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Several Classes of Additively Non-Regular Semirings

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Abstract In this paper, we introduce Green's $*$-relations on semirings and define [left, right] adequate semirings to explore additively non-regular semirings. We characterize the semirings which are strong b-lattices of [left, right] skew-halfrings. Also, as further generalization, the semirings are described which are subdirect products of an additively commutative idempotent semiring and a [left, right] skew-halfring. We extend results of constructions of generalized Clifford semirings (given by M. K. Sen, S. K. Maity, K. P. Shum, 2005) and the semirings which are subdirect products of a distributive lattice and a ring (given by S. Ghosh, 1999) to additively non-regular semirings.

Keywords Green's $*$-relations; subdirect product; adequate semiring; skew-halfring.

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A semiring S is an algebraic structure $(S,+,\cdot)$ consisting of a non-empty set S together with two binary operations $+$ and $\cdot$ on S such that $(S,+)$ and $(S,\cdot)$ are semigroups connected by distributivity, that is, $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for all $a,b,c\in S$. The additive identity (if it exists) of a semiring S is called zero and denoted by 0_S. An additively commutative semiring S with a zero satisfying $0\cdot x=x\cdot 0=0$ for all $x\in S$, is called a hemiring. A halfring is a hemiring whose additive reduct is a cancellative monoid. A skew-ring $(S,+,\cdot)$ [8] is a semiring whose additive reduct $(S,+)$ is a group, not necessarily an abelian group. To explore additively non-regular semiring, we introduce the concept of [left, right] skew-halfring which is a semiring whose additive reduct is an additively [left, right] cancellative monoid, not necessarily to be additively commutative. A semiring $(S,+,\cdot)$ is said to be a b-lattice [14] if its additive reduct $(S,+)$ is a semilattice and its multiplication reduct $(S,\cdot)$ is a band. A b-lattice is said to be commutative if the multiplication in $(S,\cdot)$ commutes. A semiring $(S,+,\cdot)$ is called an additive idempotent semiring if its additive reduct $(S,+)$ is a band.

Throughout this paper we denote the set of all additive idempotents [if they exist] of a semiring S by $E^+(S)$. By a subdirect product of two semirings S_1 and S_2 we mean a semiring

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which is isomorphism to a subsemiring T of the semiring S such that the projection maps of T into both S_1 and S_2 are surjective, where S is the direct product of S_1 and S_2.

Since semirings are generalizations of distributive lattices, b-lattices, additive commutative idempotent semirings, rings, skew-rings and [left, right] skew-halfrings, it is interesting to use those semirings to establish constructions of some semirings. Bandelt and Petrich in [2] introduced Bandelt-Petrich Construction in semirings and described the semirings with regular addition which is a subdirect products of a distributive lattice and a ring. Ghosh in [5] established the constructions "strong distributive lattice of semirings" which include the Bandelt-Petrich Construction, and characterized all semirings which are subdirect products of a distributive lattice and a ring. In particular the author introduced the semirings, Clifford semirings [additively commutative inverse semirings such that the set of its additive idempotents is a distributive sublattice as well as a k-ideal], and verified that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of rings, and if and only if it is an inverse subdirect product of a distributive lattice and a ring. Later, in [14], Sen, Maity and Shum defined the Clifford semiring which is a completely regular and an additively inverse semiring such that the set of its additive idempotents is a distributive sublattice as well as a k-ideal [without assuming that its additive reduct is commutative] and proved that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of skew-rings. Moreover, they introduced generalized Clifford semirings which are completely regular and inverse semirings such that its additive idempotent set is a k-ideal, and obtained that a semiring is a generalized Clifford semiring if and only if it is a strong b-lattice of skew-rings, and if and only if it is an additively inverse semiring and is a subdirect product of a b-lattice and a skew-ring. All semirings studied in [2], [5] and [14] are additively regular.

Let $(S, +, \cdot)$ be a semiring. We denote the Green's $\mathcal{H}$-relation on the additive reduct $(S, +)$ by $\overset{+}{\mathcal{H}}$. By Theorems 1.4, 2.5 and 3.3 in [14] we know that $\overset{+}{\mathcal{H}}$ is a congruence on both Clifford semirings and generalized Clifford semirings. McAlister [10] and Pastijn [13] introduced Green's $*$-relations on semigroups. In this paper we introduce Green's $*$-relations on semirings and consider non-regular semirings, [left,right] adequate semirings on which Green's $*$-relation [$\overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}}{}^*$] $\overset{+}{\mathcal{H}}{}^*$ is a congruence. Our purpose is to extend results of generalized Clifford semirings in [14] and the semirings which are subdirect products of a distributive lattice and a ring in [5] to non-regular semirings.

To study the [left, right] adequate semirings in Section 1, we first introduce Green's $*$-relations on semirings and then recall some results of right adequate semigroups and adequate semigroups. In Section 2 we introduce [left, right] adequate semirings and discuss adequate semirings on which Green's $*$-relation $\overset{+}{\mathcal{H}}{}^*$ is a congruence. Section 3 explores right adequate semirings on which Green's $*$-relation $\overset{+}{\mathcal{L}}{}^*$ is a congruence. We do not discuss the left adequate semiring on which Green's $*$-relation $\overset{+}{\mathcal{R}}{}^*$ is a congruence since symmetrically we can obtain the results of left adequate semirings. In Section 4 we characterize the semirings which are a strong b-lattice of [left] skew-halfrings. Section 5 describes the semirings which are subdirect products of an additively commutative idempotent semiring and a [left] skew-halfring.

In this paper, we refer to [3], [4], [6] and [9] for the undefined notions and notations about semigroups.

1. Preliminaries

Let $(S,+,\cdot)$ be any semiring. We denote the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ on additive reduct $(S,+)$ by $\overset{+}{\mathcal{L}}$, $\overset{+}{\mathcal{R}}$, $\overset{+}{\mathcal{H}}$, respectively. These are also equivalence relations on semiring $(S,+,\cdot)$. Now, we introduce Green's $*$-relations $\overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}}{}^*$, $\overset{+}{\mathcal{H}}{}^*$ on the semiring S which are given by: for $a,b \in S$,

$$a\overset{+}{\mathcal{L}}{}^*b \Leftrightarrow \text{for all } x,y \in S^1, a+x=a+y \text{ if and only if } b+x=b+y$$

$$a\overset{+}{\mathcal{R}}{}^*b \Leftrightarrow \text{for all } x,y \in S^1, x+a=y+a \text{ if and only if } x+b=y+b$$

$$\overset{+}{\mathcal{H}}{}^* = \overset{+}{\mathcal{L}}{}^* \cap \overset{+}{\mathcal{R}}{}^*.$$

It is clear that $\overset{+}{\mathcal{L}} \subseteq \overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}} \subseteq \overset{+}{\mathcal{R}}{}^*$ on $(S,+,\cdot)$. In particular, if S is an additively regular semiring, $\overset{+}{\mathcal{L}} = \overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}} = \overset{+}{\mathcal{R}}{}^*$ (see [3]). In general, since the Green's equivalence relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are not congruences on a semigroup [6], Green's $*$-equivalence relations $\overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}}{}^*$ and $\overset{+}{\mathcal{H}}{}^*$ are not congruences on $(S,+,\cdot)$.

