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On the Case of Equalities in Comparison Results for Elliptic Equations Related to Gauss Measure

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Abstract In this paper, we deal with a Dirichlet problem for linear elliptic equations related to Gauss measure. For this problem, we study the converse of some inequalities proved by other authors, in the sense that we study the case of equalities and show that equalities are achieved only in the "symmetrized" situations. In addition, under other assumptions, we give a different form of comparison results and discuss the corresponding case of equalities.

Keywords comparison results; equalities; rearrangements; Gauss measure; elliptic equation.

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1. Introduction

In this paper, we discuss the following problem

$$\text{(P1)}\quad \begin{cases} -\sum\limits_{i,j=1}^{n} D_j(a_{ij}D_iu)+\sum\limits_{i=1}^{n} D_i(b_iu)+\sum\limits_{i=1}^{n} d_iD_iu+cu=f\varphi, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases}$$

where Ω is an open subset of $\mathbf{R}^n$ ($n\ge 2$) with Gauss measure less than one, $\varphi(x)=(2\pi)^{-\frac{n}{2}}\exp(-\frac{|x|^2}{2})$ is the density of Gauss measure, a_{ij}, b_i, d_i, c and f are measurable functions on Ω such that

(i) a_{ij}/φ, $c/\varphi\in L^\infty(\Omega)$, $a_{ij}(x)=a_{ji}(x)$, a.e. $x\in\Omega$;
(ii) $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\ge\varphi(x)|\xi|^2$, a.e. $x\in\Omega$, $\forall\xi\in\mathbf{R}^n$;
(iii) $(\sum_{i=1}^n|b_i(x)+d_i(x)|^2)^{\frac12}\le R\varphi(x)$, a.e. $x\in\Omega$, $R>0$;
(iv) $\sum_{i=1}^n D_ib_i(x)+c(x)\ge c_0(x)\varphi(x)$ in $D'(\Omega)$, $c_0\in L^\infty(\Omega)$;
(v) $f\in L^2(\varphi,\Omega)$.

When Ω is bounded, by means of Schwarz symmetrization it is possible to compare the solutions of an elliptic equation with the solutions of a simpler one which is defined on a ball and has spherical symmetric data [2–4, 27, 28]. A comprehensive bibliography on this issue can be found in [16, 26, 31]. In [1, 19, 20], Kesavan, Alvino, Lions and Trombetti have studied the case of

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equalities in some comparison results for uniformly elliptic equations without lower order terms. They showed that equalities are achieved only in spherical symmetric situations. Ferone and Posteraro have extended their results to more general elliptic equations [18]. The case of equalities in some comparison results for L^1-norm or L^∞-norm of the solutions has been discussed in [8] for degenerate Dirichlet elliptic problems or Hamilton-Jacobi equations. However, by observing the proofs in the articles mentioned above, we find that it is essential for the distribution functions of solutions to "symmetrized" problems to be absolutely continuous. That is why until now there are no conclusions on the case of equalities in comparison results for general elliptic equations [4].

In recent years, by using Gauss symmetrization, some comparison results on an (possibly unbounded) open subset of $\mathbf{R}^n$ have been obtained for some elliptic and parabolic equations [6, 12, 14, 15]. In this paper, we study the case of equalities in these comparison results. Compared with Schwarz symmetrization, the excellent property of Gauss symmetrization (see Lemma 4.1) allows us to deal with this kind of problems for general elliptic equations. Actually, we get a conclusion that if equalities hold in the comparison results, the original problem is equivalent to its "symmetrized" problem in the sense of weak form modulo a rotation. Moreover, we show a different form of comparison results under other assumptions and discuss the corresponding case of equalities.

This paper is organized as follows: In Section 2, we give some notations and preliminary results; In Section 3, the main results of this paper are stated; In Section 4, we finish the proof of the main results.

2. Notations and some preliminary results

In this section, we recall some definitions and some preliminary results which we shall need in the following proof of the main results.

Definition 2.1 *We say γ_n is the n-dimensional Gauss measure on $\mathbf{R}^n$, if*

$$\mathrm{d}\gamma_n = \varphi(x)\mathrm{d}x = (2\pi)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{2})\mathrm{d}x, \quad x \in \mathbf{R}^n,$$

normalized by $\gamma_n(\mathbf{R}^n) = 1$.

Set

$$\Phi(\tau) = \gamma_n(\{x \in \mathbf{R}^n : x_1 > \tau\}) = (2\pi)^{-\frac{1}{2}} \int_\tau^{+\infty} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t, \quad \forall \tau \in \mathbf{R} \cup \{-\infty, +\infty\}.$$

We observe in [21] that

$$\lim_{t \to 0^+, 1^-} (2\pi)^{-\frac{1}{2}} \frac{\exp(-\frac{\Phi^{-1}(t)^2}{2})}{t(2\log\frac{1}{t})^{\frac{1}{2}}} = 1. \tag{1}$$

Remark 2.1 By virtue of $\lim_{t\to 0^+} \frac{t(2\log\frac{1}{t})^{\frac{1}{2}}}{t(1-\log t)^{\frac{1}{2}}} = \sqrt{2}$, $\lim_{t\to 1^-} \frac{t(2\log\frac{1}{t})^{\frac{1}{2}}}{t(1-\log t)^{\frac{1}{2}}} = 0$ and observing that

$\exp(-\frac{\Phi^{-1}(t)^2}{2})$ is a continuous function on $(0,1)$, it follows from (1) that

$$\exp\left(-\frac{\Phi^{-1}(t)^2}{2}\right) \leq C_1 t(1-\log t)^{\frac{1}{2}}, \quad t\in(0,1) \tag{2}$$

and

$$\exp\left(-\frac{\Phi^{-1}(t)^2}{2}\right) \geq C_2 t(1-\log t)^{\frac{1}{2}}, \quad t\in(0,t_0), \tag{3}$$

where t_0 is any value of $(0,1)$, C_1 is a positive constant and C_2 is a positive constant depending on t_0. Note that $\gamma_n(\Omega)<1$, thus (2) and (3) hold on $(0,\gamma_n(\Omega))$.

