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Cover a Tree by Induced Matchings

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Abstract The induced matching cover number of a graph G without isolated vertices, denoted by $\mathrm{imc}(G)$, is the minimum integer k such that G has k induced matchings $M_1, M_2, \ldots, M_k$ such that, $M_1 \cup M_2 \cup \cdots \cup M_k$ covers $V(G)$. This paper shows if G is a nontrivial tree, then $\mathrm{imc}(G) \in \{\Delta_0^*(G), \Delta_0^*(G)+1, \Delta_0^*(G)+2\}$, where $\Delta_0^*(G) = \max\{d_0(u) + d_0(v) : u, v \in V(G), uv \in E(G)\}$.

Keywords induced matching; induced matching cover; tree.

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1. Introduction

Graphs considered in this paper are finite and simple. For a graph G, $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively. The degree of a vertex x in G is denoted by $d_G(x)$. $\Delta(G)$ is used to denote the *maximum degree* of G. For $X \subseteq V(G)$, the neighbor set $N_G(X)$ of X is defined by

$$N_G(X) = \{y \in V(G) \setminus X : \text{ there is } x \in X \text{ such that } xy \in E(G)\}.$$

$N_G(\{x\})$ is written in shorter form as $N_G(x)$ for $x \in V(G)$. The pendant neighbor set $N_0(X)$ of X is defined by

$$N_0(X) = \{y \in N_G(X) : d_G(y) = 1\}.$$

$N_0(\{x\})$ is written in shorter form as $N_0(x)$ for $x \in V(G)$. And we write

$$N_0[X] = N_0(X) \cup X, \quad N_0[x] = N_0(x) \cup \{x\}.$$

For a vertex u in G, the degree, the pendant neighborhood, and the pendant degree of u are denoted by $d(u) = |N(u)|$, $N_0(u) = \{x \in V(G) : xu \in E(G) \text{ and } d_G(x) = 1\}$ and $d_0(u) = |N_0(u)|$, respectively. For $X \subseteq V(G)$, $G[X]$ is used to denote the subgraph of G induced by X. If $I \subseteq V(G)$ such that $E_G(I) = \emptyset$, I is called an independent set of G. For $M \subseteq E(G)$, set

$$V(M) = \{v \in V(G) : \text{ there is } x \in V(G) \text{ such that } vx \in M\}.$$

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$V(\{e\})$ is written in shorter form as $V(e)$ for $e \in E(G)$. $M \subseteq E(G)$ is a matching of G if $V(e) \cap V(f) = \emptyset$ for every two distinct edges $e, f \in M$. A matching M of G is perfect if $V(M) = V(G)$. M is a maximum matching if G has no matching M' with $|M'| > |M|$. A matching M of G is induced if $E_G(V(M)) = M$ (see [1,2]).

A k-partition of a set X is a k-tuple $(X_1, X_2, \ldots, X_k)$ such that $X_1, X_2, \ldots, X_k$ are mutually disjoint subsets of X such that $\cup_{1\le i\le k} X_i = X$. A k-induced-matching partition of a graph G which has a perfect matching is a k-partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that, for each i $(1 \le i \le k)$, the subgraph $G[V_i]$ of G induced by V_i is 1-regular. The induced matching partition number of a graph G, denoted by $\mathrm{imp}(G)$, is the minimum integer k such that G has a k-induced-matching partition. The k-induced-matching partition problem asks whether a given graph G has a k-induced-matching partition or not.

Let $M_1, M_2, \ldots, M_k$ be k induced matchings of G. We say $\{M_1, M_2, \ldots, M_k\}$ is a k-induced-matching cover of G if $V(M_1) \cup \cdots \cup V(M_k) = V(G)$. The induced matching cover number of G, denoted by $\mathrm{imc}(G)$, is defined to be the minimum number k such that G has a k-induced-matching cover. Clearly, $\mathrm{imc}(G)$ is defined on the graphs without isolated vertices. The k-induced-matching cover problem asks whether a given graph G has a k-induced-matching cover or not. Terminologies and notations not defined here can be found in [3].