Before starting our discussion of several classes of non-regular semirings, we need some concepts for semigroups and some results of [left, right] adequate semigroups. Let $(S,+)$ be a semigroup. We call $(S,+)$ [a right, a left] adequate semigroup if its idempotents commute and every [$\overset{+}{\mathcal{L}}{}^*$-class, $\overset{+}{\mathcal{R}}{}^*$-class] $\overset{+}{\mathcal{L}}{}^*$-class and $\overset{+}{\mathcal{R}}{}^*$-class contain an idempotent (which is, in fact, unique [4]). For an element a of such a semigroup, the idempotent in the $\overset{+}{\mathcal{L}}{}^*$-class [$\overset{+}{\mathcal{R}}{}^*$-class] containing a is denoted by $a^*[a^+]$. A [right, left] adequate semigroup S is called [right, left] type A if $[e+a=a+(e+a)^*, a+e=(a+e)^+ + a]e+a=a+(e+a)^*$ and $a+e=(a+e)^+ + a$ for $a \in S$ and $e \in E^+(S)$. In [9] an equivalent relation $[\sigma_r, \sigma_l]\sigma$ on a [right, left] type A semigroup $(S,+)$ is defined by : for $a,b \in S$, $[a\sigma_r b, a\sigma_l b]a\sigma b$ if and only if there exists an idempotent e in S such that $[a+e=b+e, e+a=e+b]a+e=b+e$. A [right, left] type A semigroup S is said to be proper if and only if $[\sigma_l \cap \overset{+}{\mathcal{L}}{}^* = \iota$, $\sigma_r \cap \overset{+}{\mathcal{R}}{}^* = \iota]\sigma \cap \overset{+}{\mathcal{L}}{}^* = \sigma \cap \overset{+}{\mathcal{R}}{}^* = \iota$, where ι is the identity mapping on S.

Lemma 1.1 *(1) Let $(S,+)$ be a right adequate semigroup with semilattice of idempotents E. If $\mathcal{L}^*$ is a congruence on S and $S/\mathcal{L}^*$ is a semilattice, then $(S/\mathcal{L}^*,+) \cong (E,+)$.*

(2) Let $(S,+)$ be an adequate semigroup with semilattice of idempotents E. If $\mathcal{H}^*$ is a congruence on S and $S/\mathcal{H}^*$ is a semilattice, then every $\mathcal{H}^*$-class contains an idempotent and $(S/\mathcal{H}^*,+) \cong (E,+)$.

Proof The conclusion follows from Lemma 2.5 and Corollary 2.6 in [4]. $\square$

Lemma 1.2 (Lemma 1.12 [4]) *Let H^* be an $\mathcal{H}^*$-class of a semigroup S. If $\mathcal{H}^*$ contains an* *idempotent, then H^* is a cancellative subsemigroup of S with an identity.*

Armstrong [1] defined two partial orders $\leq_l$ and $\leq_r$ on an adequate semigroup as follows:

$$a \leq_l b \text{ if and only if } b = a + e$$

$$a \leq_r b \text{ if and only if } b = e + a$$

for some idempotent e. The partial order $\leq = \leq_l \cap \leq_r$ is called the natural partial order on an adequate semigroup.

2. Adequate semirings on which $\overset{+}{\mathcal{H}^*}$ is a congruence

In this section we first introduce the concepts of [right, left] adequate semirings, then consider adequate semirings on which $\overset{+}{\mathcal{H}^*}$ is a congruence.

Definition 2.1 *Let $(S, +, \cdot)$ be a semiring.*

(1) $(S, +, \cdot)$ is an [a right, a left] adequate if its additive reduct $(S, +)$ is an [a right, a left] adequate semigroup.

(2) $(S, +, \cdot)$ is [right, left] type A if its additive reduct $(S, +)$ is a [right, left] type A semigroup.

(3) $(S, +, \cdot)$ is [right, left] proper type A if its additive reduct $(S, +)$ is a [right, left] proper type A semigroup.

For an element a of a semiring $(S, +, \cdot)$, the additive idempotent in the $\overset{+}{\mathcal{L}^*}$-class containing a will be denoted by a^* and the additive idempotent in the $\overset{+}{\mathcal{R}^*}$-class containing a will be denoted by a^+. For an adequate semiring $(S, +, \cdot)$, $\overset{+}{\leq}$ denotes the partial order determined by $\leq$ on the additive reduct $(S, +)$.

We now give two examples of adequate semirings, one is an adequate semiring on which $\overset{+}{\mathcal{H}^*}$ is a multiplication congruence and the other on which $\overset{+}{\mathcal{H}^*}$ is not.

Example 2.1 *Let $(A, +)$ be the infinite cyclic monoid generated by a with identity 1 and let $(B, +)$ be the free monoid generated by b, c with identity 0. The mapping $\alpha : \{b, c, 0\} \to A$ given by $\alpha(b) = \alpha(c) = a$ and $\alpha(0) = 1$, extends uniquely to a homomorphism $\alpha : B \to A$. Let $S = A \cup B$ with additive operation: for $x \in A, y \in B$, $x + y = x + \alpha(y) = \alpha(y) + x = y + x$. Thus we have an addition on S which extends those on A and B. By Example 2.3 in [9], S is an adequate semigroup with $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^*$.*

(1) Define a multiplication on S as follows:

$$xy = \begin{cases} 0 & \text{for } x, y \in B, \text{ or } x \in A, y \in B, \text{ or } y \in A, x \in B, \\ 1 & \text{for } x, y \in A. \end{cases}$$

Then, $(S, +, \cdot)$ is an adequate semiring on which $\overset{+}{\mathcal{H}^*}$ is a semiring congruence on S.

(2) Define a multiplication on S as follows:

$$xy = \begin{cases} 0 & \text{for any } x, y \in S, x = 0 \text{ or } y = 0, \\ 1 & \text{for any } x, y \in S, x \neq 0, y \neq 0. \end{cases}$$

Then S is an adequate semiring with $\overset{+}{\mathcal{H}}{}^* = \overset{+}{\mathcal{L}}{}^* = \overset{+}{\mathcal{R}}{}^*$. It is easy to check that A and B are the $\overset{+}{\mathcal{L}}{}^*$-classes. Since $x^* = 1$, $y^* = 0$ for $x \in A$, $y \in B$, $x^*y^* = 1 \cdot 0 = 0$. But $(xy)^* = 1$ and $(xy)^* = 1 \neq x^*y^*$. Hence, $\overset{+}{\mathcal{H}}{}^*$ is not a multiplication congruence on S.

Let $(S, +, \cdot)$ be a semiring. We denote the largest congruence on $(S, +, \cdot)$ contained in $\overset{+}{\mathcal{L}}{}^*[\overset{+}{\mathcal{R}}{}^*]$ by $\mu_{\overset{+}{\mathcal{L}}{}^*}[\mu_{\overset{+}{\mathcal{R}}{}^*}]$, and the largest semiring congruence contained in $\overset{+}{\mathcal{H}}{}^*$ is denoted by $\overset{+}{\mu}{}^*$. A congruence δ on a semiring $(S, +, \cdot)$ is said to be a [left, right] skew-halfring congruence if the additive reduct $(S/\delta, +)$ is a [left, right] cancellative monoid.

The semiring $(S, +, \cdot)$ may satisfy some of the following axioms:

(A_1) $a + e = e + a$ for $e \in E^+(S)$ and $a \in S$;

(A_2) $(ab)^* + a^*b^* = a^*b^*$ for $a, b \in S$;

(A_3) $aa^* = a^*$ for $a \in S$;

(A_4) $ab^* = b^*a$ for $a, b \in S$;

(A_5) $a + a^*b = a$ for $a, b \in S$;

(A_6) if $a^* = b^*$ and $a + e = b + e$ for $a, b \in S$ and some $e \in E$, then $a = b$;

(A_6') if there exists c in S for $a, b \in S$ such that $a \overset{+}{\leq} c, b \overset{+}{\leq} c$ and $a^* = b^*$, then $a = b$;

(B_1) $(a + e, e + a) \in \overset{+}{\mathcal{L}}{}^*$ for all $a \in S, e \in E^+(S)$;

(B_1') $(a + e, e + a) \in \overset{+}{\mathcal{R}}{}^*$ for all $a \in S, e \in E^+(S)$.