Definition 2.2 *The perimeter in the sense of De Giorgi with respect to Gauss measure of* Ω *will be*

$$P_\varphi(\Omega)=\sup\Big\{\int_\Omega \operatorname{div}(\varphi(x)\psi(x))\mathrm{d}x : \psi=(\psi_1,\ldots,\psi_n)\in(C_0^1(\mathbf{R}^n))^n, \sup_{x\in\mathbf{R}^n}|\psi|\leq 1\Big\}.$$

For exhaustive treatment of weighted perimeter in the sense of De Giorgi we refer to [23]–[25] and the references therein. We just mention that as $\partial\Omega$ is $(n-1)$-rectifiable,

$$P_\varphi(\Omega)=\int_{\partial\Omega}\varphi(x)H_{n-1}(\mathrm{d}x),$$

where H_{n-1} denotes the $(n-1)$-dimensional Hausdorff measure.

Moreover, it follows from [23]–[25] and [11] that if $u\in W_0^{1,1}(\varphi,\Omega)$,

$$\int_{\{x\in\Omega:|u(x)|>t\}}|\nabla u(x)|\varphi(x)\mathrm{d}x=\int_t^{+\infty}P_\varphi\{x\in\Omega:|u(x)|>\eta\}\mathrm{d}\eta. \tag{4}$$

The following isoperimetric inequality with respect to Gauss measure [9, 17, 21] can be proved that for every measurable subset Ω of $\mathbf{R}^n$,

$$P_\varphi(\Omega)\geq P_\varphi(\Omega^\sharp),$$

where $\Omega^\sharp$ denotes the set $\{x=(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n : x_1>\lambda\}$ with λ chosen such that $\gamma_n(\Omega)=\gamma_n(\Omega^\sharp)$. Clearly, $\lambda=\Phi^{-1}(\gamma_n(\Omega^\sharp))$.

Definition 2.3 *If* u *is a measurable function in* Ω*, we denote by*

(a) $u^\star$ *the decreasing rearrangement of* u *with respect to Gauss measure, i.e.,*

$$u^\star(s)=\inf\{t\geq 0:\mu(t)\leq s\}, \quad s\in[0,\gamma_n(\Omega)],$$

where $\mu(t)=\gamma_n(\{x\in\Omega:|u|>t\})$ *is the distribution function of* u*.*

(b) $u^\sharp$ *the increasing Gauss symmetrization of* u*, i.e.,*

$$u^\sharp(x)=u^\star(\Phi(x_1)), \quad x\in\Omega^\sharp,$$

where $\Omega^\sharp=\{x=(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n : x_1>\lambda\}$ *is the half-space such that* $\gamma_n(\Omega)=\gamma_n(\Omega^\sharp)$*.*

(c) The decreasing Gauss symmetrization of u *will be*

$$u_\sharp(x)=u_\star(\Phi(x_1)), \quad x\in\Omega^\sharp,$$

where

$$u_\star(s)=u^\star(\gamma_n(\Omega)-s), \quad s\in(0,\gamma_n(\Omega))$$

is the increasing rearrangement of u with respect to Gauss measure.

General results about the properties of rearrangement with respect to a positive measure can be found in [13, 23–25, 29]. We just recall that

(d) If u and v are measurable functions, Hardy-Littlewood inequality

$$\int_0^{\gamma_n(\Omega)} u_\star(s)v^\star(s)\mathrm{d}s = \int_{\Omega^\sharp} u_\sharp(x)v^\sharp(x)\mathrm{d}\gamma_n \le \int_\Omega |u(x)v(x)|\mathrm{d}\gamma_n$$
$$\le \int_{\Omega^\sharp} u^\sharp(x)v^\sharp(x)\mathrm{d}\gamma_n = \int_0^{\gamma_n(\Omega)} u^\star(s)v^\star(s)\mathrm{d}s$$

holds.

(e) The weighted L^p-norm is invariant under Gauss symmetrization

$$\|u\|_{L^p(\varphi,\Omega)} = \|u^\sharp\|_{L^p(\varphi,\Omega^\sharp)} = \|u^\star\|_{L^p(0,\gamma_n(\Omega))}, \quad 1 \le p \le +\infty.$$

(f) Polya-Szëgo principle

$$\|\nabla u^\sharp\|_{L^p(\varphi,\Omega^\sharp)} \le \|\nabla u\|_{L^p(\varphi,\Omega)}, \quad 1 \le p < +\infty \tag{5}$$

holds. This result can be found in various papers, for example, [11] and [17] for Gauss measure, [23]–[25] and [29] for all the measures which enjoy an isoperimetric inequality.

3. Statement of the main results

In this section, we state the main results of this paper.

Definition 3.1 *u is a weak solution of problem (P1), if $u \in H_0^1(\varphi,\Omega)$ and*

$$\int_\Omega a_{ij}D_iuD_j\psi\mathrm{d}x - \int_\Omega b_iuD_i\psi\mathrm{d}x + \int_\Omega d_iD_iu\psi\mathrm{d}x + \int_\Omega cu\psi\mathrm{d}x = \int_\Omega f\varphi\psi\mathrm{d}x \tag{6}$$

holds for all $\psi \in H_0^1(\varphi,\Omega)$.

At first, we give a comparison result between the solution of problem (P1) and a simpler Dirichlet problem which is defined on a half-space and whose coefficients depend only on the first variable [6, 12, 14, 15].