Some work about k-induced-matching partition problem have been done by investigators [4,5]. For the k-induced-matching cover problem, [6] has shown that 2-induced-matching cover problem of graphs with diameter 6 and 3-induced-matching cover problem of graphs with diameter 2 are NP-complete, and 2-induced-matching cover problem of graphs with diameter 2 is polynomially solvable. The motivation of this paper is originated from the relation between partition and cover. Note that, the induced matching partition problem is based on the condition that graphs have perfect matchings. But the induced matching cover problem is defined on any graph without isolated vertices. So the induced matching cover problem is applicable to more kinds of graphs than the induced matching partition problem. In fact, some graphs have perfect matchings such that induced matching cover number is less than induced matching partition number. For example, the Peterson graph G such that $\mathrm{imp}(G) = 5$ and $\mathrm{imc}(G) = 3$, the Heawood graph H such that $\mathrm{imp}(H) = 4$ and $\mathrm{imc}(H) = 3$ (see [7]). So the research of the induced matching cover problem is necessary.

In this paper we show that $\mathrm{imc}(G) \in \{\Delta_0^*(G), \Delta_0^*(G)+1, \Delta_0^*(G)+2\}$ if G is a nontrivial tree, where $\Delta_0^*(G) = \max\{d_0(u) + d_0(v) : u, v \in V(G), uv \in E(G)\}$.

2. Preliminaries

Let G be a tree with at least three vertices. We define

$$\Delta_0(G) = \max\{d_0(u) : u \in V(G)\},$$

$$\Delta_0^*(G) = \max\{d_0(u) + d_0(v) : u, v \in V(G), uv \in E(G)\}.$$

Lemma 2.1 *Let G be a tree with at least three vertices. Then* $\mathrm{imc}(G) \ge \Delta_0^*(G)$.

Proof By the definition of $\Delta_0^*(G)$, there exist at least two vertices $u, v \in V(G)$ such that

$uv \in E(G)$ and $d_0(u) + d_0(v) = \Delta_0^*(G)$. Since every $x \in N_0(u)$ and $y \in N_0(v)$ cannot belong to the same induced matching, at least $\Delta_0^*(G)$ induced matchings are needed to cover $N_0(u) \cup N_0(v)$. So $\mathrm{imc}(G) \geq \Delta_0^*(G)$. $\square$

Lemma 2.2 *For any tree G with $\Delta_0(G) = 1$, there is a 3-induced-matching cover $\{M_1, M_2, M_3\}$ of G such that each vertex of G is covered by at most two matchings in $\{M_1, M_2, M_3\}$.*

Proof By contradiction. If possible, let G be a tree with $\Delta_0(G) = 1$ and minimum number of vertices such that the conclusion of Lemma 2.2 does not hold. Then G is not a path and so $\Delta(G) \geq 3$. Write $S = \{x \in V(G) : d_G(x) = 1\}$. For each $x \in S$, let $p(x)$ be a pending path such that x is an end-vertex of $p(x)$, and the another end-vertex is the unique vertex in $p(x)$ with degree at least 3 in G. For each x, the length of $p(x)$ is denoted by $|p(x)|$, $|p(x)| = |E_G(p(x))|$. Since $\Delta_0(G) = 1$, we can observe that there exists a vertex $v \in S$ such that $|p(v)| \geq 2$. Let $uv \in E(G)$ be the unique edge incident to v, and $N_G(u) = \{v, w\}$. Set

$$T = \begin{cases} G - v, & \text{if } |p(v)| \geq 3, \\ G - \{u, v\}, & \text{if } |p(v)| = 2. \end{cases}$$

Then T is a tree with $\Delta_0(T) = 1$. By the minimality of G, T has a 3-induced-matching cover $\{M_1, M_2, M_3\}$ such that each vertex of T is covered by at most two matchings in $\{M_1, M_2, M_3\}$. We can choose the three induced matchings such that $|M_1| + |M_2| + |M_3|$ is minimum. Then every vertex x with $d_T(x) = 1$ is covered exactly by one matching in $\{M_1, M_2, M_3\}$. Since either $d_T(u) = 1$ or $u \notin V(T)$, there are two matchings, say, M_1 and M_2, in $\{M_1, M_2, M_3\}$ such that $u \notin V(M_1 \cup M_2)$. Since w is covered by at most two matchings in $\{M_1, M_2, M_3\}$, we can suppose that $w \in V(M_2) \cap V(M_3)$. It follows that $\{M_1 \cup \{uv\}, M_2, M_3\}$ for $|p(x)| \geq 3$ or $\{M_1 \cup \{uw\}, M_2 \cup \{uv\}, M_3\}$ for $|p(x)| = 2$ is a 3-induced-matching cover of G that has the required property. $\square$

