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# On $p$-Cover-Avoid and $S$-Quasinormally Embedded Subgroups in Finite Groups

**Xuan Li HE**[1,2,*], **Yan Ming WANG**[3]
1. *Department of Mathematics, Zhongshan University, Guangdong* 510275*, P. R. China;*
2. *College of Mathematics and Information Science, Guangxi University, Guangxi* 530004*, P. R. China;*
3. *Lingnan College and Department of Mathematics, Zhongshan University, Guangdong* 510275*, P. R. China*

**Abstract** Let $G$ be a finite group, $p$ the smallest prime dividing the order of $G$ and $P$ a Sylow $p$-subgroup of $G$. If $d$ is the smallest generator number of $P$, then there exist maximal subgroups $P_1, P_2, \ldots, P_d$ of $P$, denoted by $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$, such that $\bigcap_{i=1}^d P_i = \Phi(P)$, the Frattini subgroup of $P$. In this paper, we will show that if each member of some fixed $\mathcal{M}_d(P)$ is either $p$-cover-avoid or $S$-quasinormally embedded in $G$, then $G$ is $p$-nilpotent. As applications, some further results are obtained.

**Keywords** $p$-cover-avoid subgroup; $S$-quasinormally embedded subgroup; $p$-nilpotent group.

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## 1. Introduction

All groups considered in this paper are finite and $G$ always denote a finite group. Let $L$ and $K$ be normal subgroups of $G$ with $L < K$. A subgroup $H$ of $G$ covers the normal factor $K/L$ of $G$ if $HK = HL$ holds; $H$ avoids $K/L$ provided that $H \cap K = H \cap L$. If $H$ either covers or avoids each chief factor of $G$, then $H$ is said to possess the cover-avoiding property. Such a subgroup of $G$ is called a CAP-subgroup of $G$. This concept was introduced by Gaschütz [2] in 1962 and studied by many authors [1, 3–6]. For convenience, we use the notation $\mathcal{M}(G)$ to denote the set of all maximal subgroups of all Sylow subgroups of $G$ and let $\mathcal{F}$ be a saturated formation containing all supersoluble groups. In 1993, Ezquerro [1] showed that:

**Theorem 1.1** ([1]) *$G$ is supersolvable if and only if all members of $\mathcal{M}(G)$ are cover-avoid subgroups of $G$.*

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* Corresponding author
E-mail: xuanlihe@163.com (X. L. HE); stswym@mail.sysu.edu.cn (Y. M. WANG)

Recently, Fan, Guo and Shum [18] introduced the semi-$p$-cover-avoiding property, which generalizes not only the semi-cover-avoiding property [19] but also the $c$-normality [20], and obtained some characterizations of solvability of groups.

As another generalization of the normality, Kegel in [8] introduced the following concept: A subgroup $H$ of $G$ is called $S$-quasinormal in $G$ provided that $H$ permutes with all Sylow subgroups of $G$, i.e., $HS = SH$ for any Sylow subgroup $S$ of $G$. This concept has been studied extensively by Deskins [9] and Schmid [10]. Asaad [7] in 1998 obtained further results in the formation universe. Ballester-Bolinches and Pedraza-Aquilera [11] generalized $S$-quasinormal subgroups to $S$-quasinormally embedded subgroups, and showed that:

**Theorem 1.2** ([11]) *If every subgroup in $\mathcal{M}(G)$ is $S$-quasinormally embedded in $G$, then $G$ is supersolvable.*

Assad and Heliel [12] showed that a group $G$ is $p$-nilpotent for the smallest prime $p$ dividing $|G|$ if and only if all members of $\mathcal{M}(G_p)$ are $S$-quasinormally embedded in $G$. In the paper [13], Li and He continued the research in this direction by a new way and showed that:

**Theorem 1.3** ([13, Theorem 3.1]) *Let $p$ be the smallest prime dividing the order of $G$ and $G_p$ a Sylow $p$-subgroup of $G$. Then the following statements are equivalent:*

*(a) $G$ is $p$-nilpotent;*

*(b) Every member in $\mathcal{M}_d(G_p)$ is $S$-quasinormally embedded in $G$.*

By a $p$-chief factor we shall mean a chief factor each of whose elements has finite order a power of $p$. Recall that a normal factor $H/K$ is said to be a $pd$-chief factor if $p| \mid H/K \mid$.

