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Iterative Algorithms of Common Solutions for Quasi-Variational Inclusion and Fixed Point Problems

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Abstract By introducing the resolvent operator associated with a maximal monotone mapping, the author obtains a strong convergence theorem of a generalized iterative algorithm for a class of quasi-variational inclusion problems, which extends and unifies some recent results.

Keywords variational inclusion; inverse strongly monotone; maximal monotone; viscosity approximation method.

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1. Introduction

Throughout this paper, we assume, H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and C is a nonempty closed convex subset of H. We always denote the fixed points set of $T: H\to H$ by $F(T)=\{x\in H: x=Tx\}$, the natural number set by $\mathbb{N}=\{1,2,\ldots\}$, and the identity mapping by I. In addition, we denote by $\rightharpoonup$ weak convergence and by $\to$ strong convergence.

Now we introduce the so called quasi-variational inclusion problem:

Finding $x\in H$ such that

$$A(x)+M(x)\ni\theta,\ \text{the zero element in } H. \tag{1.1}$$

Here, $A: H\to H$ is a single-valued nonlinear mapping, and $M: H\to 2^H$ is a multi-valued mapping. Denote by $\mathrm{VI}(H,A,M)$ the set of solutions for the variational inclusion (1.1). From the references [1–3], we learn, studying this kind of variational inclusions is helpful to solve many problems arising in structural analysis, mechanics, economics, and so on. Some special cases of the quasi-variational inclusion problem (1.1) were studied in the papers [1–12].

In the case that $M=\partial\phi: H\to 2^H$ is the sub-differential of $\phi: H\to(-\infty,+\infty]$, a proper convex lower semi-continuous function, the variational inclusion problem (1.1) is equivalent to finding $x\in H$ such that

$$\langle Ax, y-x\rangle+\phi(y)-\phi(x)\geq 0,\quad \forall\, y\in H, \tag{1.2}$$

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which is called the mixed quasi-variational inequality [4].

If $M = \partial\delta_C$ is the sub-differential of $\delta_C : H \to [0, \infty]$, the indicator function of C:

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then the variational inclusion problem (1.1) is equivalent to finding $x \in C$ such that

$$\langle A(x), y - x\rangle \geq 0, \quad \forall\, y \in C, \tag{1.3}$$

which is called Hartman-Stampacchia variational inequality problem [1, 5, 6]. To find a common element of $F(S) \cap \mathrm{VI}(C, A)$, i.e., a common element of the fixed points set of nonexpansive mapping $S : C \to C$ and the set of solutions for variational inequality (1.3), Takahashi-Toyoda [7] introduced the following iterative scheme in 2003:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n x_n), \quad \forall\, n \geq 0, \tag{1.4}$$

and proved that $\{x_n\}$ generated by (1.4) converges weakly to an element of $F(S) \cap \mathrm{VI}(C, A)$, where P_C is the metric projection of H onto C.

In 2005, Iiduka-Takahashi [8] introduced again the following iteration:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) SP_C(x_n - \lambda_n x_n), \quad \forall\, n \geq 0, \tag{1.5}$$

and proved that $\{x_n\}$ generated by (1.4) converges strongly to the element $P_{F(S)\cap\mathrm{VI}(C,A)} u \in F(S) \cap \mathrm{VI}(C, A)$. Since then, many of authors are interested in the problems. For example, Chen [17] extended the iteration (1.5) from u to $f(x_n)$.

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) SP_C(x_n - \lambda_n x_n), \quad \forall\, n \geq 0. \tag{1.6}$$

Chen [17] proved that $\{x_n\}$ generated by (1.6) converges strongly to a common element of the fixed points set of a nonexpansive mapping and the set of solutions for a variational inequality. In 2006, Nadezhkina and Takahashi [10] studied the following composite iteration:

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A y_n), \\ \quad y_n = P_C(x_n - \lambda_n A x_n), \end{cases} \qquad \forall\, n \geq 0.$$

In [10] they introduced the so called extragradient method motivated by the idea of Korpelevich [11]. In 2006, Zeng [12] also studied the similar problem by way of the extragradient method. In 2008, Zhang [9] introduced the following composite iteration:

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) S y_n, \\ \quad y_n = J_{M,\lambda}(x_n - \lambda A x_n), \end{cases} \qquad \forall\, n \geq 0, \tag{1.7}$$

and proved that $\{x_n\}$ generated by (1.7) converges strongly to $P_{F(S)\cap\mathrm{VI}(H,A,M)} u \in F(S) \cap \mathrm{VI}(H, A, M)$, where $\mathrm{VI}(H, A, M)$ is the set of solutions for variational inclusion (1.1), $J_{M,\lambda}$ is a resolvent operator associated with M. Some special cases of the iteration (1.7) were studied in many papers [9].

In this paper, the author introduces the following iteration:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sy_n, \\ \quad y_n = J_{M,\lambda_n}(x_n - \lambda_n Ax_n), \end{cases} \quad \forall\, n \geq 0. \tag{1.8}$$

Particularly, letting $\lambda_n \equiv \lambda$ and $f(x_n) \equiv u$ in (1.8), we see, (1.8) is reduced to (1.7), which implies that the strong convergence results on the iteration (1.8) include those on (1.7). Moreover, it will be proved below that the strong convergence results on the iteration (1.8) extend those reselts on (1.6) from self-mappings to nonself-mappings. By using viscosity approximation methods, we shall extend and improve the main results of [17].

2. Preliminaries

Definition 2.1 *(1) A mapping $T : H \to H$ is called nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, $\forall\, x, y \in H$.*

(2) $f : H \to H$ is said to be a contractive mapping with a contractive constant $\beta \in (0,1)$, if $\|f(x) - f(y)\| \leq \beta\|x - y\|$, $\forall\, x, y \in H$.

