Journal of Mathematical Research & Exposition
Sept., 2010, Vol. 30, No. 5, pp. 849–854
DOI:10.3770/j.issn:1000-341X.2010.05.011
Http://jmre.dlut.edu.cn

On X-Semipermutability of Some Subgroups of Finite Groups

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Abstract Let A be a subgroup of a group G and X a nonempty subset of G. A is said to be X-semipermutable in G if A has a supplement T in G such that A is X-permutable with every subgroup of T. In this paper, we try to use the X-semipermutability of some subgroups to characterize the structure of finite groups.

Keywords finite groups; X-semipermutable subgroups; supersoluble groups; p-nilpotent groups.

Document code A

MR(2000) Subject Classification 20D10; 20D15; 20D20

Chinese Library Classification O152.1

1. Introduction

All groups in this paper will be finite.

Let A and B be subgroups of a group G. A is said to be permutable with B if $AB = BA$. A is said to be a permutable subgroup (or quasinormal subgroup) of G if A is permutable with all subgroups of G (see [13]). The permutable subgroups have many interesting properties. For example, Ore [13] proved that every permutable subgroup of a group G is subnormal in G. Ito and Scép [10] proved that for every permutable subgroup H of a group G, H/H_G is nilpotent. However, two subgroups H and T of a group G may not be permutable in G, but G may contain an element x such that $HT^x = T^xH$. Basing on the observation, Guo, Shum and Skiba [4, 5, 7] recently introduced the following generalized permutable subgroups. Let A and B be subgroups of a group G and X a nonempty subset of G. Then A is said to be X-permutable with B if there exists some $x \in X$ such that $AB^x = B^xA$; A is said to be X-permutable (or X-quasinormal) in G if for every subgroup K of G there exists some $x \in X$ such that $AK^x = K^xA$; A is said to be X-semipermutable in G if A is X-permutable with all subgroups of some supplement T of A in G (see [4]). By using the generalized permutable subgroups, one has obtained some important results [4–7, 9, 11, 12, 15].

Received October 3, 2008; Accepted May 16, 2009
Supported by the National Natural Science Foundation of China (Grant Nos. 10771172; 10771180).
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As a continuation, we try to use some $F(G)$-semipermutable subgroups to determine the structure of finite groups, where $F(G)$ is the Fitting subgroup of a group G. Some new interesting results are obtained.

For notations and terminologies not given in this paper, the reader is referred to [3] and [14].

2. Preliminaries

We cite here some known results which are useful in the sequel.

Lemma 2.1 ([4, Lemma 2.4]) *Let A and X be subgroups of a group G. Then the following statements hold:*

(1) If N is a permutable subgroup of G and A is X-semipermutable in G, then NA is an X-semipermutable subgroup of G.

(2) If N is normal in G, A is X-semipermutable in G and T is a supplement of A in G such that every subgroup of T is X-permutable with A, then AN/N is XN/N-semipermutable in G/N and TN/N is a supplement of AN/N in G/N such that every subgroup of TN/N is XN/N-permutable with AN/N.

(3) If A/N is XN/N-semipermutable in G/N and T/N is a supplement of A/N in G/N such that every subgroup of T/N is XN/N-permutable with A/N, then A is X-semipermutable in G and T is a supplement of A in G such that every subgroup of T is X-permutable with A.

(4) If A is X-semipermutable in G and $A \leq D \leq G$, $X \leq D$, then A is X-semipermutable in D.

(5) If A is a maximal subgroup of G, T is a minimal supplement of A in G and every subgroup of T is G-permutable with A, then $T = \langle a \rangle$ is a cyclic p-subgroup for some prime p and $a^p \in A$.

(6) If A is X-semipermutable in G, T is a supplement of A in G such that A is X-permutable with every subgroup of T and $A \leq N_G(X)$, then T^x is a supplement of A in G such that every subgroup of T^x is X-permutable with A, for all $x \in G$.

(7) If A is X-semipermutable in G and $X \leq D$, then A is D-semipermutable in G.