Similar to semi-$p$-cover-avoid subgroups, we give the following definition:

**Definition 1.4** *Let $H$ be a subgroup of $G$. If $H$ either covers or avoids each $pd$-chief factor of $G$, then $H$ is said to possess the $p$-cover-avoiding property, and $H$ is called a $p$-CAP-subgroup of $G$.*

More recently, Li [23] unified two independent concepts, $c$-normal and $S$-quasinormal, and improved some known results. We note that $p$-CAP-subgroups and $S$-quasinormally embedded subgroups are also two independent concepts.

**Example 1.5** Let $G = S_4$ and $X = \langle(12)\rangle$. The normal series $G \trianglerighteq A_4 \trianglerighteq K_4 \trianglerighteq 1$ is the unique chief series of $G$. We can easily show that $X$ satisfies the 2-cover-avoiding property in $G$, but it is not $S$-quasinormally embedded in $G$.

**Example 1.6** Let $G = A_5$ and $Y$ any Sylow subgroup of $G$. It is easy to see that $Y$ is $S$-quasinormally embedded in $G$, but $Y$ does not satisfy the $p$-cover-avoiding property in $G$.

Following [13], we use the following notation.

**Definition 1.7** ([13]) *Let $d$ be the smallest generator number of a $p$-group $P$. We consider the set $\mathcal{M}_d(P) = \{M_1, \ldots, M_d\}$ of all elements of $\mathcal{M}(P)$ such that $\bigcap_{i=1}^{d} M_i = \Phi(P)$, the Frattini subgroup of $P$.*

In this paper, we unify $p$-CAP-subgroups and $S$-quasinormally embedded subgroups, which are two independent concepts, and generalize some results of [1] and [13].

The following notations are used in the paper. If $H$ is a subgroup of $G$, then $H_G$ always denotes the core of $H$ in $G$, the largest normal subgroup of $G$ contained in $H$. Also, $H_p$ always denotes a Sylow $p$-subgroup of $H$. All unexplained notation and terminology are standard. The reader can refer to [14] and [15].

## 2. Preliminaries

In this section we collect some known results which are needed in the sequel.

**Lemma 2.1** ([22, P180]) *Let $G$ be a $\pi$-separate group. If $O_{\pi'}(G) = 1$, then $C_G(O_\pi(G)) \leq O_\pi(G)$.*

**Lemma 2.2** ([9]) *If $H$ is an $S$-quasinormal subgroup of $G$, then $H/H_G$ is nilpotent.*

**Lemma 2.3** ([10]) *For a nilpotent subgroup $H$ of $G$, the following two statements are equivalent:*

*(a) $H$ is $S$-quasinormal in $G$;*

*(b) The Sylow subgroups of $H$ are $S$-quasinormal in $G$.*

**Lemma 2.4** ([12]) *Let $G_p$ be a Sylow $p$-subgroup of $G$ and $P$ a maximal subgroup of $G_p$. Then following two statements are equivalent:*

*(a) $P$ is normal in $G$;*

*(b) $P$ is $S$-quasinormal in $G$.*

**Lemma 2.5** ([11]) *Suppose that $U$ is an $S$-quasinormally embedded subgroup of $G$ and $K$ is a normal subgroup of $G$. Then*

*(a) $U$ is $S$-quasinormally embedded in $H$ whenever $U \leq H \leq G$.*

*(b) $UK$ is $S$-quasinormally embedded in $G$ and $UK/K$ is $S$-quasinormally embedded in $G/K$.*

By definition of $p$-CAP-subgroups and Lemma 2.2 [18], it is easy to have the following result:

