Journal of Mathematical Research & Exposition
Jul., 2010, Vol. 30, No. 4, pp. 734–742
DOI:10.3770/j.issn:1000-341X.2010.04.018
Http://jmre.dlut.edu.cn

Quasi-Armendariz Modules

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Abstract For a right R-module N, we introduce the quasi-Armendariz modules which are a common generalization of the Armendariz modules and the quasi-Armendariz rings, and investigate their properties. Moreover, we prove that N_R is quasi-Armendariz if and only if $M_m(N)_{M_m(R)}$ is quasi-Armendariz if and only if $T_m(N)_{T_m(R)}$ is quasi-Armendariz, where $M_m(N)$ and $T_m(N)$ denote the $m\times m$ full matrix and the $m\times m$ upper triangular matrix over N, respectively. N_R is quasi-Armendariz if and only if $N[x]_{R[x]}$ is quasi-Armendariz. It is shown that every quasi-Baer module is quasi-Armendariz module.

Keywords Armendariz modules; quasi-Armendariz rings; quasi-Armendariz modules; quasi-Baer modules.

Document code A
MR(2000) Subject Classification 16S36; 16N60; 16P60
Chinese Library Classification O153.3

1. Introduction

Throughout this paper R denotes an associative ring with identity, N denotes a right R-module and $r_R(X)$ denotes the annihilator of the subset X of N in R. Let $R[x]$ and $N[x]$ be the polynomials over R and N, respectively. Rege and Chhawchharia [1] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x)=a_0+a_1x+\cdots+a_px^p$, $g(x)=b_0+b_1x+\cdots+b_qx^q\in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j=0$ for each i and j. The name "Armendariz ring" was chosen because Armendariz [2] had noted that a reduced ring satisfies this condition. Some properties, examples and counterexamples of the Armendariz rings were given in Rege and Chhawchharia [1], Armendariz [2], Anderson and Camillo [3], Huh et al. [4], and Kim and Lee [5]. Following Anderson and Camillo [3], N_R is called Armendariz if, whenever $n(x)g(x)=0$ where $n(x)=n_0+n_1x+\cdots+n_px^p\in N[x]$ and $g(x)=b_0+b_1x+\cdots+b_qx^q\in R[x]$, then $n_ib_j=0$ for all i and j. In [6], Lee and Zhou studied some properties of this module. According to Hirano [7], a ring R is called quasi-Armendariz if whenever polynomials $f(x)=a_0+a_1x+\cdots+a_px^p$, $g(x)=b_0+b_1x+\cdots+b_qx^q\in R[x]$ satisfy

Received June 12, 2008; Accepted October 8, 2008
Supported by the National Natural Science Foundation of China (Grant No. 10571026), the Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20060286006) and Science Foundation for Youth Scholars of Northwest Normal University (Grant No. NWNU-LKQN-08-1).
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$f(x)R[x]g(x) = 0$, then $a_iRb_j = 0$ for each i and j. He showed that the class of quasi-Armendariz rings is Morita invariant, and if R is a quasi-Armendariz ring, then $R[x]$ is quasi-Armendariz.

Motivated by the results above, in this paper, we introduce the notion of quasi-Armendariz modules which are a generalization of Armendariz modules and quasi-Armendariz rings. We give the equivalent characterizations of the quasi-Armendariz rings and study the relations between the set of annihilators of the submodules of N in R and the set of annihilators of the submodules of $N[x]$ in $R[x]$. We also show that N_R is quasi-Armendariz if and only if $M_m(N)_{M_m(R)}$ is quasi-Armendariz if and only if $T_m(N)_{T_m(R)}$ is quasi-Armendariz. Furthermore, it is showed that the polynomial modules over the quasi-Armendariz modules are quasi-Armendariz and every quasi-Baer module is quasi-Armendariz.

2. Main Results

We start with the following:

Definition 2.1 *Let N be a right R-module. N is said to be a quasi-Armendariz module if whenever $n(x) = n_0 + n_1x + \cdots + n_px^p \in N[x]$, $g(x) = b_0 + b_1x + \cdots + b_qx^q \in R[x]$ satisfy $n(x)R[x]g(x) = 0$, then $n_iRb_j = 0$ for all i and j.*

Example 2.2 Several simple examples of quasi-Armendariz modules can be given:

(1) R is a quasi-Armendariz ring if and only if R_R is a quasi-Armendariz module.

