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Nevanlinna Class and Its Integral Representation or Factorization Theorem in Sector and Angular Domain

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Abstract In this paper we give some sufficient conditions for analytic functions which are not identically zero and belong to Nevanlinna class in the sector and angular domain. Moreover, their integral expressions or factorization theorems are obtained.

Keywords Nevanlinna class; integral representation; factorization theorem.

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1. Introduction and Theorems

Suppose $R>1$ and $\alpha\in(0,\frac{\pi}{2})$. $A_\alpha(0,R)=\{z:|z|<R,|\arg z|<\alpha\}$ is a sector with radius R, and $\partial A_\alpha(0,R)=\{z=Re^{\theta i},|\theta|<\alpha\}\cup\{z=re^{\theta i}:0<r<R;\theta\in\{+\alpha,-\alpha\}\}$ is the boundary of $A_\alpha(0,R)$; $B(0,1)$ denotes unit disk, and $\partial B(0,1)$ is its boundary; $A_\alpha=\{z:|\arg z|<\alpha\}$ denotes angular domain, $B^+(0,R)=\{z:|z|<R,Rez>0\}$ is a half-disk with radius R and $C_+=\{z:Rez>0\}$ denotes right half-plane.

Define the conformal mapping as follows: $\phi=\phi_3\circ\phi_2\circ\phi_1$, where

$$\phi_1:z\to z^\beta,\quad \phi_2:z\to-(\frac{z+iR}{z-iR})^2,\quad \phi_3:z\to-\frac{z-\phi_2(1)}{z+\phi_2(1)},$$

ϕ_1 maps $A_\alpha(0,R)=\{z:|z|<R,|\arg z|<\alpha\}$ to half-disk $B^+(0,R^\beta)$; ϕ_2 maps half-disk $B^+(0,R)$ to upper half-plane C_+; ϕ_3 maps upper half-plane C_+ to unit disk $B(0,1)$. Let $\beta=\frac{\pi}{2\alpha}$. After the calculations we obtain:

$$\omega=\phi(z)=-\frac{\phi_2(\phi_1(z))-\phi_2(1)}{\phi_2(\phi_1(z))+\phi_2(1)}=-\frac{z^{2\beta}-R^{2\beta}-z^\beta+R^{2\beta}z^\beta}{z^{2\beta}-R^{2\beta}+z^\beta-R^{2\beta}z^\beta}=-\frac{z^\beta-1}{z^\beta+1}\cdot\frac{z^\beta+R^{2\beta}}{z^\beta-R^{2\beta}},\qquad(1)$$

for $z\in A_\alpha(0,R)$ and $\omega\in B(0,1)$, then $\phi(1)=0$. f belongs to class $\mathbb{N}(B(0,1))$ if f is analytic in $B(0,1)$ and satisfies

$$\sup_{0<r<1}\Big\{\int_0^{2\pi}\log^+|f(re^{\theta i})|\mathrm{d}\theta\Big\}=\lim_{r\to1}\int_0^{2\pi}\log^+|f(re^{\theta i})|\mathrm{d}\theta<\infty.$$

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Assume that D is a simply connected domain in the complex plane C and its boundary ∂D is piecewise smooth curve . According to Riemann Mapping Theorem there exists a conformal mapping $\psi(z)$ that maps D to $B(0,1)$. Moreover, $\psi(z)$ can be extended to the closure $\overline{D}$ of D such that $\psi(z)$ is homeomorphism from $\overline{D}$ to $\overline{B(0,1)}$. Let $f(\omega) = F(\psi^{-1}(\omega))$ for $\omega \in B(0,1)$. If F is analytic in D, f is in $B(0,1)$. $F \in \mathbb{N}(D)$ if $f \in \mathbb{N}(B(0,1))$. f is inner or outer function, Blaschke product or singular inner function in $B(0,1)$, so F is in D. Therefore, for almost all boundary points $\xi \in \partial D$, there exists a nontangential limit function $F(\xi)$ when $F \in \mathbb{N}(D)$. A necessary and sufficient condition for $F \in \mathbb{N}(D)$ is that there exists a harmonic and majorant function of $\log|F(z)|$ (see [3]). Nevanlinna class is a classical topic in the complex analysis field that many authors are interested in. Rosenblum, Rovnyak [1] and Duren [3] stated Nevanlinna class and its property in the unit disk; Horowitz [4] generalized a result from Nevanlinna class to generalized Nevanlinna class; Moreover, Dijksma, Langer, Luger and Shondin [6] considered a factorization of functions in generalized Nevanlinna class N_k; Rovnyak and Sakhnovich [5] characterized some generalized Nevanlinna classes in terms of their integral representations; Lai [2] generalized some theorems in half-disk with radius R and half-plane; On the basis of the research in [1] and [3],we mainly consider Nevanlinna class in the sector and angular domain and obtain a series of results such that [2] is parallel with the particular case in this paper. Our primary outcomes are the following

Theorem 1 *Suppose $R' > R > 1$ and $F(z)$ is analytic in $A_\alpha(0,R')$. Set $\beta = \frac{\pi}{2\alpha}$. If*

$$\begin{aligned}&\int_{-\alpha}^{\alpha} \cos\beta\theta \log^+|F(Re^{\theta i})|\mathrm{d}\theta + \varliminf_{\epsilon\to 0}\int_0^R (\log^+|F(re^{\alpha i}+\epsilon)| + \log^+|F(re^{-\alpha i}+\epsilon)|)\frac{r^{\beta-1}}{r^{2\beta}+1}\mathrm{d}r \\ &< \infty, \end{aligned}\tag{2}$$

then $F(z) \in \mathbb{N}(A_\alpha(0,R))$.