**Lemma 2.6** *Let $H$ be a $p$-CAP-subgroup of $G$ and $N \trianglelefteq G$. Then the following statements are true:*

*(a) $N$ is a $p$-CAP-subgroup of $G$.*

*(b) If $N \leq H$, then $H/N$ is a $p$-CAP-subgroup of $G/N$.*

*(c) Let $\pi$ be a set of primes, $H$ a $\pi$-subgroup of $G$ and $N$ a $\pi'$subgroup of $G$. Then $HN/N$ is a $p$-CAP-subgroup of $G/N$.*

**Lemma 2.7** ([15]) *Let $N$ be a normal subgroup of $G$ and $H \leq G$. If $N \leq \Phi(H)$, then $N \leq \Phi(G)$.*

**Lemma 2.8** ([17]) *If $P$ is a Sylow $p$-subgroup of $G$ and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then $N$ is $p$-nilpotent.*

**Lemma 2.9** ([15]) *The following statements are equivalent:*

*(a) $G$ is $p$-nilpotent;*

*(b) If $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{r-1} \lhd G_r = G$ is any chief series of $G$ and $p \mid |G_i/G_{i-1}|$, then $G_i/G_{i-1} \le Z(G/G_{i-1})$.*

**Lemma 2.10** ([16]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Suppose that there is a cyclic normal subgroup $Q$ of $G$ such that $G/Q \in \mathcal{F}$. Then $G \in \mathcal{F}$.*

**Lemma 2.11** *Let $p$ be the smallest prime of $|G|$ and $P$ a maximal subgroup of the Sylow $p$-subgroup $G_p$ of $G$. If $P$ is a $p$-CAP subgroup of $G$, then $G$ is $p$-soluble.*

**Proof** Take any $pd$-chief factor $H/K$ of $G$. By Definition 1.4, we have that $PH = PK$ or $P \cap H = P \cap K$. If $PH = PK$, then $PH/K = PK/K \cong P/(P \cap K)$ is $p$-group. Therefore, $|H/K| = \frac{|PH/K|}{|PH/H|}$ is a $p$-number. Let $P \cap H = P \cap K$. Note that $|H/K|_p = |H \cap G_p : K \cap G_p|$. If $H \cap G_p \le P$, then $H \cap G_p \le P \cap H = P \cap K \le K \cap G_p$, a contradiction. So $(H \cap G_p)P = G_p$. Hence, $p = |G_p/P| = |(H \cap G_p)P/P| = |H \cap G_p : H \cap P| = |H \cap G_p : K \cap G_p||K \cap G_p : K \cap P|$. Thus, $|H/K|_p = |H \cap G_p : K \cap G_p| = p$. We can get that $H/K$ is $p$-nilpotent since $p$ is the smallest prime of $|G|$. Therefore, $|H/K| = p$ and then $G$ is $p$-soluble. □

**Lemma 2.12** ([21]) *Let $p$ be a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

*(a) If $N$ is normal in $G$ of order $p$, then $N$ lies $Z(G)$.*

*(b) If $G$ has a cyclic Sylow $p$-subgroup, then $G$ is $p$-nilpotent.*

*(c) If $M \le G$ and $|G : M| = p$, then $M \trianglelefteq G$.*

**Lemma 2.13** ([15, I, Hauptsatz 17.4]) *Let $N$ be a normal abelian subgroup of $G$ and let $N \le M \le G$ such that $(|N|, |G : M|) = 1$. If a complement subgroup of $N$ in $M$ exists, then $N$ possesses a complement subgroup in $G$.*

## 3. Main results

**Theorem 3.1** *Let $G$ be a $p$-soluble group and $P$ a Sylow $p$-subgroup of $G$, where $p$ is a prime dividing $|G|$. Assume that every member in some fixed $\mathcal{M}_d(P)$ is either $p$-cover-avoid or $S$-quasinormally embedded in $G$. Then $G$ is $p$-supersoluble.*

**Proof** Assume that the theorem is not true and let $G$ be a counterexample of minimal order. Let $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$. By hypotheses, each $P_i$ is either $p$-cover-avoid or $S$-quasinormally embedded in $G$. Without loss of generality, let $1 \le k \le d$ such that

(i) Every $P_l (1 \le l \le k)$ is $p$-cover-avoid in $G$;

(ii) Every $P_j (k+1 \le j \le d)$ is $S$-quasinormally embedded in $G$.