Let

$$c_0^+(x) = \max\{c_0(x),0\},\ c_0^-(x) = \max\{-c_0(x),0\},$$
$$c_{0\sharp}^+(x) = (c_0^+(x))_\sharp,\quad c_0^{-\sharp}(x) = (c_0^-(x))^\sharp.$$

Proposition 3.1 *Assume that (i)–(v) hold. Let $u \in H_0^1(\varphi,\Omega)$ be a weak solution of problem (P1). If the following "symmetrized" problem*

$$\text{(P2)} \begin{cases} -D_1(\varphi D_1v) - R\varphi D_1v + (c_{0\sharp}^+ - c_0^{-\sharp})\varphi v = f^\sharp\varphi, & \textit{in } \Omega^\sharp, \\ v = 0, & \textit{on } \partial\Omega^\sharp \end{cases}$$

has a solution $v(x) = v^\sharp(x)$, then

(I) As $c_0(x) \le 0$, we have

$$u^\star(s) \le v^\star(s), \quad s \in [0,\gamma_n(\Omega)]. \tag{7}$$

(II) As $c_0^+(x) \not\equiv 0$, we have

$$u^\star(s) \le v^\star(s), \quad s \in [0, s_1'], \tag{8}$$

$$\int_{s_1'}^{s} \exp(R\Phi^{-1}(\sigma))u^\star(\sigma)\mathrm{d}\sigma \le \int_{s_1'}^{s} \exp(R\Phi^{-1}(\sigma))v^\star(\sigma)\mathrm{d}\sigma, \quad s \in [s_1', \gamma_n(\Omega)], \tag{9}$$

where $s_1' = \inf\{s \in [0, \gamma_n(\Omega)] : c_{0\star}^+(s) > 0\}$.

Theorem 3.1 *Under the same assumptions of Proposition 3.1, if equalities hold in (7)–(9), then*

$$\begin{cases} u^\sharp(x) = v(x), \text{ a.e. } x \in \Omega^\sharp, \\ \Omega = \Omega^\sharp, \\ u(x) = \varepsilon u^\sharp(x), \text{ a.e. } x \in \Omega^\sharp, \\ a_{i1}(x) = \delta_{i1}\varphi(x), \text{ a.e. } x \in \Omega^\sharp \backslash E, \ \forall 1 \le i \le n, \\ b_i(x) + d_i(x) = -R\delta_{i1}\varphi(x), \text{ a.e. } x \in \Omega^\sharp \backslash E, \ \forall 1 \le i \le n, \\ f(x) = \varepsilon f^\sharp(x), \text{ a.e. } x \in \Omega^\sharp, \\ \sum_{i=1}^{n} D_i b_i(x) + c(x) = [c_{0\sharp}^+(x) - c_0^{-\sharp}(x)]\varphi(x) \text{ in } D'(\Omega^\sharp) \end{cases}$$

modulo a rotation, where $E = \{x \in \Omega^\sharp : \nabla v(x) = 0\}$ *and* $\varepsilon = \pm 1$.

Remark 3.1 As $b_i(x) = 0$ in (P1), assumption (iv) turns into

$$c(x) \ge c_0(x)\varphi(x), \quad \text{a.e. } \ x \in \Omega.$$

In this case, the comparison results were discussed in [14]. However, we find that results given in [14] are incorrect. Here we show the correct results (see Proposition 3.1).

In the case $d_i(x) = 0$, to obtain a different comparison result, we need to make the following assumption:

(vi) $c(x) \ge c_0(x)\varphi(x)$, a.e. $x \in \Omega$.

Proposition 3.2 *Assume that (i)–(iii), (v) and (vi) hold and* $d_i(x) = 0$. *Let* $u \in H_0^1(\varphi, \Omega)$ *be a weak solution of problem (P1). If*

$$\text{(P3)} \quad \begin{cases} -D_1(\varphi D_1 v) + RD_1(\varphi v) + (c_{0\sharp}^+ - c_0^{-\sharp})\varphi v = f^\sharp \varphi, & \text{in } \ \Omega^\sharp, \\ v = 0, & \text{on } \ \partial\Omega^\sharp \end{cases}$$

has a solution $v(x) = v^\sharp(x)$, *then*

(I) As $c_0(x) \le 0$, we have

$$u^\star(s) \le v^\star(s), \quad s \in [0, \gamma_n(\Omega)]. \tag{10}$$

(II) As $c_0^+(x) \not\equiv 0$, assume (iv) also holds. Then we have

$$u^\star(s) \le v^\star(s), \quad s \in [0, s_1'], \tag{11}$$

$$\int_{s_1'}^{s} u^\star(\sigma)\mathrm{d}\sigma \le \int_{s_1'}^{s} v^\star(\sigma)\mathrm{d}\sigma, \quad s \in [s_1', \gamma_n(\Omega)], \tag{12}$$

where $s_1' = \inf\{s \in [0, \gamma_n(\Omega)] : c_{0\star}^+(s) > 0\}$.

Remark 3.2 As $c_0(x) \geq 0$ and Ω is bounded, under the assumptions of Proposition 3.2 (II), some comparison results have been obtained in [30] by using Schwarz symmetrization. However, as Ω is unbounded, there are no results under such assumptions up to now. Here we give a result by using Gauss symmetrization.

In addition, we have a similar result to Theorem 3.1.

Theorem 3.2 *Under the same assumptions of Proposition 3.2, if equalities hold in (10)–(12), then*

$$\begin{cases} u^\sharp(x) = v(x), \ \text{a.e. } x \in \Omega^\sharp, \\ \Omega = \Omega^\sharp, \\ u(x) = \varepsilon u^\sharp(x), \ \text{a.e. } x \in \Omega^\sharp, \\ a_{i1}(x) = \delta_{i1}\varphi(x), \ \text{a.e. } x \in \Omega^\sharp, \ \forall 1 \leq i \leq n, \\ b_i(x) = R\delta_{i1}\varphi(x), \ \text{a.e. } x \in \Omega^\sharp, \ \forall 1 \leq i \leq n, \\ f(x) = \varepsilon f^\sharp(x), \ \text{a.e. } x \in \Omega^\sharp, \\ c(x) = [c_{0\sharp}^{+}(x) - c_0^{-\sharp}(x)]\varphi(x), \ \text{a.e. } x \in \Omega^\sharp \end{cases}$$

modulo a rotation, where $\varepsilon = \pm 1$.

Remark 3.3 From Theorems 3.1 and 3.2, one can know that if equalities hold in the comparison results of Propositions 3.1 and 3.2, the original problem is equivalent to its "symmetrized" problem in the sense of weak form modulo a rotation. That is to say that the comparison results obtained in Propositions 3.1 and 3.2 are sharp.

4. Proof of the main results

In this section, we give the proofs of Theorems 3.1–3.2 and Propositions 3.1–3.2.

Proof of Proposition 3.1 We give a brief proof since the arguments are the same as in [4] and [14].