Lemma 2.1 *Let $(S, +, \cdot)$ be a type A semiring. Then,*

(1) The equivalent relation σ is the minimum cancellative monoid congruence on the additive reduce $(S, +)$;

(2) For $a, b \in S$, $a\sigma b$ if and only if there exists an element c in S such that $a \overset{+}{\leq} c$ and $b \overset{+}{\leq} c$.

(3) σ is the minimum skew-halfring congruence on $(S, +, \cdot)$.

Proof (1) It is from Proposition 1.7 in [9].

(2) It is from the second paragraph in [9, p283].

(3) If $a\sigma b$ for $a, b \in S$, there exists e in $E^+(S)$ such that $a + e = b + e$. Now, for any c in S, since ce and ec are in $E^+(S)$, $ac + ec = bc + ec$ and $ca + ce = cb + ce$. According to (1), σ is the minimum skew-halfring congruence on S. □

Similarly, from the seventh paragraph in [9, p283] we have Lemma 2.2.

Lemma 2.2 *Let $(S, +, \cdot)$ be a right type A semiring. Then,*

(1) The equivalent relation σ_r is the minimum left cancellative monoid congruence on the additive reduce $(S, +)$;

(2) σ_r is the minimum left skew-halfring congruence on $(S, +. \cdot)$.

Lemma 2.3 *Let $(S, +, \cdot)$ be an adequate [a right adequate] semiring. If S satisfies conditions (A_1) and (A_2), then $\overset{+}{\mathcal{H}}{}^* = \overset{+}{\mathcal{L}}{}^* = \overset{+}{\mathcal{R}}{}^*[\overset{+}{\mathcal{L}}{}^*]$ is a congruence on $(S, +, \cdot)$.*

Proof If adequate semiring $(S, +, \cdot)$ satisfies (A_1), by Proposition 2.7 and Proposition 2.9 in

[4], $\overset{+}{\mathcal{H}}^* = \overset{+}{\mathcal{L}}^* = \overset{+}{\mathcal{R}}^*$ is a semilattice congruence on $(S,+)$. To obtain that $\overset{+}{\mathcal{H}}^* = \overset{+}{\mathcal{L}}^* = \overset{+}{\mathcal{R}}^*$ is a multiplication congruence on S, we only prove that, for any $a,b \in S$, $(ab)^* = a^*b = ab^* = a^*b^*$. For any $a,b \in S$, we have

$$\begin{aligned}(ab)^* &= [(a+a^*)(b+b^*)]^* = (ab + a^*b + ab^* + a^*b^*)^* \\ &= (ab)^* + a^*b + ab^* + a^*b^* \quad (\text{by } a^*b, ab^*, a^*b^* \in E^+(S)).\end{aligned}$$

So $(ab)^* + a^*b^* = (ab)^*$. On the other hand, if S satisfies (A_2), that is, $(ab)^* + a^*b^* = a^*b^*$, then $(ab)^* = a^*b^*$. We also have $(ab)^* + a^*b = (ab)^*$ and $a^*b^* + a^*b = a^*b$, which means that $(ab)^* = a^*b$. Similarly, $(ab)^* = ab^*$. Consequently, $(ab)^* = a^*b = ab^* = a^*b^*$.

Similarly, we can prove the case that $(S,+,\cdot)$ is a right adequate semiring. □

Now, we can obtain analogues of Proposition 2.9 in [4] for semirings.

Proposition 2.1 *The following conditions are equivalent on an adequate semiring* $(S,+,\cdot)$ *with* $E = E^+(S)$:

(1) S satisfies (A_1), (A_2);

(2) $\overset{+}{\mu}^*$ is the largest semiring congruence contained in $\overset{+}{\mathcal{H}}^*$ (in this case $\overset{+}{\mathcal{H}}^* = \overset{+}{\mathcal{L}}^* = \overset{+}{\mathcal{R}}^* = \overset{+}{\mu}^*$);

(3) $(S/\overset{+}{\mu}^*, +, \cdot) \cong (E,+,\cdot)$.

Proof (1) $\Rightarrow$ (2). It follows from Lemma 2.3.

(2) $\Rightarrow$ (3). Define a mapping by $\theta : (S/\overset{+}{\mu}^*, +, \cdot) \to (E,+,\cdot)$, $a\overset{+}{\mu}^* \to a^*$. It is routine to check that θ is an isomorphism.

(3) $\Rightarrow$ (1). Assume that $S/\overset{+}{\mu}^* \cong E$. By Proposition 2.9 in [4], we have $\overset{+}{\mu}^* = \overset{+}{\mathcal{H}}^*$, and then $\overset{+}{\mathcal{H}}^*$ is a semiring congruence and E is central on $(S,+)$. Of course, S satisfies (A_1) and $(ab)^* = a^*b^*$, and then $(ab)^* + a^*b^* = a^*b^*$. Therefore (A_1) and (A_2) hold.

If $(S,+\cdot)$ is a right adequate semiring, the proof is in the same way. □

Theorem 2.1 *Let $(S,+,\cdot)$ be an adequate semiring. The following statements are true:*

(1) $S/\overset{+}{\mathcal{H}}^*$ is a semiring with semilattice additive reduct if and only if S satisfies (A_1), (A_2).

(2) $S/\overset{+}{\mathcal{H}}^*$ is a b-lattice if and only if S satisfies (A_1)–(A_3).

(3) $S/\overset{+}{\mathcal{H}}^*$ is a commutative b-lattice if and only if S satisfies (A_1)–(A_4).

(4) $S/\overset{+}{\mathcal{H}}^*$ is a distributive lattice if and only if S satisfies (A_1)–(A_5).

Proof (1) If $S/\overset{+}{\mathcal{H}}^*$ is a semiring with semilattice additive reduct, by Lemma 1.1, $(S/\overset{+}{\mathcal{H}}^*, +) \cong (E^+(S), +)$. Since $E^+(S)$ is an ideal of S, it is clear that $(S/\overset{+}{\mathcal{H}}^*, +, \cdot) \cong (E^+(S), +, \cdot)$ under a mapping $\theta : a\overset{+}{\mathcal{H}}^* \to a^*$. According to Proposition 2.1, S satisfies (A_1), (A_2).

Conversely, the conclusion follows from Proposition 2.1.

(2) Suppose that $S/\overset{+}{\mathcal{H}}^*$ is a b-lattice. Then (A_1) and (A_2) hold. Since $aa^* \in E^+(S)$, $aa^*\overset{+}{\mathcal{H}}^*a^*a^* = a^*$ implies that $aa^* = a^*$, that is, (A_3) holds.

Conversely, let S satisfy (A$_1$)–(A$_3$). Since $(ab)^* = a^*b^*$ and $aa^* = a^*$, we have

$$a\overset{+}{\mathcal{H}}{}^*a^* = aa^*\overset{+}{\mathcal{H}}{}^*a^2.$$

By (1), $S/\overset{+}{\mathcal{H}}{}^*$ is an idempotent semiring with semilattice additive reduct and band multiplicative reduct. Then $S/\overset{+}{\mathcal{H}}{}^*$ is a b-lattice.

