^*\equiv G^*$.

Proof From Lemma 2.9, we first suppose that $H\not\equiv 0$. Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,G^*)\le T(r,F^*)$ for $r\in I$. From Lemma 2.9 we get

$$T(r,F^*)\le\{\frac{22}{3}-\delta_{k+1}(0;F^*)-\frac{5}{3}\Theta(0,F^*)-\frac{5}{3}\delta_k(0;F^*)-\delta_{k+1}(0;G^*)-$$

$$\Theta(0,G^*) - \delta_k(0;G^*) + \varepsilon\}T(r,F^*) + S(r,F^*), \tag{10}$$

for $r \in I$ and $0 < \varepsilon < \Delta_{1l} - \frac{19}{3}$, that is, $\{\Delta_{1l} - \frac{19}{3} - \varepsilon\}T(r,F^*) \le S(r,F^*)$, i.e., $\Delta_{1l} - \frac{19}{3} \le 0$, i.e.,

$$\Delta_{1l} \le \frac{19}{3},$$

which is a contradiction to the condition of Lemma 2.10.

Therefore, we have $H \equiv 0$, then

$$\frac{(F^*)^{(k+2)}}{(F^*)^{(k+1)}} - \frac{2(F^*)^{(k+1)}}{(F^*)^{(k)}-1} \equiv \frac{(G^*)^{(k+2)}}{(G^*)^{(k+1)}} - \frac{2(G^*)^{(k+1)}}{(G^*)^{(k)}-1}. \tag{11}$$

From this equation we get

$$(G^*)^{(k)} = \frac{(b+1)(F^*)^{(k)} + (a-b-1)}{b(F^*)^{(k)} + (a-b)}, \tag{12}$$

where a ($\neq 0$), b are two constants. Then by the same argument of Lemma 4 in [3], we can deduce that $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$ or $(F^*)^{(k)} \equiv (G^*)^{(k)}$.

Suppose that $(F^*)^{(k)} \equiv (G^*)^{(k)}$. Thus, we obtain

$$F^* = G^* + p(z),$$

where $p(z)$ is a polynomial, then $T(r,F^*) = T(r,G^*) + S(r,F^*)$. If $p(z) \not\equiv 0$, then by Lemma 2.3, we have

$$\begin{aligned} T(r,F^*) &\le \overline{N}(r,0;F^*) + \overline{N}(r,p;F^*) + S(r,F^*) \\ &\le \overline{N}(r,0;F^*) + \overline{N}(r,0;G^*) + S(r,F^*). \end{aligned} \tag{13}$$

Hence, by the condition of this lemma we deduce easily that $T(r,F^*) \le S(r,F^*)$, $r \in I$, a contradiction. Therefore, we deduce that $p(z) \equiv 0$, that is, $F^* \equiv G^*$.

Thus we complete the proof of Lemma 2.10. □

Lemma 2.11 *Let F, G be two nonconstant entire functions such that $\overline{E}_{l)}(1;(F^*)^{(k)}) = \overline{E}_{l)}(1;(G^*)^{(k)})$, $E_{2)}(1;(F^*)^{(k)}) = E_{2)}(1;(G^*)^{(k)})$ and $H^* \not\equiv 0$, where $l \ge 4$. Then*

$$\begin{aligned} T(r,F^*) + T(r,G^*) \le& 2N_{k+1}(r,0;F^*) + 2\overline{N}(r,0;F^*) + 2N_k(r,0;F^*) + 2N_{k+1}(r,0;G^*) + \\ & 2\overline{N}(r,0;G^*) + 2N_k(r,0;G^*) + S(r,F^*) + S(r,G^*), \end{aligned}$$

where H^* is defined as Lemma 2.9.