Letting $h > 0$, $t \in [0, \sup |u|]$ and

$$\psi(x) = \begin{cases} \text{sign}(u(x)), & \text{if } \ |u(x)| > t + h, \\ \frac{(|u(x)|-t)\text{sign}(u(x))}{h}, & \text{if } \ t < |u(x)| \leq t + h, \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

in (6), we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} |\nabla u|^2 \varphi \mathrm{d}x \leq \exp\left(-R\Phi^{-1}(\mu(t))\right) \int_0^{\mu(t)} \exp\left(R\Phi^{-1}(\sigma)\right) [f^\star(\sigma) - c_{0\star}^{+}(\sigma)u^\star(\sigma) + c_0^{-\star}(\sigma)u^\star(\sigma)\big] \,\mathrm{d}\sigma \tag{14}$$

and

$$-u^{\star\prime}(s) \leq 2\pi \exp\left(\Phi^{-1}(s)^2\right) \exp\left(-R\Phi^{-1}(s)\right) \int_0^s \exp\left(R\Phi^{-1}(\sigma)\right) [f^\star(\sigma) - c_{0\star}^{+}(\sigma)u^\star(\sigma) + c_0^{-\star}(\sigma)u^\star(\sigma)\big] \,\mathrm{d}\sigma, \quad s \in [0, \gamma_n(\Omega)]. \tag{15}$$

Furthermore, considering the "symmetrized" problem (P2), we can proceed in the same way except for that the inequalities should be replaced by equalities. Thus we can get

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|v|>t\}}|\nabla v|^2\varphi \mathrm{d}x = \exp\left(-R\Phi^{-1}(\nu(t))\right)\int_0^{\nu(t)}\exp\left(R\Phi^{-1}(\sigma)\right)[f^\star(\sigma) - c_{0\star}^{+}(\sigma)v^\star(\sigma) + c_0^{-\star}(\sigma)v^\star(\sigma)]\,\mathrm{d}\sigma, \tag{16}$$

and

$$-v'(s) = 2\pi\exp\left(\Phi^{-1}(s)^2\right)\exp\left(-R\Phi^{-1}(s)\right)\int_0^{s}\exp\left(R\Phi^{-1}(\sigma)\right)[f^\star(\sigma) - c_{0\star}^{+}(\sigma)v^\star(\sigma) + c_0^{-\star}(\sigma)v^\star(\sigma)]\,\mathrm{d}\sigma, \quad s\in[0,\gamma_n(\Omega)], \tag{17}$$

where $\nu(t)$ is the distribution function of v.

Thus we complete the proof of Proposition 3.1 by following the same steps as in [4]. □

Before proving Theorem 3.1, we recall the following lemma.

Lemma 4.1 ([7, 11]) *Let Ω be a measurable subset of $\mathbf{R}^n$ and $u\in H_0^1(\varphi,\Omega)$. Then*

$$\|\nabla u^\sharp\|_{L^2(\varphi,\Omega^\sharp)} = \|\nabla u\|_{L^2(\varphi,\Omega)}$$

holds if and only if $\Omega=\Omega^\sharp$ and $|u|=u^\sharp$ modulo a rotation.

Proof of Theorem 3.1

Case I $c_0(x)\le 0$.

Since $u^\star(s)=v^\star(s)$, $s\in(0,\gamma_n(\Omega))$, integrating both sides of (14) and (16) between 0 and $+\infty$, we can deduce that

$$\int_\Omega |\nabla u|^2\varphi\mathrm{d}x \le \int_{\Omega^\sharp}|\nabla v|^2\varphi\mathrm{d}x. \tag{18}$$

It follows from Polya-Szëgo principle and (18) that

$$\int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x \le \int_\Omega |\nabla u|^2\varphi\mathrm{d}x \le \int_{\Omega^\sharp}|\nabla v|^2\varphi\mathrm{d}x = \int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x. \tag{19}$$

Thus

$$\int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x = \int_\Omega |\nabla u|^2\varphi\mathrm{d}x. \tag{20}$$

By Lemma 4.1, we have

$$\Omega=\Omega^\sharp \text{ and } |u|=u^\sharp \text{ modulo a rotation}, \tag{21}$$

which imply that u depends only on the first variable.

Letting $\Phi(x_1)=s$, we set $\widetilde{u}(s)=u(\Phi^{-1}(s))=u(x_1)$. Thus

$$|\widetilde{u}(s)| = u^\star(s), \quad s\in[0,\gamma_n(\Omega)]. \tag{22}$$

Since $u\in H_0^1(\varphi,\Omega)$, by Polya-Szëgo principle, we have $u^\sharp\in H_0^1(\varphi,\Omega^\sharp)$ and

$$\int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x = \frac{1}{2\pi}\int_0^{\gamma_n(\Omega^\sharp)}|\frac{\mathrm{d}u^\star}{\mathrm{d}s}(s)|^2\exp\left(-\Phi^{-1}(s)^2\right)\mathrm{d}s. \tag{23}$$

Taking into account the fact that $\exp\left(-\Phi^{-1}(0)^2\right)=0$, we obtain

$$\int_a^{\gamma_n(\Omega^\sharp)} |\frac{\mathrm{d}u^\star}{\mathrm{d}s}(s)|^2\mathrm{d}s \le C\int_a^{\gamma_n(\Omega^\sharp)} |\frac{\mathrm{d}u^\star}{\mathrm{d}s}(s)|^2 \exp\left(-\Phi^{-1}(s)^2\right)\mathrm{d}s \le C\int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x, \tag{24}$$

where $0<a<\gamma_n(\Omega)$ and C is a positive constant depending on a.

Hence $u^\star\in\bigcap_{0<a<\gamma_n(\Omega)} H^1[a,\gamma_n(\Omega)]$. From (22) we can see that $\widetilde{u}\in\bigcap_{0<a<\gamma_n(\Omega)} H^1[a,\gamma_n(\Omega)]$. By the imbedding theorem in Sobolev space, we get $\widetilde{u}\in C^0(0,\gamma_n(\Omega)]$.