Theorem 2 *Suppose $R' > R > 1$ and $F(z)$ is analytic in $A_\alpha(0,R')$. Set $\beta = \frac{\pi}{2\alpha}$ and Λ_R is the zero's set for $F(z)$ on the $A_\alpha(0,R)$, including repetitions for multiplicities. If $F(z) \in \mathbb{N}(A_\alpha(0,R))$, there exists a singular measure μ such that*

$$\begin{aligned}&\int_{-\alpha}^{\alpha} \cos\beta\theta \log^+|F(Re^{\theta i})|\mathrm{d}\theta + \int_0^R |\log|F(re^{\alpha i})||r^{\beta-1}\mathrm{d}r + \\ &\int_0^R |\log|F(re^{-\alpha i})||r^{\beta-1}\mathrm{d}r + |\mu|(\partial A_\alpha(0,R)\setminus E) < \infty\end{aligned}\tag{3}$$

and

$$\sum_{\lambda_n\in\Lambda_R} \frac{Re{\lambda_n}^\beta}{1+|\lambda_n|^{2\beta}} < +\infty, \tag{4}$$

where $E = \{z : |z| = R, |\arg z| < \alpha\}$. Moreover, for all $z \in A_\alpha(0,R)\setminus\Lambda_R$ we have

$$\begin{aligned}\log|F(z)| =&\frac{\beta}{2\pi}\int_{-\alpha}^{\alpha}\Big(\frac{R^{2\beta}-|z|^{2\beta}}{|R^\beta e^{\beta\theta i}-z^\beta|^2} - \frac{R^{2\beta}-|z|^{2\beta}}{|R^\beta e^{-\beta\theta i}+z^\beta|^2}\Big)\log|F(Re^{\theta i})|\mathrm{d}\theta + \\ &\frac{\beta}{\pi}\int_0^R\Big(\frac{Rez^\beta}{|ir^\beta - z^\beta|^2} - \frac{R^{2\beta}Rez^\beta}{|iz^\beta r^\beta + R^{2\beta}|^2}\Big)\log|F(re^{\alpha i})|\mathrm{d}r +\end{aligned}$$

$$\frac{\beta}{\pi}\int_0^R \Big(\frac{Rez^\beta}{|ir^\beta+z^\beta|^2}-\frac{R^{2\beta}Rez^\beta}{|iz^\beta r^\beta-R^{2\beta}|^2}\Big)\log|F(re^{-\alpha i})|\mathrm{d}r+$$
$$\sum_{\lambda\in\Lambda_R}\log\Big|\frac{z^\beta-\lambda^\beta}{z^\beta+\overline{\lambda}^\beta}\cdot\frac{R^{2\beta}+\lambda^\beta z^\beta}{R^{2\beta}-\overline{\lambda}^\beta z^\beta}\Big|+\frac{\beta}{\pi}\int_0^R\Big(\frac{Rez^\beta}{|ir^\beta-z^\beta|^2}-\frac{R^{2\beta}Rez^\beta}{|iz^\beta r^\beta+R^{2\beta}|^2}\Big)\mathrm{d}\mu(r)+$$
$$\frac{\beta}{\pi}\int_0^R \Big(\frac{Rez^\beta}{|ir^\beta+z^\beta|^2}-\frac{R^{2\beta}Rez^\beta}{|iz^\beta r^\beta-R^{2\beta}|^2}\Big)\mathrm{d}\mu(r). \tag{5}$$

Theorem 3 *Suppose $F(z)$ is analytic in A_α. Set $\beta=\frac{\pi}{2\alpha}$ and*

$$\begin{aligned}H_R(z)=&\frac{1}{R^\beta}\int_{-\alpha}^{\alpha}\cos\beta\theta\log^+|F(Re^{\theta i})|\mathrm{d}\theta+\\ &\lim_{\epsilon\to0}\int_0^R(\log^+|F(re^{\alpha i}+\epsilon)|+\log^+|F(re^{-\alpha i}+\epsilon)|)\frac{r^{\beta-1}}{r^{2\beta}+1}\mathrm{d}r.\end{aligned} \tag{6}$$

If

$$\lim_{R\to\infty}H_R(z)<+\infty, \tag{7}$$

then $F(z)\in\mathbb{N}(A_\alpha)$.