We prove the theorem by the following claims.

(1) $O_{p'}(G) = 1$.

Denote $N = O_{p'}(G)$. If $N = O_{p'}(G) > 1$, we consider the factor group $G/N$. Obviously, $PN/N$ is a Sylow $p$-subgroup of $G/N$, which is isomorphic to $P$, so $PN/N$ has the smallest generator number as the same as $P$, i.e., $d$ and $\mathcal{M}_d(PN/N) = \{P_1N/N, \ldots, P_dN/N\}$. Also, $P_iN/N$ is either $S$-quasinormally embedded or $p$-cover-avoid in $G/N$ by Lemmas 2.5 and 2.6. Thus, we know that $G/N$ satisfies the hypotheses of the theorem. Hence, $G/O_{p'}(G)$ is $p$-supersoluble by the choice of $G$, it follows that $G$ itself is $p$-supersoluble, a contradiction. Thus, we have that $N = O_{p'}(G) = 1$, as desired.

(2) $\Phi(P)_G = 1$, therefore, $O_p(G)$ is an elementary abelian group.

If not, take any $T \le \Phi(P)_G$ such that $T \trianglelefteq G$. We consider the factor group $G/T$. In this case, $P_i/T(1 \le i \le d)$ are maximal subgroups of $P/T$. By Lemmas 2.5 and 2.6, $P_i/T$ is either $S$-quasinormally embedded or $p$-cover-avoid in $G/T$. Thus, $G/T$ satisfies the hypotheses of the theorem. Hence, $G/T$ is $p$-supersoluble by the choice of $G$. By Lemma 2.7, $T \le \Phi(G)$, so $G/\Phi(G)$ is $p$-supersoluble. Consequently, $G$ is $p$-supersoluble, a contradiction.

(3) All minimal normal subgroups of $G$ contained in $O_p(G)$ are of order $p$.

Take any minimal normal subgroup $N$ of $G$ contained in $O_p(G)$. If there exists some $P_t(t \in \{1, 2, \ldots, k\})$ such that $N/1$ is avoided by $P_t$, then $N \cap P_t = 1$ and $|N| = p$. If $N$ is covered by $P_l(1 \le l \le k)$, then $N \le \cap_{l=1}^{k} P_{lG}$. Suppose that $N \le M_{jG}(k+1 \le j \le d)$. Denote $T = (\cap_{l=1}^{k} P_{lG}) \cap (\cap_{j=k+1}^{d} M_{jG})$, then $N \le T$ and $T \trianglelefteq G$. By (ii), there exist $S$-quasinormal subgroups $M_j(k+1 \le j \le d)$ of $G$ such that $P_j$ is a Sylow $p$-subgroup of $M_j$. It follows from Lemma 2.5 that $M_j/M_{jG}$ is $S$-quasinormal in $G/M_{jG}$ and then $M_j/M_{jG}$ is nilpotent by Lemma 2.2. So we may apply Lemma 2.3 to see that every Sylow subgroup of $M_j/M_{jG}$ is $S$-quasinormal in $G/M_{jG}$. Thus, $P_jM_{jG}/M_{jG}$ is $S$-quasinormal in $G/M_{jG}$ because $P_jM_{jG}/M_{jG}$ is a Sylow $p$-subgroup of $M_j/M_{jG}$. It follows by Lemma 2.4 that $P_jM_{jG}/M_{jG}$ is normal in $G/M_{jG}$. Hence, $P_j \le M_{jG}$ and $T \cap P \le \Phi(P)$. By Lemma 2.8, $T$ is $p$-nilpotent. By (1), $T$ is a $p$-group, so $T = T \cap P \le \Phi(P)$. By (2), $T = 1$. Consequently, $N = 1$, a contradiction. Therefore, there exists some $M_{sG}(s \in \{k+1, \ldots, d\})$ such that $N$ is not contained in $M_{sG}$, then $N \cap M_{sG} = 1$, $|N| = p$.