(2) Every submodule of a quasi-Armendariz module is quasi-Armendariz. In particular, if I is a right ideal of a quasi-Armendariz ring, then I_R is a quasi-Armendariz module.

(3) Every direct sum and direct product of quasi-Armendariz modules are quasi-Armendariz.

(4) If N_t is a quasi-Armendariz R_t-module for each $t \in \Gamma$, then $\prod_t N_t$ is a quasi-Armendariz $\prod_t R_t$-module.

An R-module N is torsionless if it is a submodule of a direct product of copies of R. If N is a faithful right R-module, then R is a submodule of a direct product of copies of N. We can obtain the following result easily.

Theorem 2.3 *Let R be a ring. The following statements are equivalent:*

(1) R is quasi-Armendariz;

(2) Every projective right R-module is quasi-Armendariz;

(3) Every finitely generated projective right R-module is quasi-Armendariz;

(4) Every cyclic projective right R-module is quasi-Armendariz;

(5) Every torsionless right R-module is quasi-Armendariz;

(6) There exists a faithful right R-module which is quasi-Armendariz.

Lemma 2.4 *Let $n(x) \in N[x]$ and $f(x) \in R[x]$. Then $n(x)Rf(x) = 0$ if and only if $n(x)R[x]f(x) = 0$.*

In the following, we use Lemma 2.4 freely without any mention.

Proposition 2.5 *A right R-module N is quasi-Armendariz if and only if every finitely generated* *submodule of N is quasi-Armendariz.*

Let $\theta : R \longrightarrow A$ be a ring homomorphism and let N be a right A-module. Regard N as a right R-module via θ. Buhphand and Rege [8, Proposition 2.5] showed that if N_A is Armendariz, then N_R is Armendariz. However, if N_A is quasi-Armendariz, N_R need not be quasi-Armendariz. For example, take S and $M_2(R)$ in Example 2.14. Let $\theta : S \longrightarrow M_2(R)$ be the inclusion homomorphism. By Example 2.14, $M_2(R)$ is a quasi-Armendariz right $M_2(R)$-module but not quasi-Armendariz right S-module.

Proposition 2.6 *Let $\theta : R \longrightarrow A$ be an onto ring homomorphism. Then N is a quasi-Armendariz A-module if and only if N is a quasi-Armendariz R-module.*

Proof Suppose that N is a quasi-Armendariz A-module. Let $n(x) = \sum_{i=0}^{p} n_i x^i \in N[x]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x]$ such that $n(x)R[x]g(x) = 0$. One can obtain that $n(x)\theta(R)\theta(g(x)) = 0$, where $\theta(g(x)) = \sum_{j=0}^{q} \theta(b_j)x^j \in A[x]$. Since θ is onto, we have $n(x)A\theta(g(x)) = 0$. By hypothesis, $n_i A\theta(b_j) = 0$ for all i and j. Thus, $n_i R b_j = n_i\theta(Rb_j) = n_i A\theta(b_j) = 0$ for all i and j. Therefore N is a quasi-Armendariz R-module. The proof of the converse is similar to that above. □

Corollary 2.7 *Let $\theta : R \longrightarrow A$ be an onto ring homomorphism. Then A is a quasi-Armendariz ring if and only if A is a quasi-Armendariz R-module.*

Remark 2.8 If N is a right R-module, $\overline{R}$ denotes the ring $R/r_R(N)$ and $E(N) = \mathrm{End}_R(N)$ denotes the ring of endomorphisms of N. With those notations, we consider the following conditions.

(1) The right R-module N is quasi-Armendariz.

(2) The right $\overline{R}$-module N is quasi-Armendariz.

(3) $\overline{R}$ is a quasi-Armendariz ring.

An application of Proposition 2.6 yields the equivalence of conditions (1) and (2); since the right R-module N is faithful as a right $\overline{R}$-module, applying (6) $\Rightarrow$ (1) of Theorem 2.3 we get (2) $\Rightarrow$ (3). The following example shows that (3) $\Rightarrow$ (1) does not hold.