Theorem 4 *Suppose $F(z)$ is analytic in A_α. Set $\beta=\frac{\pi}{2\alpha}$ and Λ is the zero's set for $F(z)$ on the A_α, including repetitions for multiplicities. If $F(z)\in\mathbb{N}(A_\alpha)$, $F(z)$ has a factorization as follows*

$$F(z)=G_1(z)B_1(z)S_1(z),$$

where

$$\begin{aligned}G_1(z)=&C\exp\Big\{\frac{-\beta}{\pi}\int_0^\infty(\frac{ir^\beta}{r^{2\beta}+1}+\frac{1}{ir^\beta-z^\beta})r^{\beta-1}\log|F(re^{\alpha i})|\mathrm{d}r\Big\}\cdot\\ &\exp\Big\{\frac{\beta}{\pi}\int_0^\infty(\frac{ir^\beta}{r^{2\beta}+1}+\frac{1}{ir^\beta+z^\beta})r^{\beta-1}\log|F(re^{-\alpha i})|\mathrm{d}r\Big\}\end{aligned} \tag{8}$$

is outer function, where $|C|=1$. Moreover

$$\int_0^\infty(|\log|F(re^{\alpha i})||+|\log|F(re^{-\alpha i})||)\frac{r^{\beta-1}}{r^{2\beta}+1}\mathrm{d}r<\infty, \tag{9}$$

here $F(re^{\gamma i})$ is nontangential boundary function of $F(z)$ for $\gamma\in\{-\alpha,\alpha\}$.

$$B_1(z)=C\prod_{\lambda\in\Lambda}\frac{\overline{\lambda}^\beta-1}{\lambda^\beta+1}\cdot\frac{\lambda^\beta-z^\beta}{\overline{\lambda}^\beta+z^\beta}\cdot\Big|\frac{\lambda^\beta+1}{\lambda^\beta-1}\Big| \tag{10}$$

is Blachke product, where $|C|=1$, and

$$\sum_{\lambda\in\Lambda}\frac{Re\lambda^\beta}{|\lambda|^{2\beta}+1}<\infty, \tag{11}$$

$$\begin{aligned}S_1(z)=&Ce^{az^\beta}\exp\Big\{\frac{-1}{2\pi}\int_0^\infty(\frac{ir^\beta}{r^{2\beta}+1}+\frac{1}{ir^\beta-z^\beta})\mathrm{d}\nu(r)\Big\}\cdot\\ &\exp\Big\{\frac{1}{2\pi}\int_0^\infty(\frac{ir^\beta}{r^{2\beta}+1}+\frac{1}{ir^\beta+z^\beta})\mathrm{d}\nu(r)\Big\}\end{aligned} \tag{12}$$

is the quotient of two singular functions, where $a\in\mathbb{R}_+$, $|C|=1$ and ν denotes signed singular measure on $\mathbb{R}_+$.

2. Proof of Theorems

Proof of Theorem 1 Set $f(\omega) = F(\phi^{-1}(\omega))$ for $\omega \in B(0,1)$ and $\epsilon + A_\alpha(0,R) = \{\epsilon + z : z \in A_\alpha(0,R)\}$ for $0 < \epsilon < R' - R$. Therefore $\forall \rho \in (0,1)$, we have

$$\begin{aligned}
&\int_{|\omega|=\rho} \log^+ |f(\omega)||\mathrm{d}\omega| \le \varliminf_{\epsilon\to 0}\int_{|\omega|=1} \log^+ |F(\phi^{-1}(\omega)+\epsilon)||\mathrm{d}\omega| \\
&= \varliminf_{\epsilon\to 0}\int_{-\alpha}^{\alpha} \log^+ |F(Re^{\theta i}+\epsilon)||\phi'(Re^{\theta i})|R\mathrm{d}\theta + \\
&\quad \varliminf_{\epsilon\to 0}\int_0^R (\log^+ |F(re^{\alpha i}+\epsilon)||\phi'(re^{\alpha i})| + \log^+ |F(re^{-\alpha i}+\epsilon)||\phi'(re^{-\alpha i})|)\mathrm{d}r.
\end{aligned}$$

Because $\log^+ |F(\phi^{-1}(\omega)+\epsilon)|$ is subharmonic in the domain containing $\overline{B}(0,1)$, the above inequality holds. $R' > R > 1$ and $F(z)$ is analytic in $A_\alpha(0,R')$, hence

$$\varliminf_{\epsilon\to 0}\int_{-\alpha}^{\alpha} \log^+ |F(Re^{\theta i}+\epsilon)||\phi'(Re^{\theta i})|R\mathrm{d}\theta = \int_{-\alpha}^{\alpha} \log^+ |F(Re^{\theta i})||\phi'(Re^{\theta i})|R\mathrm{d}\theta.$$

By (1) we get

$$\phi'(z) = \frac{2\beta(R^{2\beta}-1)(z^{2\beta}+R^{2\beta})z^{\beta-1}}{(z^\beta+1)^2(z^\beta-R^{2\beta})^2};$$

$$|\phi'(Re^{\theta i})| = \frac{4\beta(R^{2\beta}-1)R^{\beta-1}\cos\beta\theta}{|R^\beta e^{\beta\theta i}+1|^2|e^{\beta\theta i}-R^\beta|^2};$$

$$|\phi'(re^{\alpha i})| = |\phi'(re^{-\alpha i})| = \frac{2\beta(R^{2\beta}-1)(R^{2\beta}-r^{2\beta})r^{\beta-1}}{(R^{4\beta}+r^{2\beta})(r^{2\beta}+1)}.$$