(3) If $S/\overset{+}{\mathcal{H}}{}^*$ is a commutative b-lattice, then $a^*b^* = b^*a^*$, and so $ab^* = a^*b^* = b^*a^* = ba^*$. Thus, (A$_4$) holds. Conversely, assume that S satisfies (A$_1$)–(A$_4$). Since

$$\begin{aligned}(a\overset{+}{\mathcal{H}}{}^*)(b\overset{+}{\mathcal{H}}{}^*) &= (a\overset{+}{\mathcal{H}}{}^*)(b^*\overset{+}{\mathcal{H}}{}^*) = (a\overset{+}{\mathcal{H}}{}^*)(b^*\overset{+}{\mathcal{H}}{}^*)\\ &= (ab^*)\overset{+}{\mathcal{H}}{}^* = (b^*a)\overset{+}{\mathcal{H}}{}^* = (b^*\overset{+}{\mathcal{H}}{}^*)(a\overset{+}{\mathcal{H}}{}^*)\\ &= (b\overset{+}{\mathcal{H}}{}^*)(a\overset{+}{\mathcal{H}}{}^*),\end{aligned}$$

$S/\overset{+}{\mathcal{H}}{}^*$ is a commutative b-lattice.

(4) If $S/\overset{+}{\mathcal{H}}{}^*$ is a distributive lattice, then S satisfies (A$_1$)–(A$_4$). Moreover, for $a, b \in S$,

$$\begin{aligned}a + a^*b &= a + a^*b^* = (a + a^*) + a^*b^*\\ &= a + (a^* + a^*b^*) = a + a^* \quad (\text{by} a^*b^* \overset{+}{\leq} a^*)\\ &= a\end{aligned}$$

which proves (A$_5$).

Conversely, assume that S satisfies (A$_1$)–(A$_5$). Since $a + a^*b = a$ implies $(a + a^*b)^* = (a)^*$, by Proposition 1.6 (2) in [4] and $a^*b = a^*b^*$ we have $a^* + a^*b^* = a^* + a^*b = (a + a^*b)^* = a^*$ and $b^*a^* + a^* = a^*b^* + a^* = a^* + a^*b^* = a^*$. Consequently, $S/\overset{+}{\mathcal{H}}{}^*$ is a distributive lattice. □

Definition 2.2 ([14]) *A congruence ρ on a semiring S is called a b-lattice [distributive lattice] congruence if S/ρ is a b-lattice [distributive lattice]. A semiring S is called a b-lattice [distributive lattice] Y of semirings $\{S_\alpha : \alpha \in Y\}$ if S admits a b-lattice [distributive lattice] congruence ρ on S such that $Y = S/\rho$ and each S_α is a ρ-class.*

The following corollary follows from Lemma 1.12 [4], Lemma 1.1, Proposition 2.1 and Theorem 2.1.

Corollary *For an adequate semiring S satisfying (A$_1$) and (A$_2$),*

(1) If S satisfies (A$_3$), then S is a b-lattice of skew-halfrings;

(2) If S satisfies (A$_3$) and (A$_4$), then S is a commutative b-lattice of skew-halfrings;

(3) If S satisfies (A$_3$)–(A$_5$), then S is a distributive lattice of skew-halfrings.

3. Right adequate semirings on which $\overset{+}{\mathcal{L}}{}^*$ is a congruence

Let $(S, +, \cdot)$ be a right adequate semiring. In this section we can obtain the results on right adequate semirings analogous to those on adequate semigroups on which $\overset{+}{\mathcal{H}}{}^*$ is a congruence.

Proposition 3.1 *The following conditions are equivalent for a semiring $(S,+,\cdot)$ with $E = E^+(S)$:*

(1) S is right adequate and satisfies (A_1) and (A_2);

(2) S is right type A, and $\overset{+}{\mathcal{L}^*} = \mu_{\overset{+}{\mathcal{L}^*}}$;

(3) S is right type A, and $(S/\mu_{\overset{+}{\mathcal{L}^*}},+,\cdot) \cong (E,+,\cdot)$.

Proof (1) $\Rightarrow$ (2). If a right adequate semiring $(S,+,\cdot)$ satisfies (A_1) and (A_2), then

$$a+(e+a)^* = a+(a+e)^* = a+a^*+e = a+e = e+a.$$

Hence, S is right type A and by Lemma 2.3, $\overset{+}{\mathcal{L}^*} = \mu_{\overset{+}{\mathcal{L}^*}}$.

(2) $\Rightarrow$ (3). If S is right type A and $\overset{+}{\mathcal{L}^*} = \mu_{\overset{+}{\mathcal{L}^*}}$, then by Lemma 1.1 $(S/\mu_{\overset{+}{\mathcal{L}^*}},+) \cong (E,+)$, and by Proposition 2.1 $(S/\mu_{\overset{+}{\mathcal{L}^*}},+,\cdot) \cong (E,+,\cdot)$.

(3) $\Rightarrow$ (1). If S is right type A and $(S/\mu_{\overset{+}{\mathcal{L}^*}},+,\cdot) \cong (E,+,\cdot)$, then $(S,+)$ is a right type A semigroup. According to Corollary 2.8 in [4], $\mu_{\overset{+}{\mathcal{L}^*}} = \overset{+}{\mathcal{L}^*}$ and E is central in $(S,+)$, which means that S satisfies (A_1) and $(ab)^* = a^*b^*$ since $\overset{+}{\mathcal{L}^*}$ is a congruence on the semiring S. We deduce $(ab)^* + a^*b^* = a^*b^*$. Therefore, S satisfies (A_1) and (A_2). □

Following example shows that the condition that S is right type A cannot be deleted in Proposition 3.1 (2) and (3).

Example 3.1 *Let N denote the set of natural numbers and put $I = N \times N$. On $S = N \cup I$ define operations $\oplus$ and $\cdot$ as follows:*

For all $m,n,h,k \in N$. $m \oplus n = m+n$; $m \oplus (h,k) = (m+h,k)$; $(h,k) \oplus m = (h,k+m)$; $(h,k) \oplus (m,n) = (h,k+m+n)$. For all $x,y \in S, xy = 0$. Then, S is a semiring whose additive idempotents are $\{0,(0,0)\}$. $\overset{+}{\mathcal{L}^*}$-classes of S are $\{N,\ I\}$ so that S is right adequate. Notice that $(0,0) \oplus m \neq m \oplus (0,0)$ as $m \neq 0$, and so S does not satisfy (A_1). Since $(0,0) \oplus k = (0,k)$ whereas $k \oplus ((0,0) \oplus k)^* = k \oplus (0,0) = (k,0)$, we see that S is not right type A, while $\overset{+}{\mathcal{L}^*}$ is a semiring congruence on S.

By Proposition 3.1, the proof of following Theorem 3.1 can be proved similarly to Theorem 2.1.

Theorem 3.1 *Let $(S,+,\cdot)$ be a right type A adequate semiring on which $\overset{+}{\mathcal{L}^*}$ is a congruence. Then the following statements are true:*

(1) $S/\overset{+}{\mathcal{L}^*}$ is a b-lattice if and only if S satisfies (A_3);

(2) $S/\overset{+}{\mathcal{L}^*}$ is a commutative b-lattice if and only if S satisfies (A_3) and (A_4);

(3) $S/\overset{+}{\mathcal{L}^*}$ is a distributive lattice if and only if S satisfies (A_3)–(A_5).

Corollary 3.1 *For a right adequate semiring $(S,+,\cdot)$ satisfying (A_1) and (A_2), the following statements are valid.*

(1) If S satisfies (A_3), then S is right type A and is a b-lattice of left skew-halfrings.