Example 2.9 Let K be a field of characteristic 2 and $R = K[x, y]$ be a polynomial ring over K. Take the factor ring $A = K[x, y]/(x^2, y^2)$ of R by the ideal (x^2, y^2) generated by x^2 and y^2. By [7, Example 3.6], R is a quasi-Armendariz ring and A is not a quasi-Armendariz ring. Let $\theta : R \longrightarrow A$ be the natural epimorphism. By Corollary 2.7, A is not quasi-Armendariz as a right R-module. Now, we take $N = R \bigoplus A$. Then N is R-faithful, but N (which has A as a submodule) is not quasi-Armendariz as a right R-module. This shows that (3) $\Rightarrow$ (1) does not hold in Remark 2.8.

For $f \in R[x]$, the content A_f is the ideal of R generated by the coefficients of f. For any subset S of $R[x]$, A_S denotes the ideal $\sum_{f\in S} A_f$. According to [7], a ring R is called $quasi-Gaussian$ if $A_{fRg} = A_f A_g$ for all $f, g \in R[x]$. Hirano [7, Theorem 4.1] showed that R is quasi-Gaussian if and only if every homomorphic image of R is quasi-Armendsriz. A ring R is right duo if every right ideal of R is two-sided. By Proposition 2.6, we have the following result.

Proposition 2.10 *Let R be a right duo ring. R is quasi-Gaussian if and only if every cyclic right R-module is quasi-Armendariz.*

Let N be a right R-module and C be the centre of R. If S is a multiplicatively closed subset of C, then $S^{-1}N$ has an $S^{-1}R$-module structure. The module N is S-torsion free if whenever s is an element of S and n is a nonzero element of N, we have $ns \neq 0$.

Proposition 2.11 *Let N be S-torsion free. The right R-module N is quasi-Armendariz if and only if the right $S^{-1}R$-module $S^{-1}N$ is quasi-Armendariz.*

Proof Suppose that N is quasi-Armendariz. Let $n(x) = \sum_{i=0}^{p} \frac{n_i}{s} x^i \in S^{-1}N[x]$ and $f(x) = \sum_{j=0}^{q} \frac{a_j}{t} x^j \in S^{-1}R[x]$ with $n(x)S^{-1}Rf(x) = 0$. Since N is S-torsion free, it is easily obtained that $n'(x)Rf'(x) = 0$, where $n'(x) = \sum_{i=0}^{p} n_i x^i \in N[x]$ and $f'(x) = \sum_{j=0}^{q} a_j x^j \in R[x]$. Since N is quasi-Armendariz, we have $n_i R a_j = 0$ for all i and j . It follows that $\frac{n_i}{s} S^{-1}R \frac{a_j}{t} = 0$ for all i and j, and we have that $S^{-1}N$ is quasi-Armendariz.

Conversely. Let $n(x) = \sum_{i=0}^{p} n_i x^i \in N[x]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x]$ with $n(x)Rf(x) = 0$. Then we have $n(x)S^{-1}Rf(x) = 0$. Since $S^{-1}N$ is quasi-Armendariz, $n_i S^{-1} R a_j = 0$ for all i and j. As N is S-torsion free, $n_i R a_j = 0$ for all i and j. Therefore N is quasi-Armendariz.

We write $M_m(R)$ and $T_m(R)$ for the $m \times m$ full matrix ring and the $m \times m$ upper triangular matrix ring over R, respectively. For a right R-module N, let

$$M_m(N) = \left\{ \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1m} \\ n_{21} & n_{22} & \cdots & n_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ n_{m1} & n_{m2} & \cdots & n_{mm} \end{pmatrix} | n_{ij} \in N, i, j = 1, 2, \ldots, m \right\},$$

$$T_m(N) = \left\{ \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1m} \\ 0 & n_{22} & \cdots & n_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n_{mm} \end{pmatrix} | n_{ij} \in N, 1 \leq i \leq j \leq m \right\}.$$

Similar to that of Lee and Zhou [6], $M_m(N)$ and $T_m(N)$ become the right modules over $M_m(R)$ and $T_m(R)$ respectively under usual addition and multiplication of matrices.

Theorem 2.12 *Let N be a right R-module and m a positive integer ≥ 2. Then the following statements are equivalent:*

(1) N is a quasi-Armendariz right R-module;

(2) $M_m(N)$ is a quasi-Armendariz right $M_m(R)$-module;

(3) $T_m(N)$ is a quasi-Armendariz right $T_m(R)$-module.