(4) The counterexample $G$ does not exist.

Let $N_1, N_2, \ldots, N_r$ be all minimal normal subgroups of $G$ contained in $O_p(G)$. By (2) and (3), we have that every $N_i(i \in \{1, 2, \ldots, r\})$ is complemented in $P$. By Lemma 2.13, each $N_i(i \in \{1, 2, \ldots, r\})$ is complemented in $G$. Let $M$ be a supplement of $N_1 \times N_2 \times \cdots \times N_r$ to $G$ with the order as small as possible. We assume that $O_p(G) \cap M = 1$. If not, then $O_p(G) \cap M \trianglelefteq G$. Take $L \le O_p(G) \cap M$ such that $L$ is a minimal normal subgroup of $G$. Similar to the proof above, $L$ is complemented in $G$, so in $M$. Then there is a subgroup $K$ of $M$ such that $M = L \rtimes K$. Thus, $G = (N_1 \times N_2 \times \cdots \times N_r)LK = (N_1 \times N_2 \times \cdots \times N_r)K$, which contradicts the choice of $M$. Therefore, $O_p(G) \cap M = 1$. So we can get that $O_p(G) = N_1 \times N_2 \times \cdots \times N_r$, where $N_i \trianglelefteq G$ and $|N_i| = p$ for any $i \in \{1, 2, \ldots, r\}$. Hence, $G/C_G(N_i)$ is abelian, $G/C_G(O_p(G)) = G/\cap_{i=1}^{r} C_G(N_i)$ is abelian. Since $G$ is $p$-soluble, $C_G(O_p(G)) \le O_p(G)$ by (1) and Lemma 2.1. Thus, $G/O_p(G)$ is abelian and $G$ is $p$-supersolvable, a contradiction. $\square$

**Theorem 3.2** *Let $p$ be the smallest prime dividing $|G|$ and $P$ a Sylow $p$-subgroup of $G$. Assume that every member in some fixed $\mathcal{M}_d(P)$ is either $p$-cover-avoid or $S$-quasinormally embedded* *in $G$. Then $G$ is $p$-nilpotent.*

**Proof** Let $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$. By hypotheses, each $P_i$ is either $p$-cover-avoid or $S$-quasinormally embedded in $G$.

If all $P_i (1 \le i \le d)$ are $S$-quasinormally embedded in $G$, by Theorem 3.1 in [13], we have $G$ is $p$-nilpotent, a contradiction. Therefore, there exists some $P_t \in \mathcal{M}_d(P)$ such that it is $p$-cover-avoid in $G$. By Lemma 2.11, $G$ is $p$-soluble. By Theorem 3.1, $G$ is $p$-supersolvable. Take any $pd$-chief factor $M/N$ of $G$. By Lemma 2.12(a), $M/N \le Z(G/N)$. By Lemma 2.9, $G$ is $p$-nilpotent. □

**Remark 3.3** The $p$-cover-avoiding property of Theorem 3.2 cannot be replaced by semi-$p$-cover-avoiding property.

**Example 3.4** Let $V = \langle a_1, a_2, a_3 \rangle$ be an elementary abelian group of order $2^3$ and $\alpha$ an automorphism of $V$ defined by

$$\alpha = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}.$$

Put $H = \langle \alpha \rangle$. It is clear that $H \cong Z_3$ and $H$ acts on $V$ by conjugate. Denote $G = V \rtimes H$ and $\mathcal{M}_d(V) = \{\langle a_1 \rangle \times \langle a_2 \rangle, \langle a_2 \rangle \times \langle a_3 \rangle, \langle a_1 a_2 \rangle \times \langle a_2 a_3 \rangle\}$.

