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A Note on Star Chromatic Number of Graphs

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Abstract A star coloring of an undirected graph G is a proper coloring of G such that no path of length 3 in G is bicolored. The star chromatic number of an undirected graph G, denoted by $\chi_s(G)$, is the smallest integer k for which G admits a star coloring with k colors. In this paper, we show that if G is a graph with maximum degree Δ, then $\chi_s(G) \le \lceil 7\Delta^{\frac{3}{2}} \rceil$, which gets better bound than those of Fertin, Raspaud and Reed.

Keywords star coloring; star chromatic number; proper coloring.

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1. Introduction

All graphs considered here are undirected graphs. A star coloring of an undirected graph G is a proper coloring (i.e., no two neighbors are assigned the same color) of G such that any path of length 3 in G is not bicolored. The star chromatic number of undirected graph G, denoted by $\chi_s(G)$, is the smallest integer k for which G admits a star coloring with k colors. The terminologies and notations used but undefined in this paper can be found in [2, 3].

Star coloring was introduced in 1973 by Grünbaume [4]. In 2001, Nesetril et al. [5] proved that $\chi_s(G) \le O(\Delta^2)$. In 2004, Albertson et al. [6] proved that $\chi_s(G) \le \Delta(\Delta-1)+2$. In 2004, Fertin et al. [1] proved that $\chi_s(G) \le \lceil 20\Delta^{\frac{3}{2}} \rceil$. In this paper, we extend those results above and give a good bound for $\chi_s(G)$, i.e., we show that if G is a graph with maximum degree Δ, then $\chi_s(G) \le \lceil 7\Delta^{\frac{3}{2}} \rceil$, which is better than any of the above bounds.

2. Lemmas and the main result

Let G^* be a graph with vertex set X. We say that G^* on the set X is a dependency graph for the family of event $(A_x)_{x\in X}$ (i.e., any two events A_x and A_y $(x, y \in X)$ will share an edge in G^* iff they are dependent). Erdös and Lovász [7] proved the following fundamental lemma, namely,

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Lovász Local Lemma.

Lemma 2.1 (Lovász Local Lemma) *Let G^* be a dependency graph for the family of event $(A_x)_{x\in X}$, and suppose $(r_x)_{x\in X}$ are real numbers in $[0,1)$ such that, for each x,*

$$P(A_x) \leq r_x \prod_{y\in\Gamma(x)} (1-r_y),$$

where $\Gamma(x)$ is the set of vertices of G^* adjacent to x. Then,

$$P(\bigcap_{x\in X} \bar{A}_x) \geq \prod_{x\in X}(1-r_x) > 0.$$

Lemma 2.2 ([6]) *If G is a graph with maximum degree Δ, then $\chi_s(G) \leq \Delta(\Delta-1)+2$.*

Theorem 2.1 *Let G be a graph of maximum degree Δ. Then $\chi_s(G) \leq \lceil 7\Delta^{\frac{3}{2}}\rceil$.*

Proof The proof is divided into two cases according to Δ as follows.

Case 1 $\Delta \leq 49$

Suppose that $\Delta \leq 49$. By Lemma 2.2, $\chi_s(G) \leq \Delta(\Delta-1)+2 = \Delta^2-\Delta+2$. Since $\sqrt{\frac{49}{\Delta}} \geq 1$ as $\Delta \leq 49$, $\Delta^2-\Delta+2 \leq \lceil \Delta^2\sqrt{\frac{49}{\Delta}}\rceil = \lceil 7\Delta^{\frac{3}{2}}\rceil$. Thus the result holds in this case.

Case 2 $\Delta > 49$

Now suppose that $\Delta > 49$. Let $k = \lceil 7\Delta^{\frac{3}{2}}\rceil$. We color $V(G)$ with k colors, and the color is independently chosen randomly according to a uniform distribution on $\{1,\dots,k\}$ for each vertex v of $V(G)$. Let φ define this application. Usually, the labeling (or coloring) of vertex x is denoted by $\varphi(x)$. What we want to show here is that with non-zero probability, φ is a star coloring of G. We need define a family of events on which we will apply Lovász Local Lemma. This will imply that with non-zero probability, none of these events occurs. If our events are chosen so that if none of them happens, our coloring is a star coloring of G, and the theorem will be proved.