Definition 2.2 *A mapping $P_C : H \to C$ is called the nearest point projection (or metric projection) from H to C, if for any given $x \in H$, there exists $P_C x \in C$ with $\|P_C x - x\| = \inf_{y\in C} \|y - x\|$.*

Being the metric projection from H onto C, P_C has the following properties:

(1) P_C is nonexpansive;

(2) P_C is firmly nonexpansive. i.e., $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$, $\forall\, x, y \in H$.

Definition 2.3 *(1) A mapping $A : H \to H$ is called α-inverse-strongly-monotone, if there exists an $\alpha > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, $\forall\, x, y \in H$.*

(2) A multi-valued mapping $M : H \to 2^H$ is called monotone, if for all $x, y \in H$, $u \in Mx$ and $v \in My$ implies that $\langle u - v, x - y\rangle \geq 0$.

(3) A multi-valued mapping $M : H \to 2^H$ is called maximal monotone, if it is monotone and if for any $(x,u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in \mathrm{Graph}(M)$ (the graph of mapping M) implies that $u \in Mx$.

Definition 2.4 *A single-valued mapping $A : H \to H$ is called hemi-continuous, if for any $x, y, z \in H$, the function $t \to \langle A(x + ty), z\rangle$ is continuous as $t \to 0^+$.*

Definition 2.5 ([9]) *Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by*

$$J_{M,\lambda} = (I + \lambda M)^{-1}u, \quad u \in H$$

is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

From [9], we have the following proposition and Lemmas:

Proposition 2.6 *(1) The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and* *nonexpansive for all $\lambda > 0$.*

(2) The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly-monotone.

Proof The conclusion (1) is obvious [18]. On the other hand, one can prove the conclusion (ii) by Definition 2.5 and the maximal monotonicity of M.

Proposition 2.7 *Let $A : H \to H$ be an α-inverse-strongly-monotone mapping. Then*

(1) A is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping;

(2) If λ is any constant in $(0, 2\alpha]$, then the mapping $I - \lambda A$ is nonexpansive, where I is the identity mapping on H.

Proof Two conclusions can be proved by the definition of α-inverse-strongly-monotone mapping and the property of the norm in the setting of Hilbert spaces.

Lemma 2.8 ([13]) *Let E be a real Banach space, E^* the dual space of E, $T : E \to 2^{E^*}$ a maximal monotone mapping, and $P : E \to E^*$ a hemi-continuous bounded monotone mapping with $D(T) = E$. Then the mapping $S = T + P : E \to 2^{E^*}$ is a maximal monotone mapping.*

Lemma 2.9 ([14]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and P_C be the nearest point projection (or metric projection) from H to C. Then for any given $x \in H$ and $y \in C$, we have*

(1) $\langle z - P_C x, x - P_C x\rangle \le 0, \quad \forall\, z \in C$;

(2) $\langle z - y, x - y\rangle \le 0$, for all $z \in C$, then $y = P_C x$.

Lemma 2.10 ([15]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences, satisfying*

$$a_{n+1} \le (1 - \lambda_n)a_n + b_n + c_n, \quad n \ge n_0,$$

where n_0 is some nonnegative integer, $\lambda_n \in [0, 1]$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $a_n \to 0$ $(n \to \infty)$.

Lemma 2.11 ([16]) *Let E^* be the dual space of a real Banach space E, and $J : E \to 2^{E^*}$ be the normalized duality mapping from E into 2^{E^*} given by*

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\| \cdot \|f\| = \|x\|^2 = \|f\|^2\}, \quad x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. Then for any $x, y \in E$, we have

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall\, j(x + y) \in J(x + y).$$

Lemma 2.12 ([14]) *Let H be a real Hilbert space, and a mapping $T : E \to E$ be nonexpansive. Then the mapping $I - T$ is demi-closed at zero, i.e.,*

$$x_n \rightharpoonup x \quad \text{and} \quad x_n - Tx_n \to y \quad \text{implies } y = Ty.$$

From [9], we also have

Lemma 2.13 *(1) $u \in H$ is a solution of variational inclusion (1.1) if and only if $u = J_{M,\lambda}(I -$*

$\lambda A)u$ for all $\lambda > 0$, i.e.,

$$VI(H, A, M) = F(J_{M,\lambda}(I - \lambda A)), \quad \forall\, \lambda > 0.$$

(2) If $\lambda \in (0, 2\alpha]$, then $\mathrm{VI}(H, A, M)$ is a closed convex subset in H.

Proof Indeed, the conclusion (1) can be proved by variational inclusion (1.1) and the definition of $J_{M,\lambda}$. On the other hand, we can show the conclusion (2) by the conclusion (1), for the set of fixed points of every nonexpansive mapping defined on H is closed convex.

3. Main results

For the purpose of proving the forthcoming main results of this paper, we need firstly prove a lemma:

Lemma 3.1 *Let H be a real Hilbert space, $A : H \to H$ be an α-inverse-strongly-monotone mapping, $M : H \to 2^H$ be a maximal monotone mapping, and $S : H \to H$ be a nonexpansive mapping. Suppose that $\lambda \in (0, 2\alpha]$. Then we have the following conclusions:*

(1) The mapping $I - SJ(M, \lambda)(I - \lambda A)$ is monotone.

(2) $\|J_{M,\lambda_{n+1}}(x) - J_{M,\lambda_n}(x)\| \le \frac{|\lambda_{n+1}-\lambda_n|}{a} \cdot \|x - J_{M,\lambda_n}(x)\|$ for all $x \in H$, where the real sequence $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha]$.