Proof (1) $\Rightarrow$ (2). It is easy to see that there exists an isomorphism of abelian groups:

$$M_m(N)[x] \to M_m(N[x]) \text{ via } \sum_i A_i x^i \mapsto (\sum_i n_{st}^i x^i), \text{ where } A_i = (n_{st}^i) \in M_m(N).$$

Let $f(x) = \sum_{i=0}^{p} A_i x^i \in M_m(N)[x]$ and $g(x) = \sum_{j=0}^{q} B_j x^j \in M_m(R)[x]$ satisfy $f(x)M_m(R)[x]$

$g(x)=0$, where $A_i=(n^i_{st})\in M_m(N)$ and $B_j=(b^j_{st})\in M_m(R)$. Then, by the isomorphism above, we have

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ f_{21} & f_{22} & \cdots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mm} \end{pmatrix} M_m(R[x]) \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mm} \end{pmatrix} = 0,$$

where $f_{st}=\sum_{i=0}^{p} n^i_{st}x^i\in N[x]$, $g_{st}=\sum_{j=0}^{q} b^j_{st}x^j\in R[x]$. Since $ce_{uv}\in M_m(R)$ for any $c\in R$ and any matrix unit $e_{uv}\in M_m(R)$, we have

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ f_{21} & f_{22} & \cdots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{mm} \end{pmatrix} ce_{uv} \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mm} \end{pmatrix} = 0.$$

It follows that $f_{su}cg_{vt}=0$ for all $1\le s,u,v,t\le m$, and so $f_{su}Rg_{vt}=0$. Since N is quasi-Armendariz, $n^i_{su}Rb^j_{vt}=0$ for all $0\le i\le p$, $0\le j\le q$ and $1\le s,u,v,t\le m$. Now we can easily conclude that $A_iM_m(R)B_j=0$ for all i,j. Therefore, $M_m(N)$ is a quasi-Armendariz right $M_m(R)$-module.

(2) $\Rightarrow$ (1). Let $n(x)=\sum_{i=0}^{p} n_ix^i\in N[x]$ and $g(x)=\sum_{j=0}^{q} b_jx^j\in R[x]$ such that $n(x)R[x]g(x)=0$. Let

$$\alpha(x)=\begin{pmatrix} n(x) & 0 & \cdots & 0 \\ 0 & n(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n(x) \end{pmatrix},\quad \beta(x)=\begin{pmatrix} g(x) & 0 & \cdots & 0 \\ 0 & g(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g(x) \end{pmatrix}.$$

It follows that $\alpha(x)M_m(R[x])\beta(x)=0$. By the hypothesis, we have that

$$\begin{pmatrix} n_i & 0 & \cdots & 0 \\ 0 & n_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n_i \end{pmatrix} M_m(R) \begin{pmatrix} a_j & 0 & \cdots & 0 \\ 0 & a_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_j \end{pmatrix} = 0$$

for all i and j. So $n_iRa_j=0$ for all i and j. Hence the assertion holds.

The proof of (1) $\Leftrightarrow$ (3) is similar to that of (1) $\Leftrightarrow$ (2).

Corollary 2.13 *Let R be a ring and m a positive integer ≥ 2. Then the following statements are equivalent:*

(1) R is quasi-Armendariz;

(2) $M_m(R)$ is quasi-Armendariz;

(3) $T_m(R)$ is quasi-Armendariz.

Clearly, Armendariz modules are quasi-Armendariz. But the converse need not be true by [1, Remark 3.1] and Corollary 2.13. Let R be a subring of a ring S with $1_S\in R$ and $N_R\subseteq L_S$.

According to Lee and Zhou [6, Remark 1.11], if L_S is Armendariz, then N_R is also Armendariz. One may conjecture that if L_S is quasi-Armendariz, then N_R is also quasi-Armendariz. However the following example erases the possibility.

Example 2.14 Let T be a reduced ring. Then $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in T \right\}$ is quasi-Armendariz by [9, Proposition 1.2]. By Corollary 2.13, $M_2(R)$ is quasi-Armendariz ring, but $S = R \ltimes R$ is not a quasi-Armendariz ring.

Let $S = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} | A, B \in R \right\}$. Clearly, $S_S \subseteq M_2(R)_{M_2(R)}$.

Let

$$f(x) = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} N & -I \\ 0 & N \end{pmatrix} x, \quad g(x) = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} N & I \\ 0 & N \end{pmatrix} x \in S[x],$$

where $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R$. Then $f(x)S[x]g(x) = 0$, but

$$\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} N & I \\ 0 & N \end{pmatrix} \neq 0,$$

implying that $\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} S \begin{pmatrix} N & I \\ 0 & N \end{pmatrix} \neq 0$. Thus S is not quasi-Armendariz.

