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Shared Sets and Normal Families

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Abstract This paper deals with the problem of normal families concerning share sets. Moreover, the examples show that the conditions of theorem are necessary.

Keywords meromorphic function; normal family; shared set.

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1. Introduction and main results

In this paper, it is assumed that the reader is familiar with the notations of Nevanlinna theory of meromorphic functions, for instance,

$$T(r,f), N(r,f), m(r,f), \overline{N}(r,f), \ldots .$$

We denote by $S(r,f)$ any function satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to+\infty$, possibly outside of a set with finite measure in $\mathbf{R}$.

Let D be a domain in C, and let $\mathfrak{F}$ be a family of meromorphic functions defined in D. The family $\mathfrak{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\}\subset\mathfrak{F}$ contains a subsequence $\{f_{n_j}\}$ that converges, spherically locally uniformly in D, to a meromorphic function or ∞ ([2, 6, 8]).

Now let $\mathfrak{F}$ be a family of meromorphic functions on D. Schwick proved in [7] that if there exist three distinct finite values a_1, a_2, $a_3\in C$ such that $f(z)$ and $f'(z)$ share a_j $(j=1,2,3)$ for each $f(z)\in\mathfrak{F}$, then $\mathfrak{F}$ is normal in D. The corresponding statement in which $f(z)$ and $f'(z)$ share two distinct finite values a_1, $a_2\in C$ remains valid, as is shown by Pang and Zalcman [4]. Chang, Fang and Zalcman [10] gave a simplified proof of a result of Pang and Zalcman.

Recently, Zhang and Qin [9] have proved the following theorem.

Theorem A *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain D, and let a, b and c be three distinct finite complex numbers. If, for every $f(z)\in\mathfrak{F}$, $f(z)=a$ whenever $f'(z)=a$, $f(z)=b$ whenever $f'(z)=b$, $f'(z)=c$ whenever $f(z)=c$, then $\mathfrak{F}$ is normal in D.*

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On the other hand, Fang [1] extended Schwick's result in the view of shared sets. Actually, he proved the following theorem.

Theorem B *Let $\mathfrak{F}$ be a family of holomorphic functions in a domain D, and let a_1, a_2, and a_3 be three distinct finite complex numbers. If, for every $f(z) \in \mathfrak{F}$, $f(z)$ and $f'(z)$ share the set $S = \{a_1, a_2, a_3\}$, then $\mathfrak{F}$ is normal in D.*

Very recently, generalizing Theorem A from families of holomorphic functions to families of meromorphic functions, Liu and Pang [3] obtained the following result.

Theorem C *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2, and a_3 be three distinct finite complex numbers. If, for every $f(z) \in \mathfrak{F}$, $f(z)$ and $f'(z)$ share the set $S = \{a_1, a_2, a_3\}$, then $\mathfrak{F}$ is normal in D.*

It is natural to ask what can be stated if $f'(z)$ is replaced by $f^{(k)}(z)$ in Theorem C. In this paper, we prove the following theorem.

Theorem 1 *Let $\mathfrak{F}$ be a family of meromorphic functions in D, k be a positive integer, M be a real constant, a, b and c be three distinct finite complex numbers and $S = \{a,\ b\}$. If, for every $f(z) \in \mathfrak{F}$, all zeros of $f - c$ are of multiplicity at least k, $f(z) \in S$ whenever $f^{(k)}(z) \in S$, and $|f^{(k)}(z)| \le M$ whenever $f(z) = c$, then $\mathfrak{F}$ is normal in D.*

As immediate consequences of Theorem 1, we have the following sharp results.

Corollary 1 *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain D, a, b and c be three distinct finite complex numbers and $S = \{a, b\}$. If, for every $f(z) \in \mathfrak{F}$, $f(z) \in S$ whenever $f'(z) \in S$, and $f'(z) = c$ whenever $f(z) = c$, then $\mathfrak{F}$ is normal in D.*

Corollary 2 *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain D, a and b be two nonzero distinct finite complex numbers and $S = \{a, b\}$. If, for every $f(z) \in \mathfrak{F}$, $f(z) \in S$ whenever $f'(z) \in S$, and all zeros of $f(z)$ are of multiplicity at least 2, then $\mathfrak{F}$ is normal in D.*

Remark Notice that the conditions $f(z) = a$ whenever $f'(z) = a$, and $f(z) = b$ whenever $f'(z) = b$ imply $f(z) \in S$ whenever $f'(z) \in S$. Therefore, our results improve the Theorem A.