Proof For any $x, y \in H$, we get by Propositions 2.6 and 2.7

$$\begin{aligned}
&\langle (I - SJ(M, \lambda)(I - \lambda A))x - (I - SJ(M, \lambda)(I - \lambda A))y, x - y\rangle \\
&\quad \ge \|x - y\|^2 - \|SJ(M, \lambda)(I - \lambda A))x - SJ(M, \lambda)(I - \lambda A))y\| \cdot \|x - y\| \\
&\quad \ge \|x - y\|^2 - \|x - y\|^2 = 0,
\end{aligned}$$

which has proved the conclusion (1).

Next, we begin to prove the conclusion (2), newly given in this paper.

For any given $x \in H$, let

$$u = J_{M,\lambda_{n+1}}(x) \quad \text{and} \quad v = J_{M,\lambda_n}(x).$$

Since $M : H \to H$ is maximal monotone, we have

$$0 \le \langle \frac{x-u}{\lambda_{n+1}} - \frac{x-v}{\lambda_n},\, u - v\rangle \le -\frac{1}{\lambda_{n+1}}\|u - v\|^2 + |\frac{1}{\lambda_{n+1}} - \frac{1}{\lambda_n}| \cdot \|x - v\| \cdot \|u - v\|,$$

which deduces

$$\|J_{M,\lambda_{n+1}}(x) - J_{M,\lambda_n}(x)\| \le \frac{|\lambda_{n+1} - \lambda_n|}{a} \cdot \|x - J_{M,\lambda_n}(x)\|.$$

Now, we study the convergence of the implicit composite iteration

$$\begin{cases} x_n = \alpha_n f(x_n) + (1 - \alpha_n)Sy_n, \\ y_n = J_{M,\lambda}(x_n - \lambda A x_n), \end{cases} \quad \forall\, n \ge 0. \tag{3.1}$$

Theorem 3.2 *Let H be a real Hilbert space, $A : H \to H$ an α-inverse-strongly-monotone mapping, and $M : H \to 2^H$ a maximal monotone mapping. Assume, $S : H \to H$ is a nonexpan-*

sive mapping, and $f : H \to H$ is a contractive mapping with a contractive constant $\beta \in (0, 1)$. Suppose that $\mathrm{VI}(H, A, M)$ is the set of solutions for the variational inclusion (1.1), and the set $F(S) \cap \mathrm{VI}(H, A, M) \neq \emptyset$.

Then we have the following two conclusions:

(1) There exists the unique solution $p \in F(S) \cap \mathrm{VI}(H, A, M)$ for the following variational inequality in $F(S) \cap \mathrm{VI}(H, A, M)$:

$$\langle (f - I)p, x - p\rangle \le 0, \quad \text{for all } x \in F(S) \cap VI(H, A, M). \tag{3.2}$$

(2) If the sequence $\{x_n\}$ is defined by the implicit composite iteration (3.1), then $\{x_n\}$ defined by (3.1) converges strongly to the unique solution p for the variational inequality (3.2) in $F(S) \cap \mathrm{VI}(H, A, M)$, where $\lambda \in (0, 2\alpha]$ and $\{\alpha_n\}$ is a real sequence in $(0, 1)$, satisfying $\lim\limits_{n\to\infty} \alpha_n = 0$.

Proof First, we claim that the sequence $\{x_n\}$ given by the implicit composite iterative algorithm (3.1) is well defined.

Indeed, for each $n \ge 0$, we define a mapping $T_n : H \to H$ by

$$T_n(x) = \alpha_n f(x) + (1 - \alpha_n)S(J_{M,\lambda}(x - \lambda Ax)), \quad \forall\, x \in C.$$

Since both $J_{M,\lambda}$ and $(I - \lambda A)$ are nonexpansive, it is easily known that the mapping $SJ_{M,\lambda}(I - \lambda A)$ is nonexpansive. Hence, T_n is a contractive mapping for each integer $n \ge 0$. Then Banach contractive mapping principle yields a unique fixed point $x_n \in H$ of T_n, satisfying

$$x_n = T_n(x_n) = \alpha_n f(x_n) + (1 - \alpha_n)S(J_{M,\lambda}(x_n - \lambda Ax_n)), \quad \text{for an arbitrarily given } n \ge 0.$$

Secondly, we claim that both $\{x_n\}$ and $\{y_n\}$ are bounded.

Indeed, for any $x \in F(S) \cap VI(H, A, M)$, we know from Lemma 2.13 that $x \in F(S) \cap F(J_{M,\lambda}(I - \lambda A))$. Then we get by (3.1)

$$\|x_n - x\|^2 \le (1 - (1 - \beta)\alpha_n)\|x_n - x\|^2 + \alpha_n\langle f(x) - x, x_n - x\rangle,$$

which deduces

$$\begin{aligned} \|x_n - x\|^2 &\le \frac{1}{1-\beta}\langle f(x) - x, x_n - x\rangle \\ &\le \frac{1}{1-\beta}\|f(x) - x\| \cdot \|x_n - x\|. \end{aligned} \tag{3.3}$$

This implies $\|x_n - x\| \le \frac{1}{1-\beta}\|f(x) - x\|$. Thus, $\{x_n\}$ is bounded, and hence all the sets $\{f(x_n)\}$, $\{(I - \lambda A)x_n\}$, $\{J_{M,\lambda}(I - \lambda A)x_n\}$ (or $\{y_n\}$) and $\{Sy_n\}$ are bounded.