Now we claim that $\widetilde{u}$ dose not change sign on $(0,\gamma_n(\Omega))$. In fact, by maximum principle for (P2), we get $v^\sharp>0$ a.e. on $\Omega^\sharp$. Thus $u^\star=v^\star>0$ on $(0,\gamma_n(\Omega))$, that is $|\widetilde{u}|>0$ on $(0,\gamma_n(\Omega))$. The continuity of $\widetilde{u}$ on $(0,\gamma_n(\Omega))$ implies that $\widetilde{u}>0$ on $(0,\gamma_n(\Omega))$ or $\widetilde{u}<0$ on $(0,\gamma_n(\Omega))$. Therefore, $u>0$ a.e. on $\Omega^\sharp$ or $u<0$ a.e. on $\Omega^\sharp$.

Using (21), we have

$$u=\varepsilon u^\sharp \text{ a.e. on } \Omega^\sharp, \tag{25}$$

where $\varepsilon=\pm1$. Furthermore, taking u as a test function in (6), we obtain

$$\int_\Omega a_{ij}D_iuD_ju\mathrm{d}x-\int_\Omega b_iuD_iu\mathrm{d}x+\int_\Omega d_iuD_iu\mathrm{d}x+\int_\Omega cu^2\mathrm{d}x=\int_\Omega f\varphi u\mathrm{d}x. \tag{26}$$

Thus (ii) and (20) imply that

$$\begin{aligned}\int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x=\int_\Omega|\nabla u|^2\varphi\mathrm{d}x&\le\int_\Omega a_{ij}D_iuD_ju\mathrm{d}x\\ &=\int_\Omega fu\varphi\mathrm{d}x+\int_\Omega b_iuD_iu\mathrm{d}x-\int_\Omega d_iuD_iu\mathrm{d}x-\int_\Omega cu^2\mathrm{d}x.\end{aligned} \tag{27}$$

By using (25), (iii), (iv) and Hardy-Littlewood inequality, it follows from (27) that

$$\begin{aligned}\int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x &\le\int_{\Omega^\sharp}\varepsilon fu^\sharp\varphi\mathrm{d}x+\int_{\Omega^\sharp}b_iu^\sharp D_iu^\sharp\mathrm{d}x-\int_{\Omega^\sharp}d_iu^\sharp D_iu^\sharp\mathrm{d}x-\int_{\Omega^\sharp}cu^{\sharp2\mathrm{d}x}\\ &\le\int_{\Omega^\sharp}|f|u^\sharp\varphi\mathrm{d}x-\int_{\Omega^\sharp}(b_i+d_i)u^\sharp D_iu^\sharp\mathrm{d}x+\int_{\Omega^\sharp}2b_iu^\sharp D_iu^\sharp\mathrm{d}x-\int_{\Omega^\sharp}cu^{\sharp2}\mathrm{d}x\\ &\le\int_{\Omega^\sharp}f^\sharp u^\sharp\varphi\mathrm{d}x+R\int_{\Omega^\sharp}u^\sharp D_1u^\sharp\varphi\mathrm{d}x-\int_{\Omega^\sharp}c_0u^{\sharp2}\mathrm{d}x\\ &\le\int_{\Omega^\sharp}f^\sharp u^\sharp\varphi\mathrm{d}x+R\int_{\Omega^\sharp}u^\sharp D_1u^\sharp\varphi\mathrm{d}x+\int_{\Omega^\sharp}c_0^{-\sharp}u^{\sharp2}\mathrm{d}x\\ &=\int_{\Omega^\sharp}f^\sharp v\varphi\mathrm{d}x+R\int_{\Omega^\sharp}vD_1v\varphi\mathrm{d}x+\int_{\Omega^\sharp}c_0^{-\sharp}v^2\mathrm{d}x\\ &=\int_{\Omega^\sharp}|\nabla v|^2\varphi\mathrm{d}x=\int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x.\end{aligned} \tag{28}$$

Thus equality holds through (27) and (28). In particular, we have

$$\int_{\Omega^\sharp}|\nabla u^\sharp|^2\varphi\mathrm{d}x=\int_{\Omega^\sharp}a_{ij}D_iu^\sharp D_ju^\sharp\mathrm{d}x, \tag{29}$$

$$-\int_{\Omega^\sharp}(b_i+d_i)u^\sharp D_iu^\sharp\mathrm{d}x=R\int_{\Omega^\sharp}u^\sharp|\nabla u^\sharp|\varphi\mathrm{d}x \tag{30}$$

and

$$\int_{\Omega^\sharp} \varepsilon f u^\sharp \varphi dx = \int_{\Omega} |f| u^\sharp \varphi \mathrm{d}x = \int_{\Omega^\sharp} f^\sharp u^\sharp \varphi \mathrm{d}x. \tag{31}$$

By ellipticity condition, we have

$$a_{ij} D_i u^\sharp D_j u^\sharp \geq |\nabla u^\sharp|^2 \varphi, \text{ a.e. } x \in \Omega^\sharp.$$

Thus (29) yields

$$a_{ij} D_i u^\sharp D_j u^\sharp = |\nabla u^\sharp|^2 \varphi, \text{ a.e. } x \in \Omega^\sharp. \tag{32}$$

Since $A(x) = (a_{ij}(x))$ is a symmetric matrix, ellipticity condition implies that the first eigenvalue of $A(x)$ is larger than or equal to $\varphi(x)$ for almost all $x \in \Omega^\sharp$. On the other hand, by virtue of $c_0(x) \leq 0$ in Ω, from (17) and the fact $u^\star = v^\star$ we can see that

$$-u^{\star\prime}(s) > 0, \quad s \in (0, \gamma_n(\Omega)). \tag{33}$$

Hence

$$\nabla u^\sharp(x) = \big(u^{\sharp\prime}(x_1), 0, \ldots, 0\big) \neq 0, \text{ a.e. } x \in \Omega^\sharp. \tag{34}$$

That is to say $\Omega^\sharp \backslash E = \Omega^\sharp$ up to a zero measure set.