So

$$\begin{aligned}
&\int_{|\omega|=\rho} \log^+ |f(\omega)||\mathrm{d}\omega| \le \frac{4\beta R^\beta}{R^{2\beta}-1}\int_{-\alpha}^{\alpha} \log^+ |F(Re^{\theta i})|\cos\beta\theta\mathrm{d}\theta + \\
&2\beta\varliminf_{\epsilon\to 0}\int_0^R (\log^+ |F(re^{\alpha i}+\epsilon)| + \log^+ |F(re^{-\alpha i}+\epsilon)|)\frac{r^{\beta-1}}{r^{2\beta}+1}\mathrm{d}r.
\end{aligned}$$

Therefore

$$\sup_{0<\rho<1}\int_{|\omega|=\rho} \log^+ |f(\omega)||\mathrm{d}\omega| < \infty,$$

then $f(\omega) \in \mathbb{N}(B(0,1))$, i.e., $F(z) \in \mathbb{N}(A_\alpha(0,R))$.

Proof of Theorem 2 1) If $F(z) \in \mathbb{N}(A_\alpha(0,R))$ and $f(\omega) = F(\phi^{-1}(\omega))$, $f(\omega) \in \mathbb{N}(B(0,1))$. Without loss of generality we assume $f(0) \ne 0$. Therefore $f(\omega)$ has a factorization as follows

$$f(\omega) = G(\omega)B(\omega)S(\omega),$$

where

$$G(\omega) = C\exp\Big(\frac{1}{2\pi}\int_{\partial B(0,1)} \frac{\xi+\omega}{\xi-\omega}\log|f(\xi)||\mathrm{d}\xi|\Big)$$

is outer function in unit disk $B(0,1)$, $|C| = 1$ and $\log|f(\xi)| \in L^1(\partial B(0,1))$;

$$B(\omega) = C\prod_n \frac{a_n-\omega}{1-\overline{a}_n\omega}\cdot\frac{\overline{a_n}}{|a_n|}$$

is Blaschke product in unit disk $B(0,1)$, where $|C|=1$, $\{a_n=\phi(\lambda_n): n\in N\}\subset B(0,1)$ is the zero's set for function $f(\omega)$ and satisfies

$$\sum_{n=1}^{+\infty}(1-|a_n|)<+\infty; \tag{13}$$

$$S(\omega)=C\exp\Big(\frac{1}{2\pi}\int_{\partial B(0,1)}\frac{\xi+\omega}{\xi-\omega}\mathrm{d}\mu_1(\xi)\Big)$$

is the quotient of two singular inner functions in unit disk $B(0,1)$, μ_1 is the difference of two nonnegative singular measures in unit disk $B(0,1)$ and total variation measure $|\mu_1|$ satisfies the condition $|\mu_1|\{\partial B(0,1)\}<\infty$.

For outer function $G(\omega)$,

$$\begin{aligned}\log G(\phi(z)) =&\frac{1}{2\pi}\int_{\partial B(0,1)}\frac{\xi+\phi(z)}{\xi-\phi(z)}\log|f(\xi)||\mathrm{d}\xi|\\ =&\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{\phi(Re^{\theta i})+\phi(z)}{\phi(Re^{\theta i})-\phi(z)}|\phi'(Re^{\theta i})|\log|F(Re^{\theta i})|R\mathrm{d}\theta+\\ &\frac{1}{2\pi}\int_0^R\frac{\phi(re^{\alpha i})+\phi(z)}{\phi(re^{\alpha i})-\phi(z)}|\phi'(re^{\alpha i})|\log|F(re^{\alpha i})|\mathrm{d}r+\\ &\frac{1}{2\pi}\int_0^R\frac{\phi(re^{-\alpha i})+\phi(z)}{\phi(re^{-\alpha i})-\phi(z)}|\phi'(re^{-\alpha i})|\log|F(re^{-\alpha i})|\mathrm{d}r.\end{aligned}$$

Since

$$Re\frac{\phi(Re^{\theta i})+\phi(z)}{\phi(Re^{\theta i})-\phi(z)}=\frac{|\phi(Re^{\theta i})|^2-|\phi(z)|^2}{|\phi(Re^{\theta i})-\phi(z)|^2}=\frac{1-|\phi(z)|^2}{|\phi(Re^{\theta i})-\phi(z)|^2},$$

$$Re\frac{\phi(re^{\gamma i})+\phi(z)}{\phi(re^{\gamma i})-\phi(z)}=\frac{|\phi(re^{\gamma i})|^2-|\phi(z)|^2}{|\phi(re^{\gamma i})-\phi(z)|^2}=\frac{1-|\phi(z)|^2}{|\phi(re^{\gamma i})-\phi(z)|^2},$$

where $\gamma\in\{-\alpha,\alpha\}$. Then

$$\begin{aligned}\log|G(\phi(z))| =&\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{1-|\phi(z)|^2}{|\phi(Re^{\theta i})-\phi(z)|^2}|\phi'(Re^{\theta i})|\log|F(Re^{\theta i})|R\mathrm{d}\theta+\\ &\frac{1}{2\pi}\int_0^R\frac{1-|\phi(z)|^2}{|\phi(re^{\alpha i})-\phi(z)|^2}|\phi'(re^{\alpha i})|\log|F(re^{\alpha i})|\mathrm{d}r+\\ &\frac{1}{2\pi}\int_0^R\frac{1-|\phi(z)|^2}{|\phi(re^{-\alpha i})-\phi(z)|^2}|\phi'(re^{-\alpha i})|\log|F(re^{-\alpha i})|\mathrm{d}r.\end{aligned} \tag{14}$$