The examples below show that the conditions of Theorem 1 are necessary, and show that the Corollaries 1 and 2 are sharp.

Example 1 Let $S = \{1, -1\}$, and let k be a positive odd number. Set

$$\mathfrak{F} = \{f_n(z) : n = 2, 3, 4, \ldots\},$$

where

$$f_n(z) = \frac{n^k+1}{2n^k}e^{nz} + \frac{n^k-1}{2n^k}e^{-nz}, \quad D = \{z : |z| < 1\}.$$

Then, for any $f_n \in \mathfrak{F}$, we have

$$f_n^{(k)}(z) = \frac{n^k+1}{2}e^{nz} - \frac{n^k-1}{2}e^{-nz}, \quad f_n^{(k+1)}(z) = n^{k+1}[\frac{n^k+1}{2n^k}e^{nz} + \frac{n^k-1}{2n^k}e^{-nz}],$$

and so $n^{2k}[f_n^2(z) - 1] = [f_n^{(k)}(z)]^2 - 1$. Thus f_n and $f_n^{(k)}$ share the set $S = \{1, -1\}$, but $\mathfrak{F}$ is not normal in D, which implies that the condition "$|f^{(k)}(z)| \leq M$ whenever $f(z) = c$" in Theorem 1 is necessary.

Example 2 Set

$$\mathfrak{F} = \{\frac{n + (nz-1)^2}{n(nz-1)} + 2, n = 2, 3, 4, \ldots\}, D = \{z : |z| < 1\}.$$

Then for every $f(z) \in \mathfrak{F}$,$f'(z) = \frac{(nz-1)^2 - n}{(nz-1)^2}$.

Notice that $f(z)$ and $f'(z)$ share value 2, and $f'(z) \neq 1$. But $\mathfrak{F}$ is not normal in D, which shows that Theorem 1 is not valid when $a = b$.

The following example shows that the condition "all zeros of $f - c$ are of multiplicity at least k" in Theorem 1 is necessary.

Example 3 Let $k \geq 2$ be a positive integer such that $\lambda_i^k = 1$, where $i = 1, 2$, and $\lambda_1 \neq \lambda_2$. Set $\mathfrak{F} = \{f_n(z) : n = 1, 2, 3, \ldots\}$, where

$$f_n(z) = n(e^{\lambda_1 z} - e^{\lambda_2 z}), \quad D = \{z : |z| < 1\}.$$

Then, for any $f_n \in \mathfrak{F}$, we have

$$f_n(z) = f_n^{(k)}(z), \quad f_n^{(k+1)}(z) = n(\lambda_1 e^{\lambda_1 z} - \lambda_2 e^{\lambda_2 z}).$$

However, $\mathfrak{F}$ is not normal in D.

2. Some lemmas

Lemma 1 ([5]) *Let $\mathfrak{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathfrak{F}$. Then if $\mathfrak{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) a number $0 < r < 1$,

(b) points z_n, $|z_n| < r$,

(c) functions $f_n \in \mathfrak{F}$, and

(d) positive numbers $\rho_n \to 0$

such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^\alpha} = g_n(\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C, all of whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$. Moreover, g has order at most two. In particular, if $\mathfrak{F}$ is a family of holomorphic functions, then g has order at most one.

Here, as usual, $g^\sharp(z) = |g'(z)|/(1 + |g(z)|^2)$ is the spherical derivative.