Let R be a ring and S be a subring of $M_m(R)$ such that $e_{ii}Se_{jj} \subseteq S$ for all $i, j \in \{1, \ldots, m\}$ where e_{ij} denotes the (i, j)-matrix unit. In [7, Theorem 3.12], Hirano showed that if R_R is quasi-Armendariz, then S_S is quasi-Armendariz. In fact, this need not be true in general. For example, take R and S in Example 2.14. Let $T' = M_2(R)$. Then $M_m(S)$ is a subring of $M_m(T')$ satisfying the hypothesis above. By Example 2.14, $T'_{T'}$ is quasi-Armendariz. But $M_m(S)_{M_m(S)}$ is not quasi-Armendariz by Corollary 2.13. His gap lies in the fourth-last line of p.50, where he thinks the set $\{c \in R | ce_{pq} \in e_{pp}Re_{qq}\}$ as the ideal of R. In [9, Theorem 1.3], the author made the same gap. But this does not affect their main results.

From [6, Theorem1.12], we can infer that N_R is Armendariz if and only if $N[x]_{R[x]}$ is Armendariz. For the quasi-Armendariz module, we have the following result.

Theorem 2.15 *Let N be a right R-module. Then N_R is quasi-Armendariz if and only if $N[x]_{R[x]}$ is quasi-Armendariz.*

Proof Suppose that N is quasi-Armendariz. Let $n(T) \in N[x][T]$ and $g(T) \in R[x][T]$ with $n(T)R[x][T]g(T) = 0$. Write $n(T) = n_0(x) + n_1(x)T + \cdots + n_p(x)T^p$ and $g(T) = g_0(x) + g_1(x)T + \cdots + g_q(x)T^q$ where $n_i(x) = \sum_{s=0}^{u_s} a_{is}x^s \in N[x]$ and $g_j(x) = \sum_{t=0}^{v_t} b_{jt}x^t \in R[x]$ for all i and j. Let $k = \deg n_0(x) + \deg n_1(x) + \cdots + \deg n_p(x) + \deg g_0(x) + \cdots + \deg g_q(x)$, where the degree of $n_i(x)$ is as polynomial in $N[x]$, the degree of $g_j(x)$ is as polynomial in $R[x]$ and the degree of the zero polynomial is taken to be 0. Then $n(x^k) = n_0(x) + n_1(x)x^k + \cdots + n_p(x)x^{kp} \in N[x]$, $g(x^k) = g_0(x) + g_1(x)x^k + \cdots + g_q(x)x^{kq} \in R[x]$ and the set of coefficients of the $n_i(x)'s$ (resp.,

$g_i(x)'s$) equals to the set of coefficients of $n(x^k)$ (resp., $g(x^k)$). Since $n(T)R[x][T]g(T) = 0$, $n(T)R[x]g(T) = 0$. Since x commutes with the elements of R, we have $n(x^k)R[x]g(x^k) = 0$. By the hypothesis, we get $a_{is}Rb_{jt} = 0$ for all i, j, s and t. Thus $n_i(x)R[x]g_j(x) = 0$ for all i and j.

Conversely, suppose that $N[x]$ is quasi-Armendariz and let $n(x) \in N[x]$ and $g(x) \in R[x]$ with $n(x)R[x]g(x) = 0$, where $n(x) = \sum_{i=0}^{p} n_i x^i$ and $g(x) = \sum_{j=0}^{q} b_j x^j$. Thus, for any $c \in R$ we have the following equations:

$$n_0 c b_0 = 0,$$

$$n_0 c b_1 + n_1 c b_0 = 0,$$

$$\cdots$$

Hence, for any $h(x) \in R[x]$,

$$n_0 h(x) b_0 = 0,$$

$$n_0 h(x) b_1 + n_1 h(x) b_0 = 0,$$

$$\cdots$$

Now, take $\overline{n}(T) = \sum_{i=0}^{p} n_i T^i$ and $\overline{g}(T) = \sum_{j=0}^{q} b_j T^j$. By the equations above, we have $\overline{n}(T)R[x]\overline{g}(T) = 0$, and so $\overline{n}(T)R[x][T]\overline{g}(T) = 0$. By the hypothesis, $n_i R[x] b_j = 0$ for all i and j. Thus $n_i R b_j = 0$, proving the statement.