(2) If S satisfies (A_3) and (A_4), then S is right type A and is a commutative b-lattice of left

skew-halfrings.

(3) If S satisfies (A_3)–(A_5), then S is right type A and is a distributive lattice of left skew-halfrings.

Now, we consider another class of right adequate semirings on which $\overset{+}{\mathcal{L}^*}$ is also a congruence.

Proposition 3.2 *For a right adequate semiring $(S, +, \cdot)$, the following statements are equivalent:*

(1) S satisfies (B_1) and (A_2);

(2) $\overset{+}{\mathcal{L}^*} = \mu_{\overset{+}{\mathcal{L}^*}}$;

(3) $(S/\mu_{\overset{+}{\mathcal{L}^*}}, +, \cdot) \cong (E^+(S), +, \cdot)$.

Proof (1) $\Rightarrow$ (2). If $(S, +, \cdot)$ satisfies (B_1), $\overset{+}{\mathcal{L}^*}$ is the smallest semilattice congruence on $(S, +)$ by Proposition 2.7 in [4]. Since $\overset{+}{\mathcal{L}^*}$ is a congruence on additive reduce $(S, +)$ and S satisfies (A_2), $\overset{+}{\mathcal{L}^*}$ is also a multiplication congruence on S. Therefore, $\overset{+}{\mathcal{L}^*} = \mu_{\overset{+}{\mathcal{L}^*}}$.

(2) $\Rightarrow$ (3). Clearly, $(S/\mu_{\overset{+}{\mathcal{L}^*}}, +, \cdot) \cong (E^+(S), +, \cdot)$.

(3) $\Rightarrow$ (1). If $(S/\mu_{\overset{+}{\mathcal{L}^*}}, +, \cdot) \cong (E^+(S), +, \cdot)$, then $\overset{+}{\mathcal{L}^*}$ is a semiring congruence on S implies S satisfies (A_2). By Proposition 2.7 in [4] we know that $(a + e, e + a) \in \overset{+}{\mathcal{L}^*}$ for $e \in E^+(S)$ and $a \in S$. Therefore, (B_1) and (A_2) hold on S. $\square$

Theorem 3.2 *Let $(S, +, \cdot)$ be a right adequate semiring. Then the following statements are true.*

(1) If S satisfies (B_1) and (A_2), then $S/\overset{+}{\mathcal{L}^*}$ is a semiring with semilattice additive reduct.

(2) If S satisfies (B_1), (A_2) and (A_3), then $S/\overset{+}{\mathcal{L}^*}$ is a b-lattice.

(3) If S satisfies (B_1), (A_2)–(A_4), then $S/\overset{+}{\mathcal{L}^*}$ is a commutative b-lattice.

(4) If S satisfies (B_1), (A_2)–(A_5), then $S/\overset{+}{\mathcal{L}^*}$ is a distributive lattice.

Proof The proof is the same as that of Theorem 2.1. $\square$

In the dual case, we can obtain the similar results on left adequate semirings which satisfy (B'_1).

Now, let us consider the adequate semiring satisfying (B_1) and (B'_1).

Proposition 3.3 *An adequate semiring $(S, +, \cdot)$ satisfies (B_1) and (B'_1) if and only if S satisfies (A_1).*

Proof Suppose that the adequate semiring S satisfies conditions (B_1) and (B'_1). Then

$$\begin{aligned}
&(a + e, e + a) \in \overset{+}{\mathcal{H}^*} = \overset{+}{\mathcal{L}^*} \cap \overset{+}{\mathcal{R}^*} \text{ for } e \in E^+(S) \text{ and } a \in S \\
&\quad \Rightarrow (a + a^*, a^* + a) \in \overset{+}{\mathcal{H}^*} \text{ and } (a + a^+, a^+ + a) \in \overset{+}{\mathcal{H}^*} \\
&\quad \Rightarrow (a + a^+)\overset{+}{\mathcal{L}^*}(a^* + a^+), (a^* + a)\overset{+}{\mathcal{R}^*}(a^* + a^+) \text{ and} \\
&(a^+ + a)\overset{+}{\mathcal{L}^*}(a^* + a^+), (a + a^*)\overset{+}{\mathcal{R}^*}(a^* + a^+)
\end{aligned}$$

$$\Rightarrow a^*\overset{+}{\mathcal{L}}{}^*a\overset{+}{\mathcal{L}}{}^*(a^*+a^+) \text{ and } a^+\overset{+}{\mathcal{R}}{}^*a\overset{+}{\mathcal{R}}{}^*(a^*+a^+)$$
$$\Rightarrow a^* = a^*+a^+ = a^+.$$

From Proposition 2.9 in [4], $a+e=e+a$ for $e \in E^+(S)$ and $a \in S$. Therefore, S satisfies (A_1).

Conversely, if S is an adequate semiring satisfying (A_1), then we have $a+e=e+a \in \overset{+}{\mathcal{H}}{}^* = \overset{+}{\mathcal{L}}{}^* \cap \overset{+}{\mathcal{R}}{}^*$ for $e \in E^+(S)$ and $a \in S$. Hence, S satisfies (B_1) and (B_1'). □

4. Strong b-lattices of [left] skew-halfrings

In this section, we want to extend results of inverse semirings in [2], [5] and [14] to adequate semirings and right adequate semirings.

Lemma 4.1 ([14]) *If T is a b-lattice, then $\alpha\beta \overset{+}{\leqslant} \alpha+\beta$ for all $\alpha,\beta \in T$.*

Proof Since $\alpha+\beta=(\alpha+\beta)(\alpha+\beta)=\alpha(\alpha+\beta)+\beta(\alpha+\beta)=\alpha+\alpha\beta+\beta\alpha+\beta$, $\alpha\beta \overset{+}{\leqslant} \alpha\beta+\beta\alpha \overset{+}{\leqslant} \alpha+\beta$. □

Let T be a b-lattice. From Lemma 4.1 we know that if, for $\alpha,\beta,\gamma \in T$, $\alpha+\beta \overset{+}{\leqslant} \gamma$, then $\alpha+\beta+\alpha\beta \overset{+}{\leqslant} \gamma$.

Now, let us cite the construction which was introduced in [14].

Definition 4.1 (Definition 2.3 in [14]) *Let T be a b-lattice and $\{S_\alpha : \alpha \in T\}$ be a family of pairwise disjoint semirings which are indexed by the elements of T. For each $\alpha \overset{+}{\leqslant} \beta$ in T, we now embed S_α in S_β via a semiring monomorphism $\phi_{\alpha,\beta}$ satisfying the following conditions:*

(1.1) $\phi_{\alpha,\alpha} = I_{S_\alpha}$, the identity mapping on S_α.

(1.2) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \overset{+}{\leqslant} \beta \overset{+}{\leqslant} \gamma$.

(1.3) $S_\alpha\phi_{\alpha,\gamma}S_\beta\phi_{\beta,\gamma} \subseteq S_{\alpha\beta}\phi_{\alpha\beta,\gamma}$ if $\alpha+\beta \overset{+}{\leqslant} \gamma$, i. e., $\alpha+\beta+\alpha\beta \overset{+}{\leqslant} \gamma$.