On the other hand, for any $x \in F(S) \cap \mathrm{VI}(H, A, M)$, we can see it by (3.1) and the inverse strong monotonicity of A that

$$\begin{aligned} \|x_n - x\|^2 &\le \alpha_n\|f(x_n) - x\|^2 + (1 - \alpha_n)\|y_n - x\|^2 \\ &\le \alpha_n\|f(x_n) - x\|^2 + (1 - \alpha_n)\|x_n - \lambda Ax_n - (x - \lambda Ax)\|^2 \\ &\le \alpha_n\|f(x_n) - x\|^2 + 1 \cdot (\|x_n - x\|^2 + \lambda(\lambda - 2\alpha)\|Ax_n - Ax\|^2), \end{aligned}$$

which together with $\lim_{n\to\infty}\alpha_n = 0$ deduces

$$\|Ax_n - Ax\|^2 \le \frac{\alpha_n\|f(x_n)-x\|^2}{\lambda(2\alpha-\lambda)} \to 0, \quad \text{as } n\to\infty. \tag{3.4}$$

For any $x \in F(S)\cap \mathrm{VI}(H,A,M)$, it follows from Propositions 2.6, 2.7 and (3.1) that

$$\begin{aligned}
&\|y_n - x\|^2 \\
&= \|J_{M,\lambda}(x_n-\lambda Ax_n) - J_{M,\lambda}(x-\lambda Ax)\|^2 \\
&\le \langle (x_n-\lambda Ax_n)-(x-\lambda Ax), y_n - x\rangle \\
&= \frac{1}{2}(\|(x_n-\lambda Ax_n)-(x-\lambda Ax)\|^2 + \|y_n-x\|^2 - \|(x_n-\lambda Ax_n)-(x-\lambda Ax)-(y_n-x)\|^2) \\
&\le \frac{1}{2}(\|x_n-x\|^2 + \|y_n-x\|^2 - \|x_n-y_n\|^2 - \lambda^2\|Ax_n-Ax\|^2 + 2\lambda\langle x_n-y_n, Ax_n-Ax\rangle).
\end{aligned}$$

This implies

$$\|y_n-x\|^2 \le \|x_n-x\|^2 - \|x_n-y_n\|^2 - \lambda^2\|Ax_n-Ax\|^2 + 2\lambda\langle x_n-y_n, Ax_n-Ax\rangle. \tag{3.5}$$

Thus, we get by (3.1) and (3.5)

$$\begin{aligned}
\|x_n-x\|^2 &\le \alpha_n\|f(x_n)-x\|^2 + (1-\alpha_n)\|y_n-x\|^2 \\
&\le 1\cdot(\|x_n-x\|^2 - \|x_n-y_n\|^2 - \lambda^2\|Ax_n-Ax\|^2 + 2\lambda\langle x_n-y_n, Ax_n-Ax\rangle)+ \\
&\quad \alpha_n\|f(x_n)-x\|^2,
\end{aligned}$$

which together with (3.4) implies

$$\|x_n-y_n\|^2 \le -\lambda^2\|Ax_n-Ax\|^2 + 2\lambda\langle x_n-y_n, Ax_n-Ax\rangle + \alpha_n\|f(x_n)-x\|^2 \to 0. \tag{3.6}$$

Then we get by (3.6) and Proposition 2.7

$$\|Ax_n - Ay_n\| \le \frac{1}{\alpha}\|x_n-y_n\| \to 0.$$

In addition, we get by (3.1)

$$\|x_n - Sy_n\| = \alpha_n\|f(x_n)-Sy_n\| \le \alpha_n(\|f(x_n)\| + \|Sy_n\|) \to 0,$$

which together with (3.6) deduces

$$\|y_n - Sy_n\| \le \|y_n-x_n\| + \|x_n-Sy_n\| \to 0.$$

Thus,

$$\begin{aligned}
\|x_n - Sx_n\| &\le \|x_n-y_n\| + \|y_n-Sy_n\| + \|Sy_n-Sx_n\| \\
&\le 2\|x_n-y_n\| + \|y_n-Sy_n\| \to 0, \quad \text{as } n\to\infty.
\end{aligned} \tag{3.7}$$

By the boundedness of $\{x_n\}$, there exists a weakly convergent subsequence $\{x_{n_i}\}\subset\{x_n\}$ such that $x_{n_i} \rightharpoonup q \in H$ as $i\to\infty$. Then we know by (3.7) and Lemma 2.12 that $q\in F(S)$. In virtue of (3.6) and $x_{n_i}\rightharpoonup q$, we can prove easily that $y_{n_i}\rightharpoonup q\in F(S)$ as $i\to\infty$.

Next, we claim $q\in F(S)\cap\mathrm{VI}(H,A,M)$.

Indeed, since $A: H\to H$ is α-inverse-strongly-monotone, we deduce from Proposition 2.7 that A is a hemi-continuous bounded monotone mapping with $D(A)=H$. It follows from Lemma

2.8 that $M+A$ is maximal monotone. Let $(v,g)\in \mathrm{Graph}(M+A)$, i.e., $g-Av\in M(v)$. Since $y_{n_i}=J_{M,\lambda}(x_{n_i}-\lambda Ax_{n_i})$, we get $x_{n_i}-\lambda Ax_{n_i}\in (I+\lambda M)y_{n_i}$ or

$$\frac{1}{\lambda}\left(x_{n_i}-y_{n_i}-\lambda Ax_{n_i}\right)\in My_{n_i}.$$

Since M is maximal monotone, we get

$$\langle v-y_{n_i}, g-Av-\frac{1}{\lambda}\left(x_{n_i}-y_{n_i}-\lambda Ax_{n_i}\right)\rangle \geq 0,$$

which deduces

$$\begin{aligned}\langle v-y_{n_i}, g\rangle &\geq \langle v-y_{n_i}, Av+\frac{1}{\lambda}\left(x_{n_i}-y_{n_i}-\lambda Ax_{n_i}\right)\rangle\\ &\geq 0+\langle v-y_{n_i}, Ay_{n_i}-Ax_{n_i}\rangle+\frac{1}{\lambda}\langle v-y_{n_i}, x_{n_i}-y_{n_i}\rangle \to 0\end{aligned}$$

in virtue of $\|Ax_n-Ay_n\|\to 0$ and $\|x_n-y_n\|\to 0$. Letting $i\to\infty$, we have $\langle v-q,g\rangle\geq 0$. Then the maximal monotonicity of $M+A$ yields $\theta\in(M+A)q$, and hence $q\in \mathrm{VI}(H,A,M)$, which has proved that $q\in F(S)\cap \mathrm{VI}(H,A,M)$.