For Blaschke product $B(\omega)$,

$$B(\phi(z))=C\prod_{\lambda_n\in\Lambda_R}\frac{\phi(\lambda_n)-\phi(z)}{1-\overline{\phi(\lambda_n)}\phi(z)}\cdot\frac{\overline{\phi(\lambda_n)}}{|\phi(\lambda_n)|},$$

thus

$$\log|B(\phi(z))|=C\sum_{\lambda_n\in\Lambda_R}\log|\frac{\phi(\lambda_n)-\phi(z)}{1-\overline{\phi(\lambda_n)}\phi(z)}|. \tag{15}$$

For singular inner function $S(\omega)$,

$$\log S(\phi(z))=\frac{1}{2\pi}\int_{\partial B(0,1)}\frac{\xi+\omega}{\xi-\omega}\mathrm{d}\mu_1(\xi)$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{\phi(Re^{\theta i})+\phi(z)}{\phi(Re^{\theta i})-\phi(z)}\mathrm{d}\mu_1(\phi(Re^{\theta i}))+\frac{1}{2\pi}\int_0^R\frac{\phi(re^{\alpha i})+\phi(z)}{\phi(re^{\alpha i})-\phi(z)}\mathrm{d}\mu_1(\phi(re^{\alpha i}))+$$
$$\frac{1}{2\pi}\int_0^R\frac{\phi(re^{-\alpha i})+\phi(z)}{\phi(re^{-\alpha i})-\phi(z)}\mathrm{d}\mu_1(\phi(re^{-\alpha i})).$$

Let $\mu_2=\mu_1\circ\phi$, where μ_1 is singular measure in unit disk $B(0,1)$. Then μ_2 is zero measure in the circular arc $\{z:|z|=R,|\arg z|<\alpha\}$. Write $\mathrm{d}\mu_2=|\phi'(re^{\alpha i})|d\mu$. Since $R'>R>1$, $F(z)$ is analytic $A_\alpha(0,R')$, and $|\phi'(re^{\alpha i})|=|\phi'(re^{-\alpha i})|$, we see that

$$\log|S(\phi(z))|=\frac{1}{2\pi}\int_0^R\frac{1-|\phi(z)|^2}{|\phi(re^{\alpha i})-\phi(z)|^2}|\phi'(re^{\alpha i})|\mathrm{d}\mu(r)+\frac{1}{2\pi}\int_0^R\frac{1-|\phi(z)|^2}{|\phi(re^{-\alpha i})-\phi(z)|^2}|\phi'(re^{-\alpha i})|\mathrm{d}\mu(r). \tag{16}$$

By (1) we calculate and know

$$1-|\phi(z)|^2=\frac{4Rez^\beta(R^{2\beta}-1)(R^{2\beta}-|z|^{2\beta})}{|z^\beta+1|^2|R^{2\beta}-z^\beta|^2}, \tag{17}$$

$$\phi(\xi)-\phi(z)=\frac{2(\xi^\beta-z^\beta)(R^{2\beta}-1)(\xi^\beta z^\beta+R^{2\beta})}{(z^\beta+1)(\xi^\beta+1)(z^\beta-R^{2\beta})(\xi^\beta-R^{2\beta})}, \tag{18}$$

$$\phi'(z)=\frac{2\beta z^{\beta-1}(R^{2\beta}-1)(z^{2\beta}+R^{2\beta})}{(z^\beta+1)^2(z^\beta-R^{2\beta})^2}, \tag{19}$$

$$1-\phi(\lambda_n)\phi(z)=\frac{2(R^{2\beta}-1)(z^\beta+{\lambda_n}^\beta)(R^{2\beta}-{\lambda_n}^\beta z^\beta)}{(z^\beta+1)({\lambda_n}^\beta+1)(z^\beta-R^{2\beta})({\lambda_n}^\beta-R^{2\beta})}. \tag{20}$$

Therefore, substituting (17), (18), (19) and (20) into (14), (15) and (16) gives (5).