From (32) and (34), we can observe that the first eigenvalue of $A(x)$ is indeed $\varphi(x)$ with $\nabla u^\sharp(x)$ as eigenvector for almost all $x \in \Omega^\sharp$, that is

$$A(x) \nabla u^\sharp(x) = \varphi(x) \nabla u^\sharp(x), \text{ a.e. } x \in \Omega^\sharp,$$

which shows that

$$a_{i1}(x) = \delta_{i1} \varphi(x), \text{ a.e. } x \in \Omega^\sharp, \ \forall 1 \leq i \leq n.$$

In addition, taking into account the fact that

$$-(b_i(x) + d_i(x)) u^\sharp(x) D_i u^\sharp(x) \leq R\varphi(x) u^\sharp(x) |\nabla u^\sharp(x)|, \text{ a.e. } x \in \Omega^\sharp,$$

by (30), we get

$$-(b_1(x) + d_1(x)) u^\sharp(x_1) u^{\sharp\prime}(x_1) = R\varphi(x) u^\sharp(x_1) u^{\sharp\prime}(x_1), \text{ a.e. } x \in \Omega^\sharp,$$

which gives

$$b_1(x) + d_1(x) = -R\varphi(x), \text{ a.e. } x \in \Omega^\sharp.$$

Thus the above equality and (iii) imply that

$$b_i(x) + d_i(x) = 0, \text{ a.e. } x \in \Omega^\sharp, \ 2 \leq i \leq n.$$

Moreover, since $\mu(t)$ is continuous in this case, we can proceed similarly to the Appendix in [1] and conclude from (31) that $f(x) = \varepsilon f^\sharp(x)$, a.e. $x \in \Omega^\sharp$.

Finally, by Definition 3.1, we have

$$\begin{aligned} &\int_{\Omega^\sharp} D_1 u^\sharp D_1 \psi \varphi \mathrm{d}x - R \int_{\Omega^\sharp} D_1 u^\sharp \psi \varphi \mathrm{d}x + \int_{\Omega^\sharp} \big(-b_i D_i(u^\sharp \psi) + c u^\sharp \psi\big) \,\mathrm{d}x \\ &= \int_{\Omega^\sharp} f^\sharp \psi \varphi \mathrm{d}x, \quad \forall \psi \in H_0^1(\varphi, \Omega). \end{aligned} \tag{35}$$

On the other hand, $u^\sharp$ also satisfies

$$\int_{\Omega^\sharp} D_1u^\sharp D_1\psi\varphi\mathrm{d}x - R\int_{\Omega^\sharp} D_1u^\sharp\psi\varphi\mathrm{d}x - \int_{\Omega^\sharp} c_0^{-\sharp}u^\sharp\psi\varphi\mathrm{d}x = \int_{\Omega^\sharp} f^\sharp\psi\varphi\mathrm{d}x, \quad \forall\psi\in H_0^1(\varphi,\Omega). \tag{36}$$

(35) and (36) allow us to state that

$$\int_{\Omega^\sharp} -b_iD_i(u^\sharp\psi) + cu^\sharp\psi\mathrm{d}x = -\int_{\Omega^\sharp} c_0^{-\sharp}\varphi u^\sharp\psi\mathrm{d}x, \quad \forall\psi\in H_0^1(\varphi,\Omega). \tag{37}$$

Hence

$$D_i(b_i(x)) + c(x) = -c_0^{-\sharp}(x)\varphi(x) \quad \text{in} \quad D'(\Omega^\sharp). \tag{38}$$

Thus we get the desired result.

Case II $c_0^+(x)\not\equiv 0$.

In this case, we have that

$$\int_0^s \exp\left(R\Phi^{-1}(\sigma)\right)u^\star(\sigma)\mathrm{d}\sigma = \int_0^s \exp\left(R\Phi^{-1}(\sigma)\right)v^\star(\sigma)\mathrm{d}\sigma, \quad s\in[0,\gamma_n(\Omega)]. \tag{39}$$

Moreover, we observe that $\exp\left(R\Phi^{-1}\right)u^\star$, $\exp\left(R\Phi^{-1}\right)v^\star \in L^1(0,\gamma_n(\Omega))$. In fact, it suffices to show the first one, since another is the same. By (3) and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^{\gamma_n(\Omega)} \exp\left(R\Phi^{-1}(\sigma)\right)u^\star(\sigma)\mathrm{d}\sigma\\
&= \int_0^{\gamma_n(\Omega)} \exp\Big[-\Big(\sqrt{\frac{1}{6}}\Phi^{-1}(\sigma) - \frac{\sqrt{6}R}{2}\Big)^2 + \frac{3R^2}{2}\Big]\exp\Big(\frac{\Phi^{-1}(\sigma)^2}{6}\Big)u^\star(\sigma)\mathrm{d}\sigma\\
&\le c\int_0^{\gamma_n(\Omega)} \frac{1}{t^{\frac{1}{3}}(1-\log t)^{\frac{1}{6}}}u^\star(\sigma)\mathrm{d}\sigma \le c\|u\|_{L^2(\varphi,\Omega)}.
\end{aligned}$$

Then (39) implies

$$u^\star(s) = v^\star(s), \quad s\in(0,\gamma_n(\Omega)]. \tag{40}$$

Thus, we can proceed as Case I and obtain

$$\Omega = \Omega^\sharp \text{ and } u = \varepsilon u^\sharp \text{ modulo a rotation}, \tag{41}$$

where $\varepsilon = \pm 1$.

Taking u as a test function, we also have (29)–(31) hold. Observing

$$\nabla u^\sharp(x) = \left(u^{\sharp\prime}(x_1), 0, \ldots, 0\right) \neq 0, \ \text{a.e. } x\in\Omega^\sharp\backslash E, \tag{42}$$

we obtain

$$a_{i1}(x) = \delta_{i1}\varphi(x), \ \text{a.e. } x\in\Omega^\sharp\backslash E, \quad \forall 1\le i\le n, \tag{43}$$

and

$$b_i(x) + d_i(x) = -R\delta_{i1}\varphi(x), \ \text{a.e. } x\in\Omega^\sharp\backslash E, \quad \forall 1\le i\le n. \tag{44}$$