Lemma 2 *Let f be a meromorphic function, and a, b be two finite distinct complex numbers. Suppose that $f^{(k)} \neq a, b$. Then $f^{(k)}$ is a constant.*

Proof Using the Nevanlinna's second fundamental theorem for $f^{(k)}$, we obtain

$$T(r, f^{(k)}) \le \overline{N}(r, f) + \overline{N}(r, \frac{1}{f^{(k)} - a}) + \overline{N}(r, \frac{1}{f^{(k)} - b}) + S(r, f^{(k)}).$$

Notice

$$T(r, f^{(k)}) > N(r, f^{(k)}) > (k+1)\overline{N}(r, f),$$

so

$$\overline{N}(r, f) < \frac{1}{k}\overline{N}(r, \frac{1}{f^{(k)} - a}) + \frac{1}{k}\overline{N}(r, \frac{1}{f^{(k)} - b}) + S(r, f^{(k)}).$$

Thus, we get

$$T(r, f^{(k)}) \le (1 + \frac{1}{k})\overline{N}(r, \frac{1}{f^{(k)} - a}) + (1 + \frac{1}{k})\overline{N}(r, \frac{1}{f^{(k)} - b}) + S(r, f^{(k)}).$$

By the condition, we obtain

$$T(r, f^{(k)}) \le S(r, f^{(k)}).$$

Thus the proof is completed. □

3. Proof of Theorem 1

We may assume that $D = \Delta$, the unit disc. Suppose that $\mathfrak{F}$ is not normal on Δ. Then by Lemma 1 we can find $f_n \in \mathfrak{F}$, $z_n \in \Delta$, and $\rho_n \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta) - c}{\rho_n^k} \Rightarrow g(\zeta),$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C, all of whose zeros have multiplicity at least k, such that $g^\sharp(\zeta) \le g^\sharp(0) = k(M + 1) + 1$.

To begin with, we claim that

(i) $|g^{(k)}(\zeta)| \le M$ whenever $g(\zeta) = 0$.

Suppose now that $g(\zeta_0) = 0$. Clearly, $g(\zeta) \not\equiv 0$. Then by Hurwitz's theorem there exist ζ_n, $\zeta_n \to \zeta_0$, and for n sufficiently large, such that

$$0 = g(\zeta_0) = g_n(\zeta_n) = \frac{f_n(z_n + \rho_n\zeta_n) - c}{\rho_n^k},$$

then $f_n(z_n + \rho_n\zeta_n) = c$. By $|f^{(k)}(z)| \le M$ whenever $f(z) = c$, we have

$$|g^{(k)}(\zeta_0)| = \lim_{n\to\infty} |g_n^{(k)}(\zeta_n)| = \lim_{n\to\infty} |f_n^{(k)}(z_n + \rho_n\zeta_n)| \le M.$$

(ii) $g^{(k)}(\zeta) \ne a, b$.

Suppose now that $g^{(k)}(\zeta_0) = a$. Clearly, $g^{(k)}(\zeta) \not\equiv a$. Indeed, if $g^{(k)}(\zeta) \equiv a = 0$, then g would be a polynomial of degree at most $k - 1$ and so could not have zeros of multiplicity at least k. If $g^{(k)}(\zeta) \equiv a \ne 0$, then $g(\zeta)$ would be a polynomial of exact degree k, and $g(\zeta)$ has zeros. By Claim 1 there is a contradiction since a, c are two distinct complex numbers. Then by Hurwitz's theorem there exist ζ_n, $\zeta_n \to \zeta_0$, such that, for n sufficiently large,

$$a = g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n\zeta_n).$$

Because $f \in S$ whenever $f^{(k)} \in S$, we obtain $f_n(z_n + \rho_n \zeta_n) \in S$. It follows that

$$g(\zeta_0) = \lim_{n\to\infty} g_n(\zeta_n) = \lim_{n\to\infty} \frac{f_n(z_n + \rho_n \zeta) - c}{\rho_n^k} = \infty.$$

This is a contradiction. Therefore, $g^{(k)}(\zeta) \neq a$. Similarly, we also obtain that $g^{(k)}(\zeta) \neq b$.

By Claim (i), (ii) and Lemma 2, we know that $g(\zeta) = \frac{A}{k!}(\zeta - \zeta_1)^k$, where ζ_1 and $A(|A| \leq M)$ are constants. A simple calculation then shows that

$$|g^\sharp(0)| \leq \begin{cases} \dfrac{k}{2}, & |\zeta_1| \geq 1, \quad (1a) \\ |A|, & |\zeta_1| < 1, \quad (1b) \end{cases}$$

which contradicts $g^\sharp(\zeta) \leq g^\sharp(0) = k(|M|+1)+1$. This completes the proof of Theorem 1. □

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