It is easy to see that $\langle a_1 a_2 \rangle \times \langle a_2 a_3 \rangle$ and $\langle a_1 a_2 a_3 \rangle$ are both minimal normal subgroups of $G$. Therefore, $\langle a_1 a_2 \rangle \times \langle a_2 a_3 \rangle$ is $S$-quasinormal in $G$. We know that $G$ has the following normal series:

(1) $G = V\langle \alpha \rangle \trianglerighteq V \trianglerighteq \langle a_1 a_2 a_3 \rangle \trianglerighteq 1$;

(2) $G = V\langle \alpha \rangle \trianglerighteq V \trianglerighteq \langle a_1 a_2 \rangle \times \langle a_2 a_3 \rangle \trianglerighteq 1$.

We can show that $\langle a_1 \rangle \times \langle a_2 \rangle, \langle a_2 \rangle \times \langle a_3 \rangle$ are 2-cover-avoid in normal series (1), but not 2-cover-avoid in normal series (2). Thus, every member in $\mathcal{M}_3(V)$ is either semi-2-cover-avoid or $S$-quasinormally embedded in $G$, but $G$ is not 2-nilpotent. Furthermore, it is easy to see that $G$ is not 2-nilpotent even if all members in $\mathcal{M}_3(V)$ are semi-2-cover-avoid in $G$. □

**Theorem 3.5** *If, for each Sylow subgroup $P$ of $G$, every member in some fixed $\mathcal{M}_d(P)$ is either $p$-cover-avoid or $S$-quasinormally embedded in $G$, then $G$ is supersolvable.*

**Proof** Firstly, we claim that $G$ is solvable. Suppose there exists a non-solvable chief factor $N/K$ of $G$. Then there exists a prime $p$ such that $p^2$ divides $|N/K|$. By hypotheses, there is a fixed $\mathcal{M}_d(G_p)$ such that every member $P_j \in \mathcal{M}_d(G_p)$ is either $p$-cover-avoid or $S$-quasinormally embedded in $G$, where $G_p$ is a Sylow $p$-subgroup of $G$. If all $P_j \in \mathcal{M}_d(G_p)$ are $S$-quasinormally embedded in $G$, then $G$ is supersolvable by Theorem 3.2 in [13], a contradiction. Therefore, there exists some $P_i \in \mathcal{M}_d(G_p)$ such that it is $p$-cover-avoid in $G$. If $P_i$ covers $N/K$, then $P_i N = P_i K$. Since $P_i N/K = P_i K/K \cong P_i/P_i \cap K$ is $p$-group. Thus, $|N/K| = \frac{|P_i N/K|}{|P_i N/N|}$ is a $p$-number, a contradiction. Hence, $P_i$ avoids $N/K$, $|N/K|_p = 1$ or $p$ by the proof of Lemma 2.11, which is a contradiction. Therefore, we conclude that $G$ has no non-solvable chief factors and hence, $G$ is solvable. Now we are in the hypotheses of Theorem 3.1 for all primes $p$. Consequently, $G$ is

$p$-supersolvable for all primes $p$ dividing $|G|$. That is, $G$ is supersolvable. $\square$

**Theorem 3.6** *Suppose that there is a normal Sylow $q$-subgroup $Q$ of $G$ such that $G/Q$ is supersolvable for some prime $q$, every member in some fixed $\mathcal{M}_d(Q)$ is either $q$-cover-avoid or $S$-quasinormally embedded in $G$. Then $G$ is supersolvable.*

**Proof** Suppose that this is not true so that there exists a counterexample $G$ of minimal order. The proof is divided into four steps.

(1) $\Phi(Q) = 1$, $Q$ is an elementary abelian group.

If not, consider the factor group $G/\Phi(Q)$. It is easy to show that $G/\Phi(Q)$ satisfies the hypotheses of the theorem. By the choice of $G$, $G/\Phi(Q)$ is supersoluble. Hence, $G$ is supersoluble, a contradiction.

(2) Every $Q_i$ $(1 \le i \le d)$ is $q$-cover-avoid in $G$.