On $S=\cup_{\alpha\in T}S_\alpha$, we define addition $+$ and multiplication $\cdot$ for $a \in S_\alpha, b \in S_\beta$, as follows:

$$a+b = a\phi_{\alpha,\alpha+\beta} + b\phi_{\beta,\alpha+\beta}$$

and

$$a\cdot b = c \in S_{\alpha\beta} \ \text{ such that } \ c\phi_{\alpha\beta,\alpha+\beta} = a\phi_{\alpha,\alpha+\beta}\cdot b\phi_{\beta,\alpha+\beta}.$$

We denote the above system by $S=\langle T, S_\alpha, \phi_{\alpha,\alpha+\beta}\rangle$ and call it the strong b-lattice T of the semirings $\{S_\alpha : \alpha \in T\}$.

In an obvious way, we may replace "b-lattice T" in the above definition by "distributive lattice D", $S=\langle D, S_\alpha, \phi_{\alpha,\alpha+\beta}\rangle$ and call it the strong distributive lattice D of the semirings $\{S_\alpha : \alpha \in T\}$.

Lemma 4.2 (Theorem 2.4 in [14]) *The system $S=\langle T, S_\alpha, \phi_{\alpha,\alpha+\beta}\rangle$ defined above is a semiring.*

Ghosh [5] considered the concept of full subdirect products of a distributive lattice and a ring, and proved that subdirect products of a distributive lattice and a ring are full. Now, we introduce the additively full subdirect products of an additive idempotent semiring T and a [right] skew-halfring R.

Definition 4.2 *A subdirect product S of an additive idempotent semiring T and a [left] skew-halfring R is called additively full if $(e, 0_R) \in S$ for every $e \in E^+(T)$.*

Example 4.1 shows a subdirect product of an additive idempotent semiring and a skew-halfring may be not additively full.

Example 4.1 Let $B = \{e, f\}$ such that $e + e = e \cdot e = e$, $f + f = e + f = f + e = f \cdot f = e \cdot f = f \cdot e = f$. Then $(B, +, \cdot)$ is a b-lattice. Let $N^0 = N \cup \{0\}$, where N is the positive natural number set, and "+" and "·" are the usual additive and multiplication on N^0. Further, let $T = \{\{e\} \times N\} \cup \{\{f\} \times N^0\}$. Then T is the subdirect product of B and N^0, but is not full, since $(e, 0) \notin T$. It is clear that T is not the strong b-lattice B of halfrings because $\{\{e\} \times N\}$ does not contain any additive idempotent.

In the following, we want to describe the additive full subdirect products of a b-lattice and a [left] skew-halfring which are a generalization of subdirect products of a b-lattice and a skew-ring in [14].

Theorem 4.1 *The following conditions on a semiring $(S, +, \cdot)$ are equivalent:*

(1) S is an additive full subdirect product of a b-lattice and a skew-halfring.

(2) S is a strong b-lattice of skew-halfrings.

(3) S is adequate and satisfies the conditions (A_1)–(A_3) and (A_6).

(4) S is adequate and satisfies (A_1)–(A_3) and (A_6').

(5) S is proper type A with $E^+(S)$ the central of additive reduct, and satisfies (A_2) and (A_3).

Proof (1) $\Rightarrow$ (2). Assume that S is an additive full subdirect product of a b-lattice T and a skew-halfring R. To prove that S is a strong b-lattice of skew-halfrings, put $R_\alpha = (\{\alpha\} \times R) \cap S$ for $\alpha \in T$. Then R_α is a skew-halfring for each $\alpha \in T$ and $S = \cup_{\alpha \in T} R_\alpha$. Now for each pair $\alpha, \beta \in T$ with $\alpha \overset{+}{\leq} \beta$, we define a mapping $\phi_{\alpha,\beta} : R_\alpha \to R_\beta$ by $(\alpha, r)\phi_{\alpha,\beta} = (\alpha, r) + (\beta, 0_R) = (\beta, r)$. Clearly, $\phi_{\alpha,\beta}$ is a monomorphism satisfying the condition $\phi_{\alpha,\alpha} = I_{R_\alpha}$ and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \overset{+}{\leq} \beta \overset{+}{\leq} \gamma$ for $\alpha, \beta, \gamma \in T$.

Let $\alpha + \beta \overset{+}{\leq} \gamma$, $a = (\alpha, r) \in R_\alpha$ and $b = (\beta, r') \in R_\beta$. Then we have

$$a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') \in R_{\alpha+\beta}$$

and

$$ab = (\alpha, r)(\beta, r') = (\alpha\beta, rr') \in R_{\alpha\beta}.$$

Now, $(a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma}) = (\gamma, r)(\gamma, r') = (\gamma, rr') = (\alpha\beta, rr')\phi_{\alpha\beta,\gamma} = (ab)\phi_{\alpha\beta,\gamma}$ if $\alpha + \beta \overset{+}{\leq} \gamma$. Also,

$$\begin{aligned} a + b &= (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') = (\alpha + \beta, r) + (\alpha + \beta, r') \\ &= a\phi_{\alpha,\alpha+\beta} + b\phi_{\beta,\alpha+\beta} \end{aligned}$$

and

$$
\begin{aligned}
(a\phi_{\alpha,\alpha+\beta})(b\phi_{\beta,\alpha+\beta}) &= (\alpha+\beta, r)(\alpha+\beta, r') = (\alpha+\beta, rr') \\
&= (\alpha\beta, rr')\phi_{\alpha\beta,\alpha+\beta} = (ab)\phi_{\alpha\beta,\alpha+\beta}.
\end{aligned}
$$

Therefore, S is a strong b-lattice of skew-halfrings R_α, i.e., $S = \langle T, R_\alpha, \phi_{\alpha,\alpha+\beta}\rangle$.

(2) $\Rightarrow$ (3). Suppose that S is a strong b-lattice T of skew-halfrings $\{R_\alpha : \alpha \in T\}$. First, we prove that the additive reduce $(S,+)$ is adequate. If $a+x = a+y$ for $a \in R_\alpha$, $x \in R_\beta$ and $y \in R_\gamma$, then $a\phi_{\alpha,\alpha+\beta} + x\phi_{\beta,\alpha+\beta} = a\phi_{\alpha,\alpha+\gamma} + y\phi_{\gamma,\alpha+\gamma}$. Since $R_{\alpha+\beta} = R_{\alpha+\gamma}$ and $R_{\alpha+\beta}$ is cancellative, we deduce that $x\phi_{\beta,\alpha+\beta} = y\phi_{\gamma,\alpha+\gamma}$ and so $0_\alpha\phi_{\alpha,\alpha+\beta} + x\phi_{\beta,\alpha+\beta} = 0_\alpha\phi_{\alpha,\alpha+\gamma} + y\phi_{\gamma,\alpha+\gamma}$, that is, $0_\alpha + x = 0_\alpha + y$. Similarly, $0_\alpha + x = 0_\alpha + y$ implies $a + x = a + y$, which shows that $a\overset{+}{\mathcal{L}}{}^*0_\alpha$. Therefore, every $\overset{+}{\mathcal{L}}{}^*$-class contains an idempotent. Symmetrically, every $\overset{+}{\mathcal{R}}{}^*$-class contains an idempotent. Hence, the additive reduce $(S,+)$ is adequate. Since $(ab)^* \in S_{\alpha\beta}$ for $a \in S_\alpha$ and $b \in S_\beta$, we have $(ab)^* = 0_{\alpha\beta} = 0_\alpha 0_\beta = a^*b^*$. By $aa^* \in S_\alpha \cap E^+(S)$ we have $aa^* = a^*$. Finally, if $a^* = b^*$ and $a + 0_\gamma = b + 0_\gamma$ for some $\gamma \in T$, then $a\phi_{\alpha,\alpha+\gamma} + 0_\gamma\phi_{\gamma,\alpha+\gamma} = b\phi_{\alpha,\alpha+\gamma} + 0_\gamma\phi_{\gamma,\alpha+\gamma}$. Since $R_{\alpha+\gamma}$ is an additive cancellative monoid and $\phi_{\alpha,\alpha+\gamma}$ is injective, we deduce that $a = b$. Consequently, S satisfies the conditions (A_1)–(A_3) and (A_6).