Proof Let $F = (F^*)^{(k)}$ and $G = (G^*)^{(k)}$. Then $\overline{E}_{l)}(1;F) = \overline{E}_{l)}(1;G)$, $E_{2)}(1;F) = E_{2)}(1;G)$. Since $H^* \not\equiv 0$, by Lemma 2.2 and 2.6 we get

$$\begin{aligned} & T(r,F^*) + T(r,G^*) \\ & \le N_{k+1}(r,0;F^*) + N_{k+1}(r,0;G^*) + \overline{N}(r,1;(F^*)^{(k)}) + \overline{N}(r,1;(G^*)^{(k)}) - \\ & \quad N_0(r,0;(F^*)^{(k+1)}) - N_0(r,0;(G^*)^{(k+1)}) + S(r,F^*) + S(r,G^*) \\ & \le N_{k+1}(r,0;F^*) + N_{k+1}(r,0;G^*) + N(r,1;(F^*)^{(k)}| = 1) + \overline{N}(r,1;(F^*)^{(k)}| \ge 2) + \\ & \quad \overline{N}(r,1;(G^*)^{(k)}) - N_0(r,0;(F^*)^{(k+1)}) - N_0(r,0;(G^*)^{(k+1)}) + S(r,F^*) + S(r,G^*) \end{aligned}$$

$$\leq N_{k+1}(r,0;F^*)+N_{k+1}(r,0;G^*)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+$$
$$\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+\overline{N}(r,1;G)+$$
$$\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}(r,1;F|\geq 2)+S(r,F^*)+S(r,G^*).$$

Since $\overline{N}(r,1;F|=l;G|=l-1)+\cdots+\overline{N}(r,1;F|=l;G|=3)\leq\overline{N}(r,1;F|=l)$ and $\overline{N}(r,1;G|=l;F|=l-1)+\cdots+\overline{N}(r,1;G|=l;F|=3)\leq\overline{N}(r,1;G|=l)$, it is easy to see that $\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}(r,1;F|\geq 2)+\overline{N}(r,1;(G)\leq \frac{1}{2}[N(r,1;F)+N(r,1;G)]\leq\frac{1}{2}[T(r,F)+T(r,G)]$.

Since

$$T(r,F)=T(r,(F^*)^{(k)})\leq T(r,F^*)+S(r,F^*);\quad T(r,G)=T(r,(G^*)^{(k)})\leq T(r,G^*)+S(r,G^*).$$

Then by Lemma 2.4, we can get

$$\begin{aligned}T(r,F^*)+T(r,G^*)\leq& 2N_{k+1}(r,0;F^*)+2\overline{N}(r,0;F^*)+2N_k(r,0;F^*)+2N_{k+1}(r,0;G^*)+\\&2\overline{N}(r,0;G^*)+2N_k(r,0;G^*)+S(r,F^*)+S(r,G^*).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.12 *Let F^*, G^* be two transcendental entire functions and $\overline{E}_{l)}(1;(F^*)^{(k)})=\overline{E}_{l)}(1;(G^*)^{(k)})$, $E_{2)}(1;(F^*)^{(k)})=E_{2)}(1;(G^*)^{(k)})$, where $l\geq 4$. If*

$$\Theta_{F^*}(0)>\frac{5}{2},\quad \Theta_{G^*}(0)>\frac{5}{2},$$

where $\Theta_f(0)=\Theta(0;f)+\delta_k(0;f)+\delta_{k+1}(0;f)$, then $(F^*)^{(k)}(G^*)^{(k)}\equiv 1$ or $F^*\equiv G^*$.

Proof We omit the proof since the proof can be carried out in the line of proof of Lemma 2.10 by using the Lemma 2.11. This completes the proof of the lemma. □

Proof of Theorem 1.1 (i) $P(z)=a_mz^m+a_{m-1}z^{m-1}+\cdots+a_1z+a_0$. By the assumptions of Theorem 2.1 and Lemma 5 in [10], we know that f and g are transcendental entire functions.

Let $F=f^nP(f)$ and $G=g^nP(g)$. From the condition of Theorem 2.1, we have $\overline{E}_{l)}(1;F^{(k)})=\overline{E}_{l)}(1;G^{(k)})$ and $E_{1)}(1;F^{(k)})=E_{1)}(1;G^{(k)})$.