2) Suppose $R'>R_1>R>1$ and define the conformal mapping ϕ_{R_1} that maps $A_\alpha(0,R_1)$ to $B(0,1)$ as follows

$$\phi_{R_1}(z)=-\frac{z^\beta-1}{z^\beta+1}\cdot\frac{z^\beta+R_1^{2\beta}}{z^\beta-R_1^{2\beta}}. \tag{21}$$

For $\log|F(\zeta)|\in L^1(B(0,1))$, we have

$$\infty>\int_{|\omega|=1}|\log|f(\omega)|||\mathrm{d}\omega|=\int_{\partial A_\alpha(0,R)}|\log|F(z)|||\phi'(z)||\mathrm{d}z|$$
$$\geq\int_{-\alpha}^{\alpha}|\log|F(Re^{\theta i})||\frac{4\beta(R^{2\beta}-1)R^\beta\cos\beta\theta}{(R^{2\beta}+1)^2}\mathrm{d}\theta+$$
$$\int_0^R(|\log|F(re^{\alpha i})||+|\log|F(re^{-\alpha i})||)\frac{2\beta(R^{2\beta}-1)(R^{2\beta}-r^{2\beta})r^{\beta-1}}{(R^{4\beta}+R^{2\beta})(R^{2\beta}+1)}\mathrm{d}r.$$

Hence

$$\int_{-\alpha}^{\alpha}|\log|F(Re^{\theta i})||\cos\beta\theta\mathrm{d}\theta<+\infty;$$

$$\int_0^R(|\log|F(re^{\alpha i})||+|\log|F(re^{-\alpha i})||)\frac{2\beta(R^{2\beta}-1)(R^{2\beta}-r^{2\beta})r^{\beta-1}}{(R^{4\beta}+R^{2\beta})(R^{2\beta}+1)}\mathrm{d}r<+\infty.$$

Similarly

$$\int_0^{R_1}(|\log|F(re^{\alpha i})||+|\log|F(re^{-\alpha i})||)\frac{(R_1^{2\beta}-1)(R_1^{2\beta}-r^{2\beta})r^{\beta-1}}{(R_1^{4\beta}+R_1^{2\beta})(R_1^{2\beta}+1)}\mathrm{d}r<+\infty.$$

So

$$\begin{aligned}\infty &> \int_0^{R_1} (|\log|F(re^{\alpha i})|| + |\log|F(re^{-\alpha i})||)\frac{(R_1^{2\beta}-1)(R_1^{2\beta}-r^{2\beta})r^{\beta-1}}{(R_1^{4\beta}+R_1^{2\beta})(R_1^{2\beta}+1)}\mathrm{d}r\\ &\geq \int_0^{R} (|\log|F(re^{\alpha i})|| + |\log|F(re^{-\alpha i})||)\frac{(R_1^{2\beta}-1)(R_1^{2\beta}-R^{2\beta})r^{\beta-1}}{(R_1^{4\beta}+R_1^{2\beta})(R_1^{2\beta}+1)}\mathrm{d}r.\end{aligned}$$

Thereby

$$\int_0^R (|\log|F(re^{\alpha i})|| + |\log|F(re^{-\alpha i})||)r^{\beta-1}\mathrm{d}r < +\infty.$$

Combining with

$$|\mu_2|(\partial A_\alpha(0,R)\setminus\{z:|z|=R,|\arg z|<\alpha\}) = |\mu_1|(B(0,1)) < +\infty,$$

we get the proof of (3).

3) Combining with (13) and (21), we see

$$\sum_{\lambda_n\in\Lambda_{R_1}}(1-|\phi_{R_1}(\lambda_n)|^2) \leq 2\sum_{\lambda_n\in\Lambda_{R_1}}(1-|\phi_{R_1}(\lambda_n)|) < +\infty.$$

$$1-|\phi_{R_1}(\lambda_n)|^2 = \frac{4Re\lambda_n{}^\beta(R_1^{2\beta}-1)(R_1^{2\beta}-|\lambda_n|^{2\beta})}{|\lambda_n{}^\beta+1|^2|R_1^{2\beta}-\lambda_n{}^\beta|^2}.$$

Therefore

$$\sum_{\lambda_n\in\Lambda_R}\frac{4Re\lambda_n{}^\beta(R_1^{2\beta}-1)(R_1^{2\beta}-|\lambda_n|^{2\beta})}{|\lambda_n{}^\beta+1|^2|R_1^{2\beta}-\lambda_n{}^\beta|^2} \leq \sum_{\lambda_n\in\Lambda_{R_1}}\frac{4Re\lambda_n{}^\beta(R_1^{2\beta}-1)(R_1^{2\beta}-|\lambda_n|^{2\beta})}{|\lambda_n{}^\beta+1|^2|R_1^{2\beta}-\lambda_n{}^\beta|^2} < \infty.$$

Then we see that

$$\sum_{\lambda_n\in\Lambda_R}\frac{4Re\lambda_n{}^\beta(R_1^{2\beta}-1)(R_1^{2\beta}-R^{2\beta})}{|\lambda_n{}^\beta+1|^2|R_1^{2\beta}+R^\beta|^2} < \infty,$$

so

$$\sum_{\lambda_n\in\Lambda_R}\frac{Re\lambda_n{}^\beta}{|\lambda_n|^{2\beta}+1} \leq \sum_{\lambda_n\in\Lambda_R}\frac{2Re\lambda_n{}^\beta}{|\lambda_n{}^\beta+1|^2} < \infty,$$

thus (4) holds.