By $x_{n_i}\rightharpoonup q$, and by interchanging x with q in (3.3), we get

$$\|x_{n_i}-q\|^2\leq \frac{1}{1-\beta}\langle f(q)-q, x_{n_i}-q\rangle \to 0, \quad \text{as } i\to\infty,$$

which implies $x_{n_i}\to q\in F(S)\cap \mathrm{VI}(H,A,M)$.

Next, we claim that q solves the variational inequality (3.2), i.e.,

$$\langle (f-I)q, x-q\rangle \leq 0, \quad \text{for all } x\in F(S)\cap \mathrm{VI}(H,A,M). \tag{3.8}$$

Indeed, it follows from (3.1) that

$$x_n-f(x_n)=(1-\alpha_n)(Sy_n-x_n+x_n-f(x_n)).$$

Thus,

$$x_n-f(x_n)=-\frac{1-\alpha_n}{\alpha_n}(x_n-Sy_n).$$

Then, for any $x\in F(S)\cap VI(H,A,M)$, we get by Lemma 2.13 that $x=Sx=(SJ(M,\lambda)(I-\lambda A))x$. Thus, we get by (3.1) and Lemma 3.1

$$\begin{aligned}\langle (I-f)x_n, x_n-x\rangle &= -\frac{1-\alpha_n}{\alpha_n}\langle x_n-Sy_n, x_n-x\rangle\\ &= -\frac{1-\alpha_n}{\alpha_n}\langle (I-SJ(M,\lambda)(I-\lambda A))x_n-(I-SJ(M,\lambda)(I-\lambda A))x, x_n-x\rangle \leq 0.\end{aligned} \tag{3.9}$$

Interchanging x_n with x_{n_i} in (3.9), we can easily prove and obtain (3.8) as a result of $x_{n_i}\to q$. This also implies that $\{x_n\}$ is sequentially compact.

Next, we prove the uniqueness of the solution for the variational inequality (3.2) in $F(S)\cap \mathrm{VI}(H,A,M)$.

Indeed, if there exists another element $p\in F(S)\cap \mathrm{VI}(H,A,M)$ satisfying (3.2), we have

$$\langle (f-I)p, x-p\rangle \leq 0, \quad \text{for all } x\in F(S)\cap \mathrm{VI}(H,A,M). \tag{3.10}$$

Then we get by adding (3.8) and (3.10)

$$(1-\beta)\|p-q\|^2 \leq \langle (I-f)p-(I-f)q, p-q\rangle \leq 0,$$

which implies $p=q$. Hence, below we can denote by p the unique solution for the variational inequality (3.2) in $F(S)\cap \mathrm{VI}(H,A,M)$.

Finally, we have also proved $\lim_{n\to\infty} x_n = p$ since $\{x_n\}$ is sequentially compact and each cluster point of $\{x_n\}$ equals p. This completes the proof of Theorem 3.2. □

Now we are in the position to give the main result of this paper:

Theorem 3.3 *Let H be a real Hilbert space, $A: H\to H$ be an α-inverse-strongly-monotone mapping, $M: H\to 2^H$ a maximal monotone mapping, and $S: H\to H$ a nonexpansive mapping. Assume, $f: H\to H$ is a contractive mapping with a contractive constant $\beta\in(0,1)$, $\mathrm{VI}(H,A,M)$ is the set of solutions for the variational inclusion (1.1), and the set $F(S)\cap\mathrm{VI}(H,A,M)\neq\emptyset$. Suppose that x_0 is an arbitrarily given point in H, and the sequence $\{x_n\}$ is generated by $x_0\in H$ and the composite iteration*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sy_n, \\ \quad y_n = J_{M,\lambda_n}(x_n - \lambda_n Ax_n), \end{cases} \qquad \forall\, n\geq 0, \tag{3.11}$$

where $\{\lambda_n\}\subset[a,b]\subset(0,2\alpha]$, and $\{\alpha_n\}$ is a real sequence in $[0,1]$, satisfying the following conditions: (i) $\alpha_n\to 0$; $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ (or $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n}=1$); $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$.

Then the sequence $\{x_n\}$ generated by (3.11) converges strongly to such an element $p\in F(S)\cap\mathrm{VI}(H,A,M)$ that p is just the unique solution for the variational inequality (3.2) in $F(S)\cap\mathrm{VI}(H,A,M)$.

Proof Firstly, we know from the conclusion (1) of Theorem 3.2 that there exists the unique solution $p\in F(S)\cap\mathrm{VI}(H,A,M)$ for the variational inequality (3.2) in $F(S)\cap\mathrm{VI}(H,A,M)$.

Secondly, we point out that $\{x_n\}$ generated by (3.11) is bounded.