Suppose that there is some $Q_j$ $(j \in \{1, 2, \dots, d\})$ which is $S$-quasinormally embedded in $G$. By the definition, there exists $S$-quasinormal subgroup $H_j$ of $G$ such that $Q_j$ is a Sylow $q$-subgroup of $H_j$. It follows from Lemma 2.5 that $H_j/H_{jG}$ is $S$-quasinormal in $G/H_{jG}$ and $H_j/H_{jG}$ is nilpotent by Lemma 2.2. So we may apply Lemma 2.3 to see that every Sylow subgroup of $H_j/H_{jG}$ is $S$-quasinormal in $G/H_{jG}$. Thus, $Q_jH_{jG}/H_{jG}$ is $S$-quasinormal in $G/H_{jG}$ because $Q_jH_{jG}/H_{jG}$ is a Sylow $q$-subgroup of $H_j/H_{jG}$. It follows by Lemma 2.4 that $Q_jH_{jG}/H_{jG}$ is normal in $G/H_{jG}$. Thus, $Q_j \le H_{jG}$. Note that $Q_j \le H_{jG} \cap Q \le H_j \cap Q = Q_j$, Consequently, $Q_j = H_{jG} \cap Q \trianglelefteq G$. By Lemma 2.6, every $Q_i$ $(1 \le i \le d)$ is $q$-cover-avoid in $G$.

(3) Final contradiction.

Take any minimal normal subgroup $L$ of $G$ contained in $Q$. If all $Q_i$ cover $L/1$, then $L \le \Phi(Q) = 1$, a contradiction. Thus, there exists some $Q_j$ $(j \in \{1, 2, \dots, d\})$ such that $Q_j$ avoids $L/1$, then $Q_j \cap L = 1$, $|L| = q$. Similar to the proof of (4) of Theorem 3.1, we have that $Q = L_1 \times L_2 \times \cdots \times L_s$ by (1), where $L_t \trianglelefteq G$ and $|L_t| = q$ for any $t \in \{1, 2, \dots, s\}$. Hence, all $Q_i$ $(1 \le i \le d)$ are normal in $G$. Consequently, $(G/Q_i)/(Q/Q_i)$ is supersolvable. By Lemma 2.10, $G/Q_i$ is supersolvable. Therefore, $G \cong G/\cap_{i=1}^d Q_i$ is supersolvable, a contradiction. $\square$

**Remark 3.7** The condition that $Q$ is a Sylow subgroup of $G$ of Theorem 3.6 is necessary.

**Example 3.8** Let $V = \langle a_1, a_2, a_3\rangle$ be an elementary abelian group of order $3^3$ and $\alpha, \beta$ be two automorphisms of $V$ defined respectively by

$$\alpha = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}, \quad \beta = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_2 & a_1 \end{pmatrix}.$$

Set $H = \langle \alpha, \beta\rangle$. It is clear that $H \cong S_3$ and $H$ acts on $V$ by conjugate. Denote $G = V \rtimes H$ and $\mathcal{M}_d(V) = \{\langle a_1\rangle \times \langle a_2\rangle, \langle a_2\rangle \times \langle a_3\rangle, \langle a_1a_2\rangle \times \langle a_2a_3\rangle\}$. Then $G/V \cong S_3$ is supersolvable, but $V$ is not a Sylow subgroup of $G$. We know that $G$ has unique normal series:

$$G = V\langle \alpha, \beta\rangle \trianglerighteq V\langle\alpha\rangle \trianglerighteq V \trianglerighteq \langle a_1a_2a_3\rangle \trianglerighteq 1. \qquad (*)$$

It is easy to see that $\langle a_1a_2a_3\rangle$ is a minimal normal subgroup of $G$ and $\langle a_1a_2\rangle \times \langle a_2a_3\rangle$ is

$S$-quasinormal in $G$. We can easily show that both $\langle a_1\rangle \times \langle a_2\rangle$ and $\langle a_2\rangle \times \langle a_3\rangle$ are 3-cover-avoid in $(*)$. Thus, every member in $\mathcal{M}_d(V)$ is either 3-cover-avoid or $S$-quasinormally embedded in $G$, but we know that $G$ is not supersolvable. $\square$

## References


[1] EZQUERRO L M. *A contribution to the theory of finite supersoluble groups* [J]. Rend. Sem. Mat. Univ. Padova, 1993, **89**: 161–170.