Lemma 2.2 ([8, Lemma 2.6]) *Let H be a nilpotent normal subgroup of a group G. If $H \neq 1$ and $H \cap \Phi(G) = 1$, then H has a complement in G and H is a direct product of some minimal normal subgroups of G.*

Lemma 2.3 ([11, Lemma 3.3]) *Let G be a group and X a normal p-soluble subgroup of G. Then G is p-soluble if and only if a Sylow p-subgroup P of G is X-permutable with all Sylow q-subgroups of G, where $q \neq p$.*

Lemma 2.4 *Let G be a soluble group and N a minimal normal subgroup of G. If every minimal subgroup of N is G-semipermutable in G, then N is cyclic of prime order.*

Proof Obviously N is an elementary abelian p-group for some prime p. By Jordan-Hölder theorem, we can choose a minimal subgroup A of N such that A is normal in some Sylow p-

subgroup P of G. By the hypothesis, A has a supplement T in G such that every subgroup of T is G-permutable with A. Let D be a Hall p'-subgroup of T. Then $E = AD^x = D^x A$ for some $x \in G$. Since A is subnormal in G, A is subnormal in E. Then since D^x is a Hall p'-subgroup of G, $D^x \le N_G(A)$. It follows that $G \le N_G(A)$. The minimal choice of N implies that $N = A$ and thereby N is cyclic of order p.

3. Main results

Theorem 3.1 *Let G be a group. Then G is supersoluble if the normalizer of every Sylow subgroup of G is $F(G)$-semipermutable in G.*

Proof We first prove that G is soluble. Let p be the maximal prime dividing $|G|$ and G_p a Sylow p-subgroup of G. Assume that G_p is normal in G and Q/G_p is a Sylow q-subgroup of G/G_p, where $q \neq p$. Then $Q/G_p = G_qG_p/G_p$ for some Sylow q-subgroup G_q of G and $N_{G/G_p}(Q/G_p) = N_G(G_q)G_p/G_p$. By Lemma 2.1, we see that $N_{G/G_p}(Q/G_p)$ is $F(G/G_p)$-semipermutable in G/G_p. This shows that G/G_p satisfies the hypothesis. Hence G/G_p is soluble by induction on $|G|$. It follows that G is soluble. Now assume that $N_G(G_p) < G$. By hypothesis, $N_G(G_p)$ is $F(G)$-semipermutable in G and so there exists a supplement T of $N_G(G_p)$ in G such that $N_G(G_p)$ is $F(G)$-permutable with every subgroup of T. Let T_q be any Sylow q-subgroup of T, where $q \neq p$. Then $N_G(G_p)T_q^x = T_q^x N_G(G_p)$ for some $x \in F(G)$. Put $L = N_G(G_p)T_q^x$. Then $|L : N_G(G_p)| = q^{\beta}$. We claim that there exists some $q \neq p$ such that $\beta \neq 0$. If not, then $N_G(G_p) = G$, a contradiction. Assume that $\beta > 1$ and T_1 is a maximal subgroup of T_q. Then there exists some $y \in F(G)$ such that $L_1 = N_G(G_p)T_1^y$ is a subgroup of G. Let $|L_1 : N_G(G_p)| = q^{\beta_1}$, where $\beta_1 \le \beta$. If $\beta_1 \neq 1$, then by the same argument we see that there exists an r-maximal subgroup T_r of T_q such that $|L_r : N_G(G_p)| = q$, where $L_r = N_G(G_p)T_r^z$ for some $z \in F(G)$. Therefore L_r has exactly q Sylow p-subgroups. By Sylow's theorem, p divides $q - 1$, which is impossible since $p > q$. This contradiction shows that $N_G(G_p) = G$ and so G is soluble.

Now we prove that G is supersoluble by induction on $|G|$. Let N be a minimal normal subgroup of G and P/N a Sylow p-subgroup of G/N, where p divides $|G|$. Then $P/N = G_pN/N$ for some Sylow p-subgroup G_p of G. Hence $N_{G/N}(P/N) = N_G(G_p)N/N$. Since $N_G(G_p)$ is $F(G)$-semipermutable in G, $N_{G/N}(P/N)$ is $F(G/N)$-semipermutable in G/N. Hence G/N is supersoluble by induction. Since the class of all supersoluble groups is a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. Let M be a maximal subgroup of G such that N is not contained in M. Then $G = [N]M$ and $N = C_G(N) = F(G) = O_q(G)$ for some prime q. Let s be the largest prime dividing $|M|$ and M_s a Sylow s-subgroup of M. Then $N_G(M_s) = M$ since M is supersoluble and since N is the unique minimal normal subgroup of G. If $s \neq q$, then M_s is a Sylow s-subgroup of G. By hypothesis and Lemma 2.1(5), we have that $|G : M| = q$ and so $|N| = q$, which implies that G is supersoluble. If $s = q$, then $O_q(G/C_G(N)) = O_q(G/N) \neq 1$, which contradicts [3, Lemma 1.7.11]. Then the proof is thus completed. □

Corollary 3.2 *Let G be a group. If the normalizer of every Sylow subgroup of G is $F(G)$-*

quasinormal in G, then G is supersoluble.