Now, let us describe the two types of events we have chosen.

Type I For each pair of adjacent vertices x and y in G, let $A_{x,y}$ be the event that $\varphi(x) = \varphi(y)$.

Type II For each path of length 3 $wxyz$ in G, let $B_{w,x,y,z}$ be the event that $\varphi(w) = \varphi(y)$ and $\varphi(x) = \varphi(z)$.

For two types of the events above, now let us come back to our coloring φ. We can easily see that the following two Observations are straightforward.

Observation 2.1 *For each event A of type I, $P(A) = \frac{1}{k}$;*

Observation 2.2 *For each event B of type II, $P(B) = \frac{1}{k^2}$.*

If none of the two events A and B occurs, then φ is star coloring of G by the definition of star coloring. Now, we want to show that with strictly positive probability, none of these two events occurs. We want to apply Lováz Local Lemma, so we construct a graph G^* whose nodes are all the events of the two types, and in which two nodes R_{X_1} and W_{X_2}, $R, W \in \{A, B\}$, are adjacent iff $X_1 \cap X_2 \neq \varnothing$. Since the occurrence of each event R_{X_1} depends only on the color of the vertices in X_1, G^* is a dependency graph for these events, because even if the colors of all vertices of G but those in X_1 are known, the probability of R_{X_1} remains unchanged. Now, if a vertex of G^* corresponds to an event of type i, then it will be said to be of type $i \in \{I, II\}$.

In order to estimate the degree of a vertex of type i in G^*, we need the following lemma that was proved in Fertin et al. [1].

Lemma 2.3 *Let G be a graph with maximum degree Δ and let $u \in V(G)$. Then,*

(i) u belongs to at most Δ edges of G;

(ii) u belongs to at most $2\Delta(\Delta-1)^2$ paths of length 3 in G.

Lemma 2.4 *Let $Y_{i,j}$ be an upper bound on the number of vertices of type j which are adjacent to a vertex of type i (for $i, j \in \{I, II\}$) in the dependency graph G^*. Then*

(1) $Y_{I,I} = 2\Delta$;

(2) $Y_{I,II} = 4\Delta - 2$;

(3) $Y_{II,I} = 4\Delta(\Delta-1)^2$;

(4) $Y_{II,II} = 8\Delta(\Delta-1)^2 - 8$.

Proof (1) It was proved in [1].

(2) Take a vertex w_{II} of vertices of type II in G^*, and let us give an upper bound on the number of type I in G^* that are the neighbors of w_{II}. Since the vertex w_{II} corresponds to an event $B_{w,x,y,z}$, it implies four vertices w, x, y and z in G. Thus according to the definition of the event of the graph G^*, w_{II} is connected to all the vertices (type I) that correspond to events $A_{w,u}$, $A_{x,v}$, $A_{y,p}$, $A_{z,q}$ for all vertices u that are neighbors of w in G, all vertices v that are neighbors of x in G, all vertices p that are neighbors of y in G, and all vertices q that are neighbors of z in G. By Lemma 2.3, there are at most Δ vertices that are neighbor of w in G (resp,of x, y, z in G), and so $Y_{I,II}$ is upperly bounded by $4\Delta - 2$.

(3) Take a vertex x_I of vertices of type I in G^*, and let us give an upper bound on the number of type II in G^* that are the neighbors of x_I. Since the vertex x_I corresponds to an event $A_{x,y}$, it implies two vertices x, y in G. Thus according to the definition of the event of the graph G^*, x_I is connected to all the vertices (type II) that correspond to events $B_{x,o,p,q}$, $B_{o,x,p,q}$, $B_{o,p,x,q}$, $B_{o,p,q,x}$, $B_{y,r,s,t}$, $B_{r,y,s,t}$, $B_{r,s,y,t}$ and $B_{r,s,t,y}$ for all vertices $o, p, q, r, s, t \in V(G)$. By Lemma 2.3, $Y_{II,I}$ is upperly bounded by $Y_{II,I} = 4\Delta(\Delta-1)^2$.