Indeed, for any given $n\geq 0$, we get by (3.11), Lemma 2.13, Proposition 2.6 (1) and Proposition 2.7(2) that

$$\begin{aligned}\|x_{n+1}-p\| &\leq \alpha_n\|f(x_n)-f(p)+f(p)-p\| + (1-\alpha_n)\|Sy_n-p\| \\ &\leq \alpha_n\|f(p)-p\| + (1-(1-\beta)\alpha_n)\|x_n-p\| \\ &\leq \max\{\|x_n-p\|, \frac{1}{1-\beta}\|f(p)-p\|\}.\end{aligned}$$

Mathematical induction method yields

$$\|x_n-p\| \leq \max\{\|x_0-p\|, \frac{1}{1-\beta}\|f(p)-p\|\}, \quad \text{for all } n\geq 0.$$

Hence, $\{x_n\}$ generated by (3.11) is bounded, and so are all the sequences $\{f(x_n)\}$, $\{Ax_n\}$, $\{(I-\lambda_n A)x_n\}$, $\{J_{M,\lambda_n}(I-\lambda_n A)x_n\}$, $\{y_n\}$ and $\{Sy_n\}$. Thus, there exists a constant $M>0$ such that

$$\|Ax_n\|+\|f(x_n)\|+\|(I-\lambda_n A)x_n\|+\|J_{M,\lambda_n}(I-\lambda_n A)x_n\|+\|y_n\|+\|Sy_n\|+\|x_n\|+\|p\| \leq M,\ \forall\, n\geq 0.$$

Next, we claim that

$$\|x_{n+1}-x_n\|\to 0 \quad \text{and} \quad \|y_n-y_{n-1}\|\to 0. \tag{3.12}$$

Indeed, it follows from (3.11), Proposition 2.7 and Lemma 3.1(2) that

$$\begin{aligned}
&\|y_{n+1}-y_n\|\\
&\quad\le \|J_{M,\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}Ax_{n+1})-J_{M,\lambda_n}(x_{n+1}-\lambda_{n+1}Ax_{n+1})\|+\\
&\qquad \|J_{M,\lambda_n}(x_{n+1}-\lambda_{n+1}Ax_{n+1})-J_{M,\lambda_n}(x_n-\lambda_nAx_n)\|\\
&\quad\le \frac{2M}{a}|\lambda_{n+1}-\lambda_n|+\|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|\cdot\|Ax_n\|\\
&\quad\le (\frac{2M}{a}+M)|\lambda_{n+1}-\lambda_n|+\|x_{n+1}-x_n\|, \quad \forall\, n\ge 0.
\end{aligned}$$

Thus, we can get by (3.11) and the equality above

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&\quad=\|(\alpha_n-\alpha_{n-1})(f(x_{n-1})-Sy_{n-1})+(1-\alpha_n)(Sy_n-Sy_{n-1})+\alpha_n(f(x_n)-f(x_{n-1}))\|\\
&\quad\le (1-(1-\beta)\alpha_n)\|x_n-x_{n-1}\|+(\frac{2M}{a}+M)|\lambda_{n-1}-\lambda_n|+2M|\alpha_n-\alpha_{n-1}|, \quad \forall\, n\ge 1.
\end{aligned}$$

Now we know from the conditions (i), (ii) and Lemma 2.10 that $\|x_n-x_{n-1}\|\to 0$ as $n\to\infty$. Hence, $\|y_{n+1}-y_n\|\le(\frac{2M}{a}+M)|\lambda_{n+1}-\lambda_n|+\|x_{n+1}-x_n\|\to 0$, and (3.12) is proved.

Then we get by (3.11) and (3.12)

$$\begin{aligned}
\|x_n-Sy_n\|&\le\|x_n-Sy_{n-1}\|+\|Sy_{n-1}-Sy_n\|\\
&\le\alpha_{n-1}\|f(x_{n-1})-Sy_{n-1}\|+\|y_{n-1}-y_n\|\to 0.
\end{aligned}$$

From (3.11) and the inverse-strong-monotone mapping A, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\
&\le\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-\lambda_nAx_n-(p-\lambda_nAp)\|^2\\
&\le\alpha_n\|f(x_n)-p\|^2+1\cdot(\|x_n-p\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Ap\|^2),
\end{aligned}$$

which implies

$$\begin{aligned}
\|Ax_n-Ap\|^2&\le\frac{\alpha_n\|f(x_n)-p\|^2+(\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|)}{\lambda_n(2\alpha-\lambda_n)}\\
&\le\frac{\alpha_n\|f(x_n)-p\|^2+(\|x_n-x_{n+1}\|)(\|x_n-p\|+\|x_{n+1}-p\|)}{a(2\alpha-b)}\longrightarrow 0.
\end{aligned}$$

Since J_{M,λ_n} is 1-inverse-strongly-monotone, we get by (3.11)

$$\begin{aligned}
&\|y_n-p\|^2\\
&\quad=\|J_{M,\lambda_n}(I-\lambda_nA)x_n-J_{M,\lambda_n}(I-\lambda_nA)p\|^2\\
&\quad\le\langle(I-\lambda_nA)x_n-(I-\lambda_nA)p,y_n-p\rangle\\
&\quad=\frac12(\|(I-\lambda_nA)x_n-(I-\lambda_nA)p\|^2+\|y_n-p\|^2-\|(I-\lambda_nA)x_n-(I-\lambda_nA)p-(y_n-p)\|^2)\\
&\quad\le\frac12(\|x_n-p\|^2+\|y_n-p\|^2-\|x_n-y_n\|^2+2\lambda_n\langle x_n-y_n,Ax_n-Ap\rangle-\lambda_n^2\|Ax_n-Ap\|^2),
\end{aligned}$$

which deduces

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\langle x_n - y_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2.$$

Then we get by (3.11)

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le 1\cdot(\|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\langle x_n - y_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2) + \\
&\quad \alpha_n\|f(x_n) - p\|^2,
\end{aligned}$$

which deduces

$$\begin{aligned}
&\|x_n - y_n\|^2 \\
\le&(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) + 2\lambda_n\langle x_n - y_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2 + \alpha_n\|f(x_n) - p\|^2 \\
&\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2b(\|x_n\| + \|y_n\|)\|Ax_n - Ap\| + \alpha_n(\|f(x_n)\| + \|p\|)^2 \to 0.
\end{aligned}$$

Thus, we get by $\|x_n - Sy_n\| \to 0$ that

$$\|x_n - Sx_n\| \le \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \le \|x_n - Sy_n\| + \|y_n - x_n\| \to 0. \tag{3.13}$$

Since the sequence $\{\langle f(p) - p, x_n - p\rangle\}$ is bounded, $\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle$ exists, and hence there exists a subsequence $\{x_i\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle = \lim_{i\to\infty}\langle f(p) - p, x_i - p\rangle.$$

Then we know from the boundedness of $\{x_i\}$ that there exists a subsequence $\{x_j\} \subset \{x_i\}$ such that $x_j \rightharpoonup w$ as $j \to \infty$. Now we can see it by (3.13) and Lemma 2.12 that $w \in F(S)$.