However, we cannot get $f(x) = \varepsilon f^\sharp(x)$ a.e. $x\in\Omega^\sharp$ from (31) as before because $\mu(t)$ may not be continuous in this case. Instead, we take εw as a test function in (6), where

$$w(x) = w^\sharp(x) = \int_\lambda^{x_1} \exp\Big(\frac{\tau^2}{2}\Big)\int_\tau^{+\infty} \exp\Big(-\frac{\sigma^2}{2}\Big)\mathrm{d}\sigma\mathrm{d}\tau$$

is the weak solution of

$$\text{(P4)} \begin{cases} -D_1(\varphi D_1 w) = \varphi, & \text{in } \Omega^\sharp, \\ w = 0, & \text{on } \partial\Omega^\sharp. \end{cases}$$

By (43), (44), (iv) and Hardy-Littlewood inequality, we get

$$\begin{aligned} &\int_{\Omega^\sharp} \varphi D_1 u^\sharp D_1 w \mathrm{d}x - R\int_{\Omega^\sharp} D_1 u^\sharp w\varphi \mathrm{d}x = \int_{\Omega^\sharp} a_{11} D_1 u^\sharp D_1 w \mathrm{d}x + \int_{\Omega^\sharp} (b_1 + d_1) D_1 u^\sharp w \mathrm{d}x \\ &= \int_{\Omega^\sharp} [b_i D_i(u^\sharp w) - c u^\sharp w] \mathrm{d}x + \int_{\Omega^\sharp} \varepsilon f w\varphi \mathrm{d}x \le -\int_{\Omega^\sharp} c_0 u^\sharp w\varphi \mathrm{d}x + \int_{\Omega^\sharp} |f| w\varphi \mathrm{d}x \\ &\le \int_{\Omega^\sharp} (-c_{0\sharp}^+ + c_0^{-\sharp}) u^\sharp w\varphi \mathrm{d}x + \int_{\Omega^\sharp} f^\sharp w\varphi \mathrm{d}x = \int_{\Omega^\sharp} \varphi D_1 u^\sharp D_1 w \mathrm{d}x - R\int_{\Omega^\sharp} D_1 u^\sharp w\varphi \mathrm{d}x. \quad (45) \end{aligned}$$

Then

$$\int_{\Omega^\sharp} \varepsilon f w\varphi \mathrm{d}x = \int_{\Omega^\sharp} |f| w\varphi \mathrm{d}x = \int_{\Omega^\sharp} f^\sharp w\varphi \mathrm{d}x. \quad (46)$$

Since $\gamma_n(\{w > t\})$ is continuous on $[0, \operatorname{ess\,sup} w]$, it follows that $f(x) = \varepsilon f^\sharp(x)$ a.e. $x \in \Omega^\sharp$. On the other hand, we also have

$$D_i b_i(x) + c(x) = [c_{0\sharp}^+(x) - c_0^{-\sharp}(x)]\varphi(x) \ \text{ in } \ \mathrm{D}'(\Omega^\sharp).$$

Thus we complete the proof of Theorem 3.1. □

Proof of Proposition 3.2 Taking $\psi(x)$ (see (13)) in (6) and letting h tend to 0, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} |\nabla u|^2 \varphi \mathrm{d}x \le -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} b_i u D_i u \mathrm{d}x + \int_{\{|u|>t\}} (|f|\varphi - c|u|) \mathrm{d}x. \quad (47)$$

As $c_0(x) \le 0$, we have $c_0(x) = -c_0^-(x)$. Thus

$$\int_{\{|u|>t\}} (|f|\varphi - c_0\varphi|u|)\, \mathrm{d}x = \int_{\{|u|>t\}} \big(|f|\varphi + c_0^- \varphi|u|\big)\, \mathrm{d}x \ge 0. \quad (48)$$

As $c_0^+(x) \not\equiv 0$, using (iv) gives

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} b_i u D_i u \mathrm{d}x = \lim_{h\to 0} \frac{1}{h}\int_{\{t<|u|\le t+h\}} b_i u D_i u \mathrm{d}x \\ &= \lim_{h\to 0} \Big(\frac{1}{h}\int_{\{t<|u|\le t+h\}} b_i D_i u(|u| - t)\operatorname{sign} u \mathrm{d}x + t\frac{1}{h}\int_{\{t<|u|\le t+h\}} b_i \operatorname{sign} u D_i u \mathrm{d}x\Big) \\ &= \lim_{h\to 0} \Big(t\int_\Omega (b_i D_i(|\psi|) - c|\psi|)\mathrm{d}x + t\int_\Omega c|\psi| \mathrm{d}x\Big) \\ &\le \lim_{h\to 0} t\Big(-\int_\Omega c_0|\psi|\varphi \mathrm{d}x + \int_\Omega c|\psi| \mathrm{d}x\Big) \\ &= t\int_{\{|u|>t\}} (c - c_0\varphi)\mathrm{d}x \le \int_{\{|u|>t\}} (c - c_0\varphi)|u| \mathrm{d}x. \quad (49) \end{aligned}$$

It follows from (47) and (49) that

$$\int_{\{|u|>t\}} (|f|\varphi - c|u|)\mathrm{d}x \ge -\int_{\{|u|>t\}} (c - c_0\varphi)|u| \mathrm{d}x \quad (50)$$

which yields

$$\int_{\{|u|>t\}} (|f|\varphi - c_0\varphi|u|)\mathrm{d}x \ge 0. \quad (51)$$

Thus we have proved that under the assumptions of (I) or (II), (51) holds whatever the sign of $c_0(x)$.