Hirano [7, Theorem 3.16] showed that if R is quasi-Armendariz, then $R[x]$ is quasi-Armendariz. By Theorem 2.15, the converse is also true.

Corollary 2.16 *R is quasi-Armendariz if and only if $R[x]$ is quasi-Armendariz.*

For a right R-module N, we put $r\mathrm{Ann}_R(\mathrm{sub}(N)) = \{r_R(S) | S \text{ is a submodule of } N\}$. The following result is a generalization of that of [5, Proposition 3.4].

Proposition 2.17 *Let N be a right R-module. Then the following statements are equivalent:*

(1) N is quasi-Armendariz;

(2) $\psi : r\mathrm{Ann}_R(\mathrm{sub}(N)) \to r\mathrm{Ann}_{R[x]}(\mathrm{sub}(N[x]))$ defined by $A \to AR[x]$ is bijective.

Proof (1) $\Rightarrow$ (2). Let $A \in r\mathrm{Ann}_R(\mathrm{sub}(N))$. Then there exists a submodule N' of N such that $A = r_R(N')$. Clearly, $N'R[x]$ is a submodule of $N[x]$ and $N'R[x]AR[x] = 0$. Thus, $AR[x] \subseteq r_{R[x]}(N'R[x])$. Let $g(x) = \sum_{j=0}^{q} b_j x^j \in r_{R[x]}(N'R[x])$. Then $N'R[x]g(x) = 0$. Hence $N'Rg(x) = 0$, and so $N'Rb_j = 0$, $b_j \in r_R(N'R) = r_R(N')$ for all j. Thus $g(x) \in AR[x]$ and $r_{R[x]}(N'R[x]) = AR[x]$. Consequently, ψ is a well-defined map. Assume that $B \in r\mathrm{Ann}_{R[x]}(\mathrm{sub}(N[x]))$. Then there exists a submodule S of $N[x]$ such that $B = r_{R[x]}(S)$. Let B_1 and S_1 denote the set of coefficients of elements of B and S, respectively. We claim that $r_R(S_1 R) = B_1 R$. Let $n(x) = \sum_{i=0}^{p} n_i x^i \in S$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in B$. Then $n(x)R[x]g(x) = 0$. Hence $n_i R b_j = 0$ for all i, j, since N is quasi-Armendariz. Thus $b_j \in r_R(S_1 R)$ for all j, and so $B_1 R \subseteq r_R(S_1 R)$. Clearly $r_R(S_1 R) \subseteq B_1 R$, hence $r_{R[x]}(S) = B_1 R[x]$.

(2) $\Rightarrow$ (1). Let $n(x) = \sum_{i=0}^{p} n_i x^i \in N[x]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x]$ satisfy $n(x)R[x]g(x) = 0$. Then $g(x) \in r_{R[x]}(n(x)R[x]) = AR[x]$, where A is an ideal of R. Hence $b_0, b_1, \ldots, b_q \in A$, and so $n(x)Rb_j = 0$ for all j. Thus $n_i R b_j = 0$ for all i, j. Therefore N is quasi-Armendariz.

A submodule S of a right R-module N is called a pure submodule if $S_R \bigotimes L \to N_R \bigotimes L$ is a monomorphism for every left R-module L. Following Tominaga [10], an ideal I of R is said to be left s-unital if, for each $a \in I$, there is an $x \in I$ such that $xa = a$. If an ideal I of R is left s-unital, then for any finite subset F of I, there exists an element $e \in I$ such that $ex = x$ for all $x \in F$. By [11, Proposition 11.3.13], for an ideal I, the following conditions are equivalent:

(1) I is pure as a right ideal of R;

(2) R/I is flat as a right R-module;

(3) I is left s-unital.

Theorem 2.18 *Let N be a right R-module. Then the following statements are equivalent:*

(1) $r_R(nR)$ is pure as a right ideal in R for any element $n \in N$.

(2) $r_{R[x]}(n(x)R[x])$ is pure as a right ideal in $R[x]$ for any element $n(x) \in N[x]$.

In this case, N is quasi-Armendariz.