Next, since $\|x_n - y_n\| \to 0$ has been proved, we can see it by the Lipschitz continuity of the map A that

$$\|Ax_n - Ay_n\| \to 0, \qquad \text{as} \quad n \to \infty.$$

Now, similarly to the proof of Theorem 3.2, we can also prove by Lemma 2.8 that $w \in \mathrm{VI}(H, A, M)$. Hence, $w \in F(S) \cap \mathrm{VI}(H, A, M)$. It follows from the definition of p that

$$\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle = \lim_{i\to\infty}\langle f(p) - p, x_i - p\rangle = \lim_{j\to\infty}\langle f(p) - p, x_j - p\rangle = \langle f(p) - p, w - p\rangle \le 0.$$

Set

$$\gamma_n = \max\{\langle f(p) - p, x_n - p\rangle, 0\}, \quad \forall\, n \ge 0.$$

Then it is easily known that $\gamma_n \ge 0$ and $\lim_{n\to\infty}\gamma_n = 0$.

Finally, we prove $x_n \to p$ as $n \to \infty$.

In fact, we get by (3.11) and Lemma 2.11

$$\|x_{n+1} - p\|^2 \le (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle.$$

In addition,

$$2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle = 2\alpha_n\langle f(x_n) - f(p) + f(p) - p, x_{n+1} - p\rangle$$

$$\leq \alpha_n\beta(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + 2\alpha_n\gamma_{n+1}.$$

Thus, we have

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)^2\|x_n - p\|^2 + \alpha_n\beta(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + 2\alpha_n\gamma_{n+1},$$

$$(1-\alpha_n\beta)\|x_{n+1} - p\|^2 \leq ((1-\alpha_n)^2 + \alpha_n\beta)\|x_n - p\|^2 + 2\alpha_n\gamma_{n+1}.$$

We get by $\lim_{n\to} \alpha_n = 0$ that there exists a positive integer $n_1 > 0$ satisfying

$$2(1-\beta)\alpha_n \in [0,1) \quad \text{and} \quad 1 - \alpha_n\beta \geq \frac{1}{2}, \quad \forall\, n \geq n_1.$$

Hence,

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - \frac{2\alpha_n(1-\beta)}{1-\alpha_n\beta})\|x_n - p\|^2 + \frac{\alpha_n^2}{1-\alpha_n\beta}\|x_n - p\|^2 + \frac{2\alpha_n\gamma_{n+1}}{1-\alpha_n\beta} \\ &\leq (1 - 2(1-\beta)\alpha_n)\|x_n - p\|^2 + 2\alpha_n(M^2\alpha_n + 2\gamma_{n+1}), \quad \forall\, n > n_1.\end{aligned}$$

Now, taking $\lambda_n = 2(1-\beta)\alpha_n$, $a_n = \|x_n - p\|^2$, $b_n = 2\alpha_n(M^2\alpha_n + 2\gamma_{n+1})$, and $c_n = 0$ for all $n \geq n_1$, we can get by Lemma 2.10 that $x_n \to p$ as $n \to \infty$. This completes the proof. □

Particularly in Theorem 3.3, if we set $\lambda_n \equiv \lambda \in (0, 2\alpha]$, then Theorem 3.3 yields the following immediate corollary:

Corollary 3.4 *Let H be a real Hilbert space, $A : H \to H$ an α-inverse-strongly-monotone mapping, and $M : H \to 2^H$ a maximal monotone mapping. Assume, $S : H \to H$ is a nonexpansive mapping, and $f : H \to H$ is a contractive mapping with a contractive constant $\beta \in (0,1)$. Suppose that $VI(H, A, M)$ is the set of solutions for the variational inclusion (1.1), and the set $F(S) \cap VI(H, A, M) \neq \emptyset$. x_0 is an arbitrarily given point in H, and the sequence $\{x_n\}$ is generated by $x_0 \in H$ and the composite iteration*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sy_n, \\ \quad y_n = J_{M,\lambda}(x_n - \lambda Ax_n), \end{cases} \qquad \forall\, n \geq 0,$$

where $\lambda \in (0, 2\alpha]$ and $\{\alpha_n\}$ is a real sequence in $[0,1]$, satisfying the following conditions:

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then the sequence $\{x_n\}$ converges strongly to such an element $p \in F(S) \cap \mathrm{VI}(H, A, M)$ that p is the unique solution for the variational inequality (3.2) in $F(S) \cap \mathrm{VI}(H, A, M)$.