(3) $\Rightarrow$ (4). Assume that an adequate semiring S satisfies the conditions (A_1)–(A_3) and (A_6), we only need to prove that S satisfies the condition (A_6'). By Corollary 2.1, S is a b-lattice T of skew-halfrings $\{R_\alpha : \alpha \in T\}$. Let $a, b \in S$ with $a^* = b^*$. This means that there exists α in T such that a, b is in R_α. If there exists c in R_β such that $a \overset{+}{\leqslant} c$ and $b \overset{+}{\leqslant} c$, then there exist γ, δ in T such that $a + 0_\gamma = b + 0_\delta$, where $\alpha \overset{+}{\leqslant} \gamma$ and $\alpha \overset{+}{\leqslant} \delta$. Hence, $a + (0_\gamma + 0_\delta) = b + (0_\gamma + 0_\delta)$. Since $0_\gamma + 0_\delta = 0_{\gamma+\delta}$, immediately $a = b$ by (A_6).

(4) $\Rightarrow$ (5). We need only to prove that S is proper type A. Since S satisfies (A_1), S is type A with $E^+(S)$ as central on additive reduct. According to Lemma 2.3, $\overset{+}{\mathcal{R}}{}^* = \overset{+}{\mathcal{L}}{}^* = \overset{+}{\mathcal{H}}{}^*$. If $a(\sigma \cap \overset{+}{\mathcal{H}}{}^*)b$ for $a, b \in S$, then $a^* = b^*$ and there exists e in $E^+(S)$ such that $a + e = b + e$. Put $c = a + e$. Then $a \overset{+}{\leqslant} c$ and $b \overset{+}{\leqslant} c$. According to (A_6'), $a = b$, which means that $\sigma \cap \overset{+}{\mathcal{L}}{}^* = \sigma \cap \overset{+}{\mathcal{R}}{}^* = \iota_S$. Therefore, S is proper type A.

(5) $\Rightarrow$ (1). If a semiring S is proper type A with $E^+(S)$ the central of additive reduct, and satisfies (A_2) and (A_3), then S is adequate and satisfies the condition (A_1), (A_2) and (A_3). By Theorem 2.1 (2), $\overset{+}{\mathcal{R}}{}^* = \overset{+}{\mathcal{L}}{}^* = \overset{+}{\mathcal{H}}{}^*$ is a b-lattice congruence on S. Since S is proper type A, $\sigma \cap \overset{+}{\mathcal{H}}{}^* = \iota_S$. In view of Lemma 2.1 (3), σ is a congruence on the semiring S. Then S can be embedded in direct product $S/\sigma \times S/\overset{+}{\mathcal{H}}{}^*$ under the mapping $a \to (a\sigma, a\overset{+}{\mathcal{H}}{}^*)$ and both projections from S to S/σ and S to $S/\overset{+}{\mathcal{H}}{}^*$ are onto. Consequently, S is a subdirect product of a b-lattice and a skew-halfring. $\square$

Clearly, it is valid for additive full subdirect products of a commutative b-lattice and a [left] skew-halfring and additive full subdirect products of a distributive lattice and a [left] skew-halfring.

Corollary 4.1 *The following statements are equivalent on a semiring S:*

(1) *S is an additive full subdirect product of a commutative b-lattice and a skew-halfring.*

(2) *S is adequate and satisfies (A_1)–(A_4) and (A_6).*

(3) *S is adequate and satisfies (A_1)–(A_4) and (A'_6).*

(4) *S is proper type A with $E^+(S)$ as central on additive reduct, and it satisfies (A_2)–(A_4).*

Proof The conclusions follow from Theorem 2.1 (3) and Theorem 4.1. □

Corollary 4.2 *The following statements are equivalent on a semiring S:*

(1) *S is an additive full subdirect product of a distributive lattice and a skew-halfring.*

(2) *S is adequate and satisfies (A_1)–(A_6).*

(3) *S is adequate and satisfies (A_1)–(A'_6).*

(4) *S is proper type A with $E^+(S)$ as central on additive reduct, and it satisfies (A_2)–(A_5).*

Proof The conclusions follow from Theorem 2.1 (4) and Theorem 4.1. □

Let S be a semiring. In [2], [5] and [14] that $E^+(S)$ is a k-ideal plays an important role in studying the structure of Clifford semirings and Generalized Clifford semirings. Recall that an ideal I of a semiring S is a k-ideal if $a \in I$ and either $a + x \in I$ or $x + a \in I$ for some $x \in S$ implies $x \in I$.

Proposition 4.1 *Let $(S, +, \cdot)$ be a type A semiring with $E^+(S)$ as central on additive reduct. If $(S, +, \cdot)$ is proper, then $E^+(S)$ is a k-ideal.*

Proof Assume that $(S, +, \cdot)$ is proper type A with $E^+(S)$ as central on additive reduct. It is clear that $E^+(S)$ is an ideal of S. If $a + x = e \in E^+(S)$ for $a \in S$ and $x \in E^+(S)$, then $a + (x + e) = a + a^* + (x + e) = a^* + (a + x) + e = a^* + e$ from which we get $a\sigma a^*$. But $a\overset{+}{\mathcal{L}^*}a^*$ together with the fact that S is proper yields $a = a^* \in E^+(S)$. Hence, $E^+(S)$ is a k-ideal of S.□

Example 4.2 shows that the converse of Proposition 4.1 is not true. Hence, the condition of Theorem 4.1 (5) that type A semiring S is proper cannot be replaced by that $E^+(S)$ is a k-ideal.

Example 4.2 Let $S = A \cup B$ as Example 2.1. We give the new multiplication $\circ$ on S:

$$x \circ y = \begin{cases} 1 & \text{for } x, y \in A, \\ 0 & \text{for } x, y \in B, \\ 0 & \text{for } x \in A, y \in B \text{ or } y \in A, x \in B. \end{cases}$$

It is not difficult to check that S is a type A semiring with $E^+(S)$ as central on additive reduct and satisfies (A_2) and (A_3). Clearly, $E^+(S) = \{1, 0\}$ is an ideal of S. If $x + e \in E^+(S)$ for $e \in E^+(S)$, then $x \in E^+(S)$, that is, $E^+(S)$ is a k-ideal. On the other hand, since $\alpha(b) = a = \alpha(c)$ and $b\overset{+}{\mathcal{L}^*}c$ imply that $b(\sigma \cap \overset{+}{\mathcal{L}^*})c$, we know that S is not proper.

By using Lemmas 2.2 and 2.3 it is not difficult to prove the following results of right adequate semirings.

Theorem 4.2 *The following conditions on a semiring $(S, +, \cdot)$ are equivalent:*

(1) *S is an additive full subdirect product of a b-lattice and a left skew-halfring.*

(2) S is a strong b-lattice of left skew-halfrings.

(3) S is right adequate and satisfies the conditions (A_1)–(A_4) and (A_6).

(4) S is proper right type A with $E^+(S)$ as central on additive reduct, and satisfies (A_2) and (A_3).

Corollary 4.3 *The following statements are equivalent on a semiring S:*

(1) S is an additive full subdirect product of a commutative b-lattice and a left skew-halfring.

(2) S is right adequate and satisfies (A_1)–(A_4) and (A_6).