Proof of Theorem 3 Without loss of generality we assume $F(1)\neq 0$, and there exists an $\epsilon_0 > 0$ such that $F(1+\epsilon)\neq 0$ when $0<\epsilon<\epsilon_0$. Although $F(z+\epsilon)$ is analytic in $A_\alpha(0,R)$, by (6) and (7) we get

$$\varliminf_{\epsilon\to 0}\int_0^R(\log^+|F(re^{\alpha i}+\epsilon)| + \log^+|F(re^{-\alpha i}+\epsilon)|)\frac{r^{\beta-1}}{r^{2\beta}+1}\mathrm{d}r < \infty,$$

so there exists $\{\epsilon_n\}$, $\epsilon_n\to 0$ and positive measure τ that satisfies for all g

$$\lim_{n\to\infty}\int_0^\infty g(t)\log^+|F(re^{\gamma i}+\epsilon_n)|\frac{r^{\beta-1}}{r^{2\beta}+1}\mathrm{d}r = \int_0^\infty g(t)\frac{r^{\beta-1}}{r^{2\beta}+1}\mathrm{d}\tau, \tag{22}$$

where g is continuous in $\mathbb{R}_+$, $\lim_{t\to\infty} g(t) = 0$ and $\gamma\in\{-\alpha,\alpha\}$.

Since (6) and (7) hold, there exists R large enough to satisfy $H_R(z)<\infty$. By Theorem 1 we know $F(z)\in\mathbb{N}(A_\alpha)$. Therefore we applied Theorem 2 to $F(z+\epsilon_n)$ and have

$$\log|F(z+\epsilon_n)| = \frac{\beta}{2\pi}\int_{-\alpha}^{\alpha}\Big(\frac{R^{2\beta}-|z|^{2\beta}}{|R^\beta e^{\beta\theta i}-z^\beta|^2} - \frac{R^{2\beta}-|z|^{2\beta}}{|R^\beta e^{-\beta\theta i}+z^\beta|^2}\Big)\log|F(Re^{\theta i}+\epsilon_n)|\mathrm{d}\theta +$$

$$\frac{\beta}{\pi}\int_0^R\Big(\frac{Rez^\beta}{|ir^\beta-z^\beta|^2}-\frac{R^{2\beta}Rez^\beta}{|iz^\beta r^\beta+R^{2\beta}|^2}\Big)\log|F(re^{\alpha i}+\epsilon_n)|\mathrm{d}r+$$
$$\frac{\beta}{\pi}\int_0^R\Big(\frac{Rez^\beta}{|ir^\beta+z^\beta|^2}-\frac{R^{2\beta}Rez^\beta}{|iz^\beta r^\beta-R^{2\beta}|^2}\Big)\log|F(re^{-\alpha i}+\epsilon_n)|\mathrm{d}r+$$
$$\sum_{\lambda\in\Lambda_R}\log|\frac{z^\beta-\lambda^\beta}{z^\beta+\overline{\lambda}^\beta}\cdot\frac{R^{2\beta}+\lambda^\beta z^\beta}{R^{2\beta}-\overline{\lambda}^\beta z^\beta}|+0.$$

Write

$$g(t)=\begin{cases}R^{2\beta}-t^{2\beta}, & |t|\le R;\\ 0, & |t|>R.\end{cases}$$

Since

$$\sum_{\lambda\in\Lambda_R}\log|\frac{z^\beta-\lambda^\beta}{z^\beta+\overline{\lambda}^\beta}\cdot\frac{R^{2\beta}+\lambda^\beta z^\beta}{R^{2\beta}-\overline{\lambda}^\beta z^\beta}|\le\sum_{\lambda\in\Lambda_R}\log 1\le 0,$$

according to (17), (18) and (19) we know

$$\begin{aligned}\log|F(z)|&=\varliminf_{\epsilon\to 0}\log|F(z+\epsilon_n)|\\
&\le\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{4\beta Rez^\beta(R^{2\beta}-|z|^{2\beta})R^\beta\cos\beta\theta}{|R^\beta e^{\beta\theta i}-z^\beta|^2|R^\beta e^{-\beta\theta i}+z^\beta|^2}\log|F(Re^{\theta i})|\mathrm{d}\theta+\\
&\quad\frac{1}{\pi}\varliminf_{\epsilon_n\to 0}\int_0^\infty\frac{\beta Rez^\beta(R^{2\beta}-|z|^{2\beta})g(r)r^{\beta-1}}{|ir^\beta-z^\beta|^2|z^\beta r^\beta i+R^{2\beta}|^2}\log|F(re^{\alpha i}+\epsilon_n)|\mathrm{d}r+\\
&\quad\frac{1}{\pi}\varliminf_{\epsilon_n\to 0}\int_0^\infty\frac{\beta Rez^\beta(R^{2\beta}-|z|^{2\beta})g(r)r^{\beta-1}}{|ir^\beta+z^\beta|^2|z^\beta r^\beta i-R^{2\beta}|^2}\log|F(re^{-\alpha i}+\epsilon_n)|\mathrm{d}r.\end{aligned}$$