By Lemma 2.1 we can get easily

$$\begin{aligned}\Theta(0,F)&=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,0;F)}{T(r,F)}=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,0;f^nP(f))}{(n+m)T(r,f)}\\&=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,0;f)+\overline{N}(r,0;P(f))}{(n+m)T(r,f)},\end{aligned}$$

i.e.,

$$\Theta(0,F)\geq 1-\frac{m+1}{n+m}=\frac{n-1}{n+m}. \tag{14}$$

Similarly, we have

$$\Theta(0,G)\geq\frac{n-1}{n+m}. \tag{15}$$

Next, by the definition of $N_k(r,a;f)$ we have

$$\delta_k(0,F)=1-\overline{\lim_{r\to\infty}}\frac{N_k(r,0;f^nP(f))}{T(r,F)}.$$

Therefore

$$\delta_k(0,F)\ge 1-\overline{\lim_{r\to\infty}}\frac{(m+k)T(r,f)}{(n+m)T(r,f)}=\frac{n-k}{n+m}. \tag{16}$$

Similarly we get

$$\delta_k(0,G)\ge\frac{n-k}{n+m} \tag{17}$$

and

$$\delta_{k+1}(0,F)\ge\frac{n-k-1}{n+m},\quad \delta_{k+1}(0,G)\ge\frac{n-k-1}{n+m}. \tag{18}$$

From (14)–(18), we can get

$$\begin{aligned}\Delta_{1l}&=\delta_{k+1}(0;F)+\frac{5}{3}\Theta(0,F)+\frac{5}{3}\delta_k(0;F)+\delta_{k+1}(0;G)+\Theta(0,G)+\delta_k(0;G)\\&\ge\frac{n-k-1}{n+m}+\frac{5}{3}\frac{2n-k-1}{n+m}+\frac{3n-2k-2}{n+m}.\end{aligned}$$

By $n>\frac{19}{3}m+\frac{14}{3}(k+1)$, we have

$$\Delta_{1l}=\delta_{k+1}(0;F)+\frac{5}{3}\Theta(0,F)+\frac{5}{3}\delta_k(0;F)+\delta_{k+1}(0;G)+\Theta(0,G)+\delta_k(0;G)>\frac{19}{3}.$$

Therefore by Lemma 2.10, we deduce either $F^{(k)}\cdot G^{(k)}\equiv 1$ or $F\equiv G$.

If $F^{(k)}\cdot G^{(k)}\equiv 1$, that is

$$[f^n(a_mf^m+a_{m-1}f^{m-1}+\cdots+a_0)]^{(k)}[g^n(a_mg^m+a_{m-1}g^{m-1}+\cdots+a_0)]^{(k)}\equiv 1, \tag{19}$$

then by the assumptions of Theorem 1.1 and Proposition 1 in [10] we can get a contradiction. Hence, we deduce that $F\equiv G$, that is

$$f^n(a_mf^m+a_{m-1}f^{m-1}+\cdots+a_0)=g^n(a_mg^m+a_{m-1}g^{m-1}+\cdots+a_0). \tag{20}$$

Let $h=f/g$. If h is a constant, then substituting $f=gh$ into (20), we deduce

$$a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+\cdots+a_0g^n(h^n-1)=0,$$

which implies $h^d=1$, where $d=(n+m,\dots,n+m-i,\dots,n)$, $a_{m-i}\ne 0$ for some $i=0,1,\dots,m$. Thus $f\equiv tg$ for a constant t such that $t^d=1$, where $d=(n+m,\dots,n+m-i,\dots,n)$, $a_{m-i}\ne 0$ for some $i=0,1,\dots,m$.

If h is not a constant, then we know by (20) that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(\omega_1,\omega_2)=\omega_1^n(a_m\omega_1^m+a_{m-1}\omega_1^{m-1}+\cdots+a_1\omega_1+a_0)-\omega_2^n(a_m\omega_2^m+a_{m-1}\omega_2^{m-1}+\cdots+a_1\omega_2+a_0)$. This proves (i) of Theorem 1.1.

(ii) $P(z)\equiv c_0$. From Theorem A, we can easily see that the case (ii) of Theorem 1.1 holds.

Thus, we complete the proof of Theorem 1.1. □

Proof of Theorem 1.2 By the conditions of Theorem 1.2 and Lemma 2.12, using the same argument as in Theorem 1.1, one can easily prove Theorem 1.2.

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