(4) Then as in (2) and (3), the lemma holds. $\square$

Now, in order to apply Lováz Local Lemma, there remains to choose the $r_x(i)$ for $i \in \{I, II\}$ and $x \in X$, where $0 \leq r_x(i) < 1$. For this, we choose that

$$\begin{cases} r_x(\mathrm{I}) = \frac{\sqrt{3}}{k} & \text{for type I} \\ r_x(\mathrm{II}) = \frac{\sqrt{5}}{k^2} & \text{for type II} \end{cases}$$

In order to be able to apply Lováz Local Lemma, it is necessary to prove that:

$$P(A) = \frac{1}{k} \leq \frac{\sqrt{3}}{k}(1 - \frac{\sqrt{3}}{k})^{2\Delta}(1 - \frac{\sqrt{5}}{k^2})^{4\Delta(\Delta-1)^2} \tag{$*$}$$

and

$$P(B)=\frac{1}{k^2}\leq\frac{\sqrt{5}}{k^2}(1-\frac{\sqrt{3}}{k})^{4\Delta-2}(1-\frac{\sqrt{5}}{k^2})^{8\Delta(\Delta-1)^2-8}.$$

Clearly, since $0<1-\frac{\sqrt{3}}{k}<1$ and $0<1-\frac{\sqrt{5}}{k^2}<1$,

$$(1-\frac{\sqrt{3}}{k})^{4\Delta-2}(1-\frac{\sqrt{5}}{k^2})^{8\Delta(\Delta-1)^2-8}\geq(1-\frac{\sqrt{3}}{k})^{4\Delta}(1-\frac{\sqrt{5}}{k^2})^{8\Delta(\Delta-1)^2}.$$

If the inequality $P(B)=\frac{1}{k^2}\leq\frac{\sqrt{5}}{k^2}(1-\frac{\sqrt{3}}{k})^{4\Delta}(1-\frac{\sqrt{5}}{k^2})^{8\Delta(\Delta-1)^2}$ is satisfied, then the inequality $(*)$ is satisfied too. In order to prove that it is satisfied, let

$$N=(1-\frac{\sqrt{3}}{k})^{4\Delta}(1-\frac{\sqrt{5}}{k^2})^{8\Delta^3}\leq(1-\frac{\sqrt{3}}{k})^{4\Delta}(1-\frac{\sqrt{5}}{k^2})^{8\Delta(\Delta-1)^2}.$$

We only need show that $N\geq\frac{1}{\sqrt{5}}$. Since,

$$(1-\frac{\sqrt{3}}{k})^{4\Delta}\geq1-\frac{4\sqrt{3}\Delta}{k};\ \ (1-\frac{\sqrt{5}}{k^2})^{8\Delta^3}\geq1-\frac{8\sqrt{5}\Delta^3}{k^2},$$

$N\geq(1-\frac{4\sqrt{3}\Delta}{k})(1-\frac{8\sqrt{5}\Delta^3}{k^2})$. When $k=\lceil7\Delta^{\frac{3}{2}}\rceil$,

$$N\geq(1-\frac{4\sqrt{3}}{7\sqrt{\Delta}})(1-\frac{8\sqrt{5}}{(7)^2})=(1-\frac{1}{7}\sqrt{\frac{48}{\Delta}})(1-\frac{8\sqrt{5}}{49}).$$

Since $\Delta>49$, $N\geq(1-\frac{1}{7})(1-\frac{8\sqrt{5}}{49})$. It is easy to check that in that case $N>\frac{1}{\sqrt{5}}$ for any $\Delta>49$. Hence by Lováz Local Lemma,

$$P((\bigcap_{x\in X}\bar{A}_x)\bigcap(\bigcap_{x\in X}\bar{B}_x))\geq\prod_{x\in X}(1-r_x(1))(1-r_x(2))>0,$$

which means that φ is a star coloring of G with non-zero probability. The theorem is proved. □

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