On the other hand, by the second equality in (49), we obtain

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} b_i u D_i u \mathrm{d}x &= \lim_{h\to 0} t\frac{1}{h}\int_{\{t<|u|\le t+h\}} b_i \mathrm{sign}\, u D_i u \mathrm{d}x \\
&\le \lim_{h\to 0} Rt\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|\varphi \mathrm{d}x = -Rt\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} |\nabla u|\varphi \mathrm{d}x. \qquad (52)
\end{aligned}
$$

Applying (52) and (vi) to (47), we get

$$
-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} |\nabla u|^2\varphi \mathrm{d}x \le -Rt\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} |\nabla u|\varphi \mathrm{d}x + \int_{\{|u|>t\}} (|f|\varphi - c_0\varphi|u|)\mathrm{d}x. \qquad (53)
$$

(1) and the isoperimetric inequality with respect to Gauss measure imply

$$
1 \le \sqrt{2\pi}\exp(\frac{\Phi^{-1}(\mu(t))^2}{2})\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} |\nabla u|^2\varphi \mathrm{d}x\Big)^{\frac12}(-\mu'(t))^{\frac12}. \qquad (54)
$$

By (51), (54), Hölder inequality and Hardy-Littlewood inequality, (53) turns into

$$
\begin{aligned}
&\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} |\nabla u|^2\varphi \mathrm{d}x\Big)^{\frac12} \\
&\le Rt(-\mu'(t))^{\frac12} + \sqrt{2\pi}\exp(\frac{\Phi^{-1}(\mu(t))^2}{2})(-\mu'(t))^{\frac12}\int_{\{|u|>t\}} |f|\varphi - c_0\varphi|u| \mathrm{d}x \\
&\le Rt(-\mu'(t))^{\frac12} + \sqrt{2\pi}\exp(\frac{\Phi^{-1}(\mu(t))^2}{2})(-\mu'(t))^{\frac12}\int_0^{\mu(t)} [f^\star(\sigma) - c_{0\star}^{+}(\sigma)u^\star(\sigma) + \\
&\quad c_0^{-\star}(\sigma)u^\star(\sigma)]\mathrm{d}\sigma. \qquad (55)
\end{aligned}
$$

Using (54) again, we get

$$
\begin{aligned}
\frac{1}{-\mu'(t)} \le& \sqrt{2\pi}R\exp(\frac{\Phi^{-1}(\mu(t))^2}{2})t + 2\pi\exp(\Phi^{-1}(\mu(t))^2)\times \\
&\int_0^{\mu(t)} [f^\star(\sigma) - c_{0\star}^{+}(\sigma)u^\star(\sigma) + c_0^{-\star}(\sigma)u^\star(\sigma)]\mathrm{d}\sigma. \qquad (56)
\end{aligned}
$$

Using the properties of rearrangements, we deduce that

$$
\begin{aligned}
-u^{\star\prime}(s) \le& \sqrt{2\pi}R\exp(\frac{\Phi^{-1}(s)^2}{2})u^\star(s) + 2\pi\exp\left(\Phi^{-1}(s)^2\right)\int_0^s [f^\star(\sigma) - \\
& c_{0\star}^{+}(\sigma)u^\star(\sigma) + c_0^{-\star}(\sigma)u^\star(\sigma)]\mathrm{d}\sigma, \quad s\in[0,\gamma_n(\Omega)]. \qquad (57)
\end{aligned}
$$

Then

$$
\begin{aligned}
-(\exp(-R\Phi^{-1}(s))u^\star(s))' \le& 2\pi\exp(\Phi^{-1}(s)^2)\exp(-R\Phi^{-1}(s))\int_0^s [f^\star(\sigma) - c_{0\star}^{+}(\sigma)u^\star(\sigma) + \\
& c_0^{-\star}(\sigma)u^\star(\sigma)]\mathrm{d}\sigma, \quad s\in[0,\gamma_n(\Omega)]. \qquad (58)
\end{aligned}
$$

Now let us consider "symmetrized" problem (P3). Proceeding in the same way except for that the equalities are now replaced by inequalities, we have

$$
\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{|u|>t\}} |\nabla v|^2\varphi \mathrm{d}x\Big)^{\frac12} = Rt(-\nu'(t))^{\frac12} + \sqrt{2\pi}\exp(\frac{\Phi^{-1}(\nu(t))^2}{2})(-\nu'(t))^{\frac12}\int_0^{\nu(t)} [f^\star(\sigma) -
$$

$$c_{0\star}^{+}(\sigma)v^{\star}(\sigma)+c_0^{-\star}(\sigma)v^{\star}(\sigma)]\mathrm{d}\sigma, \tag{59}$$

where $\nu(t)$ is the distribution function of v. Then

$$-v^{\star\prime}(s)=\sqrt{2\pi}R\exp(\frac{\Phi^{-1}(s)^2}{2})v^{\star}(s)+2\pi\exp(\Phi^{-1}(s)^2)\int_0^s[f^{\star}(\sigma)-$$
$$c_{0\star}^{+}(\sigma)v^{\star}(\sigma)+c_0^{-\star}(\sigma)v^{\star}(\sigma)]\mathrm{d}\sigma,\quad s\in[0,\gamma_n(\Omega)]. \tag{60}$$

That is

$$-(\exp(-R\Phi^{-1}(s))v^{\star}(s))'=2\pi\exp(\Phi^{-1}(s)^2)\exp(-R\Phi^{-1}(s))\int_0^s[f^{\star}(\sigma)-c_{0\star}^{+}(\sigma)v^{\star}(\sigma)+$$
$$c_0^{-\star}(\sigma)v^{\star}(\sigma)]\mathrm{d}\sigma,\quad s\in[0,\gamma_n(\Omega)]. \tag{61}$$

Thus we get the desired results by following the same steps as in [4].

Proof of Theorem 3.2 Firstly, by the same arguments as in Theorem 3.1, we have that

$$u^{\star}(s)=v^{\star}(s),\quad s\in[0,\gamma_n(\Omega)]. \tag{62}$$

Moreover, observing (51) in the proof of Proposition 3.2, we know that

$$\int_0^s\left[f^{\star}(\sigma)-c_{0\star}^{+}(\sigma)v^{\star}(\sigma)+c_0^{-\star}(\sigma)v^{\star}(\sigma)\right]\mathrm{d}\sigma\geq 0,\quad s\in[0,\gamma_n(\Omega)].$$

It follows by using (61) that

$$-\left(\exp\left(-R\Phi^{-1}(s)\right)v^{\star}(s)\right)'\geq 0,\quad s\in[0,\gamma_n(\Omega)]. \tag{63}$$

Since $\exp\left(-R\Phi^{-1}(s)\right)$ is a positive and strictly increasing function on $[0,\gamma_n(\Omega)]$, (63) yields

$$v^{\star\prime}(s)<0,\quad s\in(0,\gamma_n(\Omega)). \tag{64}$$

By observing (55), (59), (62) and (64), we proceed as Case I in Theorem 3.1 and complete the proof. □

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