Lemma 2.3 *Let G be a tree with at least three vertices and $\Delta_0^*(G) = 1$. Then $\mathrm{imc}(G) \in \{1, 2, 3\}$.*

Proof The conclusion follows from $1 = \Delta_0^*(G) \geq \Delta_0(G) = 1$ and Lemma 2.2. $\square$

Lemma 2.4 *Let G be a tree with at least three vertices. If $V_0 \subseteq V(G)$ such that V_0 is independent, and $d_0(v) \geq 1$ for every $v \in V_0$. Then every matching of $G[N_0[V_0]]$ is an induced matching of G.*

Proof If V_0 is independent, then every component of $G[N_0[V_0]]$ is a star. Since a matching is an induced matching in star, the result follows. $\square$

Lemma 2.5 *Let G be a tree with $\Delta_0^*(G) = 2$. Then $\mathrm{imc}(G) \in \{2, 3, 4\}$.*

Proof Let $V_0 = \{u \in V(G) : d_0(u) = 2\}$. If $V_0 = \emptyset$, then $\Delta_0(G) = 1$. By Lemma 2.2, $\mathrm{imc}(G) \leq 3$. If $V_0 \neq \emptyset$, obviously, V_0 is an independent set.

Let M be a maximum matching of $G[N_0[V_0]]$. By Lemma 2.4, M is an induced matching of G. Let $G' = G \setminus (N_0(V_0) \cap V(M))$. Then $\Delta_0(G') = 1$. By Lemma 2.2, $\mathrm{imc}(G') \leq 3$, so $\mathrm{imc}(G) \leq 3 + 1 = 4$. By Lemma 2.1, the result follows. $\square$

3. Main results

The following theorem is the main result of this paper.

Theorem 3.1 *Let G be a tree with at least three vertices. Then*

$$\mathrm{imc}(G) \in \{\Delta_0^*(G), \Delta_0^*(G)+1, \Delta_0^*(G)+2\},$$

where $\Delta_0^*(G) = \max\{d_0(u) + d_0(v) : u, v \in V(G), uv \in E(G)\}$.

Moreover, if $\Delta_0^*(G) > \Delta_0(G)$, *then* $\mathrm{imc}(G) \in \{\Delta_0^*(G), \Delta_0^*(G)+1\}$.

Proof We will prove the theorem by induction on $\Delta_0^*(G)$.

If $\Delta_0^*(G) \le 2$, the correctness of Theorem 3.1 is implied by Lemmas 2.3 and 2.5.

Let k be a positive integer with $k \ge 3$. Suppose that the results of Theorem 3.1 are true for any tree T with $\Delta_0^*(T) \le k-1$.

Let G be a tree with $\Delta_0^*(G) = k$. Set

$$V_1 = \{u \in V(G) : d_0(u) > \frac{k}{2}\}, \quad V_2 = \{u \in V(G) : d_0(u) = \frac{k}{2}\}.$$

If $V_2 \neq \emptyset$, we define $H = G[V_2]$. Since H is a bipartite graph, we can choose a bipartition (X, Y) of H such that $d_H(y) \ge 1$ for $y \in Y$. Note that, in the case $E(H) = \emptyset$, we have $X = V(H)$ and $Y = \emptyset$. We further make the convention that $X = \emptyset$ if $V_2 = \emptyset$. It can be observed that $W = V_1 \cup X$ is an independent set in G. Let M be a maximum matching in $G[N_0[W]]$. By Lemma 2.4, M is an induced matching in G. Let $T = G - N_0(W) \cap V(M)$. Then $\Delta_0^*(T) = \Delta_0^*(G) - 1 = k-1$. By the induction hypothesis, the results of Theorem 3.1 hold for T. Then $\mathrm{imc}(T) \le \Delta_0^*(T) + 2 = \Delta_0^*(G) + 1$. By adding the induced matching M in consideration, we have $\mathrm{imc}(G) \le \mathrm{imc}(T) + 1 \le \Delta_0^*(G) + 2$. By Lemma 2.1, the results of Theorem 3.1 hold for G. $\square$

It can be observed that, if G is a tree with $\Delta_0^*(G) = \Delta_0(G)$ and G is not a star, then G has no $\Delta_0(G)$-induced-matching cover. Hence, we have

Corollary 3.2 *If G is a tree with $\Delta_0^*(G) = \Delta_0(G)$ and G is not a star, then* $\mathrm{imc}(G) \in \{\Delta_0^*(G)+1, \Delta_0^*(G)+2\}$.

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