Corollary 3.5 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ an α-inverse-strongly-monotone mapping, $f : C \to C$ a contractive mapping with a contractive constant $\beta \in (0,1)$. Assume, $S : C \to C$ is nonexpansive so that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. x_0 is an arbitrarily given point in C, and the sequence $\{x_n\}$ is generated by $x_0 \in C$ and the composite iteration*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sy_n, \\ \quad y_n = P_C(x_n - \lambda_n Ax_n), \end{cases} \qquad \forall\, n \geq 0,$$

where $\{\lambda_n\} \subset [a,b] \subset (0,2\alpha]$, *and* $\{\alpha_n\}$ *is a real sequence in* $[0,1]$, *satisfying the following conditions:*

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to such an element* $p \in F(S) \cap \mathrm{VI}(C,A)$ *that* p *is the unique solution for the following variational inequality in* $F(S) \cap \mathrm{VI}(C,A)$*:*

$$\langle (f-I)p, x-p\rangle \le 0, \quad \text{for all } x \in F(S) \cap \mathrm{VI}(C,A).$$

Proof In Theorem 3.3, we take $M = \partial\delta_C : H \to 2^H$, where $\delta_C : H \to [0,\infty]$ is the indicator function of C, a nonempty closed convex subset of H. Namely,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Then the variational inclusion problem (1.1) is equivalent to the variational inequality (1.3), i.e., finding $x \in C$ such that

$$\langle A(x), y-x\rangle \ge 0, \quad \forall\, y \in C.$$

The restriction of J_{M,λ_n} on C is an identity mapping $J_{M,\lambda_n}|_C = I$ by virtue of $M = \partial\delta_C$. Thus,

$$y_n = P_C(x_n - \lambda_n A x_n) = J_{M,\lambda_n}(P_C(x_n - \lambda_n A x_n)).$$

Hence, the conclusion of Corollary 3.5 can be obtained from Theorem 3.3 immediately.

Corollary 3.6 *Let* H *be a real Hilbert space,* $A : H \to H$ *an* α*-inverse-strongly-monotone mapping, and* $f : H \to H$ *a contractive mapping with a contractive constant* $\beta \in (0,1)$*. Assume,* $S : H \to H$ *is a nonexpansive mapping such that* $F(S) \cap VI(H,A) \ne \emptyset$. x_0 *is an arbitrarily given point in* H, *and the sequence* $\{x_n\}$ *is generated by* $x_0 \in H$ *and the composite iteration*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n, \\ \quad y_n = x_n - \lambda_n A x_n, \end{cases} \qquad \forall\, n \ge 0,$$

where $\{\lambda_n\} \subset [a,b] \subset (0,2\alpha]$, *and* $\{\alpha_n\}$ *is a real sequence in* $[0,1]$, *satisfying the following conditions:*

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to such an element* $p \in F(S) \cap \mathrm{VI}(H,A)$ *that* p *is the unique solution in* $F(S) \cap \mathrm{VI}(H,A)$ *for the following variational inequality:*

$$\langle (f-I)p, x-p\rangle \le 0, \quad \text{for all } x \in F(S) \cap \mathrm{VI}(H,A).$$

Proof In Theorem 3.3, we may take $M = \partial\delta : H \to 2^H$, where $\delta : H \to [0,\infty]$ is defined by $\delta(x) \equiv 0$ for all $x \in H$. Then the variational inclusion problem (1.1) is equivalent to finding $x \in H$ such that

$$\langle A(x), y-x\rangle \ge 0, \quad \forall\, y \in H. \tag{3.14}$$

Then $J_{M,\lambda_n} = I$ in virtue of $M = \partial\delta$. Thus,

$$y_n = J_{M,\lambda_n}(x_n - \lambda_n A x_n) = x_n - \lambda_n A x_n.$$

Hence, the conclusion of Corollary 3.6 can be obtained from Theorem 3.3 immediately.

Corollary 3.7 *Let H be a real Hilbert space, $T : H \to H$ a k-strictly pseudocontractive mapping, and $f : H \to H$ a contractive mapping with a contractive constant $\beta \in (0,1)$. Assume, $S : H \to H$ is a nonexpansive mapping such that $F(S) \cap F(T) \neq \emptyset$. x_0 is an arbitrarily given point in H, and the sequence $\{x_n\}$ is generated by $x_0 \in H$ and the iteration $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)S((1-\lambda_n)x_n + \lambda_n T x_n)$, i.e.,*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n, \\ \quad y_n = x_n - \lambda_n (I - T) x_n, \end{cases} \qquad \forall\, n \geq 0,$$

where $\{\lambda_n\} \subset [a,b] \subset (0, 2\alpha]$, and $\{\alpha_n\}$ is a real sequence in $[0,1]$, satisfying the following conditions:

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to such an element $p \in F(S) \cap F(T)$ that p is the unique solution in $F(S) \cap F(T)$ for the following variational inequality:

$$\langle (f - I)p, x - p \rangle \leq 0, \quad \text{for all } x \in F(S) \cap F(T).$$

Proof We may take $A = I - T$, then A is $\frac{1-k}{2}$-inverse-strongly monotone. We claim $F(T) = \mathrm{VI}(H, A)$.

Indeed, for any $x \in F(T)$, we know, x must be a solution of the variational inequality (3.14), which implies $x \in \mathrm{VI}(H, A)$, and hence $F(T) \subset \mathrm{VI}(H, A)$.

Next, for any $x \in \mathrm{VI}(H, A)$, then x is just a solution of the variational inequality (3.14), i.e.,

$$\langle (I - T)(x), y - x \rangle \geq 0, \quad \forall\, y \in H. \tag{3.15}$$

Particularly, letting $y = Tx$ in (3.15), we get $\|x - Tx\| \leq 0$, which implies $x \in F(T)$, and hence $VI(H, A) \subset F(T)$. Now we have completed the proof immediately by Corollary 3.6.

Remark (1) Corollary 3.5 is just [17, Proposition 3.1], and [17, Theorem 3.1] can be deduced by Theorem 3.2. Moreover, Corollaries 3.6 and 3.7 extend [17, Proposition 3.1] and [17, Theorem 4.1] from self-maps to nonself-maps. These mappings involved include nonexpansive mapping, contractive mapping and k-strictly pseudocontractive mapping.

(2) Taking $f(x_n) \equiv u$ in Corollary 3.4, we can obtain [9, Theorem 2.1] in view of Lemma 2.9 and Lemma 2.12, for $p \in F(S) \cap VI(H, A, M)$ solving the variational inequality (3.2) is just $P_{F(S)\cap \mathrm{VI}(H,A,M)} u$.

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