Proof (1) $\Rightarrow$ (2). First we shall prove that N is quasi-Armendariz. Let $n(x) = \sum_{i=0}^{p} n_i x^i \in N[x]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x]$ satisfy $n(x)R[x]g(x) = 0$. We will prove that $n_i R b_j = 0$ for all i, j. Let c be an arbitrary element of R. Then we have the equation:

$$(n_0 + n_1 x + \cdots + n_p x^p)c(b_0 + b_1 x + \cdots + b_q x^q) = 0. \tag{*}$$

Thus $n_p c b_q = 0$. Hence $b_q \in r_R(n_p R)$. By hypothesis, $r_R(n_p R)$ is left s-unital, and hence there exists $e_p \in r_R(n_p R)$ such that $e_p b_q = b_q$. Replacing c by ce_p in Eq.(*), we obtain that

$$(n_0 + n_1 x + \cdots + n_{p-1} x^{p-1})ce_p(b_0 + b_1 x + \cdots + b_q x^q) = 0.$$

It follows that $n_{p-1} c e_p b_q = 0$. That is, $n_{p-1} c b_q = 0$. Hence $b_q \in r_R(n_{p-1} R)$. Since $r_R(n_{p-1}R)$ is left s-unital, there exists $f \in r_R(n_{p-1}R)$ such that $f b_q = b_q$. If we put $e_{p-1} = f e_p$, then $e_{p-1} b_q = b_q$ and $e_{p-1} \in r_R(n_p R + n_{p-1} R)$. Next, replacing c by ce_{p-1} in Eq.(*), we obtain $n_{p-2} c b_q = 0$ in the same way as above. Hence we have $b_q \in r_R(n_p R + n_{p-1} R + n_{p-2} R)$. Continuing this process, we obtain $n_k R b_q = 0$ for $k = 0, 1, \ldots, p$. Thus

$$(n_0 + n_1 x + \cdots + n_p x^p)R[x](b_0 + b_1 x + \cdots + b_{q-1} x^{q-1}) = 0.$$

Using induction on $p + q$, we have $n_i R b_j = 0$ for all i, j.

Let $g(x) = \sum_{j=0}^{q} b_j x^j \in r_{R[x]}(n(x)R[x])$, where $n(x) = \sum_{i=0}^{p} n_i x^i \in N[x]$. Then $n(x)R[x]g(x) = 0$. Since N is quasi-Armendariz, we obtain $n_i R b_j = 0$ for all $j = 0, 1, \ldots, q$. Since $r_R(n_i R)$ is left s-unital, there exists $e_i \in r_R(n_i R)$ such that $e_i b_j = b_j$ for all j. Take $e = e_0 e_1 \cdots e_p$. Then $e \in \cap_{i=0}^{p} r_R(n_i R)$ and $e b_j = b_j$ for all j. Hence $e \in r_R(n(x)R[x])$ and $g(x) = eg(x)$. Therefore, $r_{R[x]}(n(x)R[x])$ is left s-unital.

(2) $\Rightarrow$ (1). Let n be an element of N. Then $r_{R[x]}(nR[x])$ is left s-unital. Hence, for any $b \in r_R(nR)$, there exists a polynomial $f \in r_{R[x]}(nR[x])$ such that $fb = b$. Let a_0 be the constant term of f. Then $a_0 \in r_R(nR)$ and $a_0 b = b$. This implies that $r_R(nR)$ is left s-unital. Therefore the condition (1) holds. $\Box$

Corollary 2.19 ([7, Theorem 3.9]) *Let R be a ring. Then the following statements are equiva-*

lent:

(1) $r_R(aR)$ *is pure as a right ideal in* R *for any element* $a \in R$*;*

(2) $r_{R[x]}(f(x)R[x])$ *is pure as a right ideal in* $R[x]$ *for any element* $f(x) \in R[x]$.

In this case, R *is quasi-Armendariz.*

Lee and Zhou [6, Definition 2.1] called a right R-module N quasi-Baer if the right annihilator of every submodule of N in R as a right ideal is generated by an idempotent. Let N be a quasi-Baer module and $n \in N$. Then $r_R(nR) = eR$ for some $e^2 = e \in R$, and so $R/r_R(nR) \cong (1-e)R$ is projective. Therefore a quasi-Baer module satisfies the hypothesis of Theorem 2.18. Hence we have the following result.

Corollary 2.20 *Every quasi-Baer module is quasi-Armendariz.*

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