(3) S is proper right type A with $E^+(S)$ as central on additive reduct, and it satisfies (A_2)–(A_4).

Corollary 4.4 *The following statements are equivalent on a semiring S:*

(1) S is an additive full subdirect product of a distributive lattice and a left skew-halfring.

(2) S is right adequate and satisfies (A_1)–(A_6).

(3) S is proper right type A with $E^+(S)$ as central on additive reduct, and it satisfies (A_2)–(A_5).

5. Subdirect products of an additively commutative idempotent semiring and a [left] skew-halfring

A semiring is called an additively commutative idempotent semring if its additive reduct is a semilattice. In this section, we want to extend several results in section 5 to the full subdirect products of an additively commutative idempotent semiring and a [left] skew-halfring.

Theorem 5.1 *The following statements are equivalent on a semiring $(S,+,\cdot)$:*

(1) S is an additive full subdirect product of an additively commutative idempotent semiring and a skew-halfring.

(2) S is an adequate semiring and satisfies (A_1), (A_2), (A_6).

(3) S is an adequate semiring and satisfies (A_1), (A_2), (A_6').

(4) S is proper type A with $E^+(S)$ as central on additive reduct, and satisfies (A_2).

Proof (1) $\Rightarrow$ (2). Assume that S is an additive full subdirect product of an additively commutative idempotent semiring T and a skew-halfring R, then $E^+(S) = (T \times \{0_R\}) \cap S$. Since for each $a \in S$ there exist α in T and r in R such that $a = (\alpha, r)$, we have $(\alpha, r) + (\beta, 0_R) = (\beta, 0_R) + (\alpha, r)$, where $(\beta, 0_R) \in E^+(S)$. Hence, S satisfies (A_1). Now, for $x = (\beta, r_1), y = (\beta, r_2) \in S^1$,

$$\begin{aligned} a + x = a + y &\Leftrightarrow (\alpha+\beta, r+r_1) = (\alpha+\gamma, r+r_2) \\ &\Leftrightarrow \alpha+\beta = \alpha+\gamma \text{ and } r_1 = r_2 \\ &\Leftrightarrow (\alpha, 0_R) + (\beta, r_1) = (\alpha, 0_R) + (\beta, r_2) \\ &\Leftrightarrow (\alpha, 0_R) + x = (\alpha, 0_R) + y, \end{aligned}$$

which means that $a\overset{+}{\mathcal{L}}{}^*(\alpha, 0_R) = a^*$. Similarly, $a\overset{+}{\mathcal{R}}{}^*(\alpha, 0_R) = a^+$. That is, S is adequate. Now, for any $a=(\alpha,r),b=(\beta,r')\in S$, $(ab)^*=(\alpha\beta,0_R)=(\alpha,0_R)(\beta,0_R)=a^*b^*$. Hence, (A_2) holds in S. Further, for $a=(\alpha,r),b=(\beta,r')\in S$, $e=(\gamma,0_R)\in E^+(S)$,

$$\begin{aligned}a^*=b^*,\ \text{and } a+e=b+e &\Rightarrow (\alpha,r)+(\gamma,0_R)=(\beta,r')+(\gamma,0_R)\\ &\Rightarrow \alpha=\beta \text{ and } r=r' \Rightarrow (\alpha,r)=(\beta,r')\Rightarrow a=b.\end{aligned}$$

Hence, (A_6) holds in S.

(2) $\Rightarrow$ (3). It is easy to check that if S is adequate and satisfies (A_1), then (A_6) is equivalent to (A_6').

(3) $\Rightarrow$ (4). Assume that S is adequate and satisfies (A_1), (A_2) and (A_6'). Since S satisfies (A_1), S is type A with $E^+(S)$ as central on additive reduct. According to Lemma 2.3, $\overset{+}{\mathcal{R}}{}^*=\overset{+}{\mathcal{L}}{}^*=\overset{+}{\mathcal{H}}{}^*$. Now, assume that S satisfies (A_6'). If $a(\sigma\cap\overset{+}{\mathcal{H}}{}^*)b$ for $a,b\in S$, then $a^*=b^*$ and there exists e in $E^+(S)$ such that $a+e=b+e$. Put $c=a+e$. Then $a\overset{+}{\leqslant}c$ and $b\overset{+}{\leqslant}c$. Hence, $a=b$. It means that $\sigma\cap\overset{+}{\mathcal{L}}{}^*=\sigma\cap\overset{+}{\mathcal{R}}{}^*=\iota_S$. Therefore, S is proper type A.

(4) $\Rightarrow$ (1). Assume that S is proper type A with $E^+(S)$ as central on additive reduct, and satisfies (A_2). By Theorem 2.1 and Lemma 2.1, $S/\overset{+}{\mathcal{H}}{}^*$ is a semiring with semilattice additive reduct and σ is the minimum skew-halfring congruence on S. Moreover, $\sigma\cap\overset{+}{\mathcal{L}}{}^*=\sigma\cap\overset{+}{\mathcal{R}}{}^*=\sigma\cap\overset{+}{\mathcal{H}}{}^*=\iota_S$. Now, we define a mapping $\Psi:S\to S/\sigma\times S/\overset{+}{\mathcal{H}}{}^*$ by $\Psi(a)=(a\sigma,a\overset{+}{\mathcal{H}}{}^*)$. It is a routine calculation that Ψ is monomorphism and the corresponding projective mappings are surjective. It follows that S is a subdirect product of the additively commutative idempotent semiring $S/\overset{+}{\mathcal{H}}{}^*$ and the skew-halfring S/σ. By Lemma 2.1 in [9] $E^+(S)$ is a σ-class, which shows that for any $a\in S$, $a\sigma(a+a)$ implies $a\in E^+(S)$. Moreover, by Proposition 2.9 in [3] every $\overset{+}{\mathcal{H}}{}^*$-class contains an idempotent. Then, for any additive idempotent $\alpha\in E^+(S/\sigma\times S/\overset{+}{\mathcal{H}}{}^*)$ there exists $a\in E^+(S)$ such that $\Psi(a)=\alpha=(a\sigma,a\overset{+}{\mathcal{H}}{}^*)$, which means that $\Psi(E^+(S))\supseteq E^+(S/\sigma\times S/\overset{+}{\mathcal{H}}{}^*)$. Therefore, S is a full subdirect product of the additively commutative idempotent semiring $S/\overset{+}{\mathcal{H}}{}^*$ and the skew-halfring S/σ.

We complete the proof. $\square$

A semiring $(S,+,\cdot)$ can be a full subdirect product of an additively commutative idempotent semiring and a skew-halfring, but $E^+(S)$ may be not multiplicative idempotent.

Example 5.1 *Assume that $T=\{e,f\}$ is a semiring with the addition $+$ and multiplication $\cdot$ as follows: $e+e=e$, $f+f=f$, $e+f=f+e=f$; $e\cdot e=f\cdot f=e\cdot f=f\cdot e=f$. Let U be a skew-halfring. Additive idempotents $E^+(T\times U)$ of the direct product $T\times U$ is not multiplicative idempotent since $e\cdot e=f\neq e$.*

Similarly, we can prove following theorem for right adequate semirings.

Theorem 5.2 *The following statements are equivalent on a semiring $(S,+,\cdot)$:*

(1) S is an additive full subdirect product of an additively commutative idempotent semiring and a left skew-halfring.

(2) S is a right adequate semiring and satisfies (A_1), (A_2), (A_6).

(3) S is proper right type A with $E^+(S)$ as central on additive reduct, and satisfies (A_2).

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