[2] GASCHÜTZ W. *Praefrattinigruppen* [J]. Arch. Math. (Basel), 1962, **13**: 418–426. (in German)

[3] GILLAM J D. *Cover-avoid subgroups in finite solvable groups* [J]. J. Algebra, 1974, **29**: 324–329.

[4] GUO Xiuyun, SHUM K P. *Cover-avoidance properties and the structure of finite groups* [J]. J. Pure Appl. Algebra, 2003, **181**(2-3): 297–308.

[5] TOMKINSON M J. *Cover-avoidance properties in finite soluble groups* [J]. Canad. Math. Bull., 1976, **19**(2): 213–216.

[6] PETRILLO J. *CAP-subgroups in a direct product of finite groups* [J]. J. Algebra, 2006, **306**(2): 432–438.

[7] ASAAD M. *On maximal subgroups of Sylow subgroups of finite groups* [J]. Comm. Algebra, 1998, **26**(11): 3647–3652.

[8] KEGEL O H. *Sylow-Gruppen und Subnormalteiler endlicher Gruppen* [J]. Math. Z., 1962, **78**: 205–221. (in German)

[9] DESKINS W E. *On quasinormal subgroups of finite groups* [J]. Math. Z., 1963, **82**: 125–132.

[10] SCHMID P. *Subgroups permutable with all Sylow subgroups* [J]. J. Algebra, 1998, **207**: 285–293.

[11] BALLESTER-BOLINCHES A, PEDRAZA-AQUILERA M C. *Sufficient conditions for supersolubility of finite groups* [J]. J. Pure Appl. Algebra, 1998, **127**(2): 113–118.

[12] ASAAD M, HELIEL A A. *On S-quasinormally embedded subgroups of finite groups* [J]. J. Pure Appl. Algebra, 2001, **165**(2): 129–135.

[13] LI Shirong, HE Xuanli. *On normally embedded subgroups of prime power order in finite groups* [J]. Comm. Algebra, 2008, **36**(6): 2333–2340.

[14] DOERK K, HAWKES T. *Finite Soluble Groups* [M]. Walter de Gruyter & Co., Berlin, 1992.

[15] HUPPERT B. *Endliche Gruppen (I)* [M]. Springer-Verlag, Berlin-New York, 1967. (in German)

[16] LI Shirong. *On minimal subgroups of finite groups (III)* [J]. Comm. Algebra, 1998, **26**(8): 2453–2461.

[17] TATE J. *Nilpotent quotient groups* [J]. Topology, 1964, **3**(suppl. 1): 109–111.

[18] FAN Yun, GUO Xiuyun, SHUM K P. *Remarks on two generalizations of normality of subgroups* [J]. Chinese Ann. Math. Ser. A, 2006, **27**(2): 169–176.

[19] GUO Xiuyun, GUO Pengfei, SHUM K P. *On semi cover-avoiding subgroups of finite groups* [J]. J. Pure Appl. Algebra, 2007, **209**(1): 151–158.

[20] WANG Yanming. *c-normality of groups and its properties* [J]. J. Algebra, 1996, **180**(3): 954–965.

[21] WEI Huaquan, WANG Yanming. *On $c^*$-normality and its properties* [J]. J. Group Theory, 2007, **10**(2): 211–223.

[22] XU Mingyao. *An Introduce to Finite Groups* [M]. Beijing: Science Press, 2001.

[23] LI Shirong. *On s-quasinormal and c-normal subgroups of a finite group* [J]. Acta Math. Sin. (Engl. Ser.), 2008, **24**(4): 647–654.