Since $g(t)$ is continuous in $\mathbb{R}_+$ and $g(t)=0$ when $|t|>R$, by (22) we have

$$\begin{aligned}\log|F(z)|\le&\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{4\beta Rez^\beta\cos\beta\theta}{R^\beta}\cdot\frac{(R^{2\beta}-|z|^{2\beta})R^{2\beta}}{|R^\beta e^{\beta\theta i}-z^\beta|^2|R^\beta e^{-\beta\theta i}+z^\beta|^2}\mathrm{d}\tau(r)+\\
&\frac{1}{\pi}\int_0^\infty\frac{r^{\beta-1}}{r^{2\beta}+1}\cdot\frac{\beta Rez^\beta(R^{2\beta}-|z|^{2\beta})(R^{2\beta}-r^{2\beta})(r^{2\beta}+1)}{|ir^\beta-z^\beta|^2|z^\beta r^\beta i+R^{2\beta}|^2}\mathrm{d}\tau(r)+\\
&\frac{1}{\pi}\int_0^\infty\frac{r^{\beta-1}}{r^{2\beta}+1}\cdot\frac{\beta Rez^\beta(R^{2\beta}-|z|^{2\beta})(R^{2\beta}-r^{2\beta})(r^{2\beta}+1)}{|ir^\beta+z^\beta|^2|z^\beta r^\beta i-R^{2\beta}|^2}\mathrm{d}\tau(r).\end{aligned}$$

Letting $R\to+\infty$ and combining with (6) and (7), we obtain

$$\log|F(z)|\le A_1Rez^\beta+A_2\int_0^\infty\frac{Rez^\beta}{|ir^\beta-z^\beta|^2}\mathrm{d}\tau(r)+A_3\int_0^\infty\frac{Rez^\beta}{|ir^\beta+z^\beta|^2}\mathrm{d}\tau(r),$$

here A_1, A_2 and A_3 are some positive numbers independent of z. Since Rez^β, $\int_0^R\frac{Rez^\beta}{|ir^\beta-z^\beta|^2}\mathrm{d}\tau(r)$ and $\int_0^R\frac{Rez^\beta}{|ir^\beta+z^\beta|^2}\mathrm{d}\tau(r)$ are harmonic functions, there exists a harmonic majorant function for $\log|F(z)|$ in A_α, hence $F(z)\in\mathbb{N}(A_\alpha)$.

Proof of Theorem 4 From the proof of Theorem 2, we get $G(\phi(z))$, $B(\phi(z))$ and $S(\phi(z))$. Therefore, letting

$$G_1(z)=\lim_{R\to\infty}G(\phi(z))$$

yields (8).

Define the conformal mapping as follows

$$\psi(z)=\frac{z^\beta-1}{z^\beta+1},$$

then $\psi(z)$ maps A_α to $B(0,1)$. Because $f(\omega)=F(\psi^{-1}(\omega))\in\mathbb{N}(B(0,1))$ and $f(\omega)\in\mathbb{L}^1(\sigma)$,

$$\begin{aligned}\infty &> \int_{-\alpha}^{\alpha}\log|f(e^{\theta i})|\mathrm{d}\theta\\ &=\int_0^\infty(|\log|F(re^{\alpha i})|||\psi'(re^{\alpha i})|+|\log|F(re^{-\alpha i})|||\psi'(re^{-\alpha i})|)\mathrm{d}r\\ &=2\beta\int_0^\infty(|\log|F(re^{\alpha i})||+|\log|F(re^{-\alpha i})||)\frac{r^{\beta-1}}{r^{2\beta}+1}\mathrm{d}r,\end{aligned}$$

then (9) holds.

Letting

$$B_1(z)=\lim_{R\to\infty}B(\phi(z)),$$

results in (11).

By (13), we know

$$\sum_{\lambda_n\in\Lambda}(1-|\psi_1(\lambda_n)|^2)\le 2\sum_{\lambda_n\in\Lambda}(1-|\psi_1(\lambda_n)|)<+\infty,$$

$$1-|\psi_1(\lambda_n)|^2=\frac{4Re{\lambda_n}^\beta}{|{\lambda_n}^\beta+1|^2}\ge\frac{2Re{\lambda_n}^\beta}{|{\lambda_n}^\beta|^2+1},$$

then (11) holds.

Letting

$$S_1(z)=\lim_{R\to\infty}S(\phi(z)),$$

we obtain

$$\begin{aligned}S_1(z)=&Ce^{az^\beta}\exp\Big\{\frac{1}{2\pi}\int_0^\infty\frac{ir^\beta z^\beta-1}{ir^\beta-z^\beta}\mathrm{d}\mu_1(\phi(re^{\alpha i}))\Big\}\cdot\\ &\exp\Big\{\frac{1}{2\pi}\int_0^\infty\frac{ir^\beta z^\beta+1}{ir^\beta+z^\beta}\mathrm{d}\mu_1(\phi(re^{-\alpha i}))\Big\},\end{aligned}$$

where $|C|=1$, and μ_1 denotes signed singular measure on $\partial B(0,1)$. Letting $a=\nu(\partial B(0,1))$ and $\mathrm{d}\nu=(1+r^{2\beta})\mathrm{d}\mu_1(\phi)$ gives (12).

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