Theorem 3.3 *Let G be a group and M a supersoluble maximal subgroup of G. If M is $F(G)$-semipermutable in G and $F(G)$ is not contained in M, then G is supersoluble.*

Proof If $\Phi(G) \neq 1$, then by Lemma 2.1 we easily see that $G/\Phi(G)$ satisfies the hypothesis and so $G/\Phi(G)$ is supersoluble by induction on $|G|$. This implies that G is supersoluble. Now assume that $\Phi(G) = 1$. By Lemma 2.2, $F(G)$ is a direct product of some minimal normal subgroups of G. Since $F(G)$ is not contained in M, $G = [N]M$, where N is a minimal normal subgroup of G contained in $F(G)$. By Lemma 2.1(5), we have that $|G : M| = p$ for some prime p. Since $G/N \simeq M$ is supersoluble and $|N| = |G : M| = p$, we obtain that G is supersoluble. $\square$

Theorem 3.4 *Let p be an odd prime dividing the order of a group G, P a Sylow p-subgroup of G and $X = O_{p'p}(G)$. If $N_G(P)$ is p-nilpotent and every cyclic subgroup of P is X-semipermutable in G, then G is p-nilpotent.*

Proof Assume that the assertion is false and let G be a counterexample of minimal order. Then we have the following claims:

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Then by using Lemma 2.1, we see that $G/O_{p'}(G)$ satisfies the hypothesis. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Consequently G is p-nilpotent, a contradiction.

(2) If M is a proper subgroup of G such that $P \leq M \leq G$, then M is p-nilpotent.

Since $N_M(P) \leq N_G(P)$, $N_M(P)$ is p-nilpotent. Besides, by (1), $X = O_{p'p}(G) = O_p(G) \leq O_{p'p}(M)$. Hence by Lemma 2.1, we see that M satisfies the hypothesis. The minimal choice of G implies that M is p-nilpotent.

(3) $O_p(G) \neq 1$.

Assume that $O_p(G) = 1$ and $\{x_1, x_2, \ldots, x_n\}$ is a minimal generator set of P. Then by the hypothesis, for every i, $\langle x_i \rangle$ has a supplement T_i in G such that $\langle x_i \rangle$ is permutable with every subgroup of T_i. By Lemma 2.1(6), we see that $\langle x_i \rangle$ is permutable with every Sylow q-subgroup of G, where $q \neq p$. Then since $P = \langle x_1, x_2, \ldots, x_n \rangle$, P is permutable with every Sylow q-subgroup of G, where $q \neq p$. It follows from Lemma 2.3 that G is p-soluble and thereby $O_{p'}(G) > 1$ by (1), a contradiction. Hence (3) holds.

(4) $G = PQ$, where Q is a Sylow q-subgroup of G and $q \neq p$.

Clearly the hypothesis still holds on $G/O_p(G)$. Hence $G/O_p(G)$ is p-nilpotent by the choice of G. Then by [2, Ch. 6, Theorem 3.5], for any prime q dividing the order of G with $q \neq p$, there exists a Sylow q-subgroup of G such that $E = PQ$ is a subgroup of G. If $E < G$, then E is p-nilpotent by (2). This leads to that $Q \leq C_G(O_p(G)) \leq O_p(G)$ (see [3, Theorem 1.8.19]), a contradiction. Thus $G = PQ$.

(5) Final contradiction.

Let N be a minimal normal subgroup of G. Then N is an elementary abelian p-group because G is soluble by (4) and $O_{p'}(G) = 1$. It is easy to see that G/N satisfies the hypothesis. Hence

G/N is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is a saturated formation, N is the only minimal normal subgroup of G and $\Phi(G) = 1$. Thus $G = [N]M$ for some maximal subgroup M of G and $N = C_G(N) = F(G) = O_p(G) = X$. By Lemma 2.4, N is cyclic of order p. Since $M \simeq G/N = G/C_G(N)$ is isomorphic to some subgroup of $Aut(N)$, M is a p'-subgroup of G and so N is a Sylow p-subgroup of G. Then $G = N_G(N) = N_G(P)$ is p-nilpotent by hypothesis. The final contradiction completes the proof. □

Remark 3.5 The assumption "$N_G(P)$ is p-nilpotent" in Theorem 3.4 is essential. For example, let $G = S_3$ and $p = 3$. Then every subgroup of Sylow 3-subgroup of G is $O_{p'p}(G)$-semipermutable in G, but G is not 3-nilpotent.

However, if p is the smallest prime dividing the order of G, then we have the following result.

Theorem 3.6 *Suppose that p is the smallest prime dividing the order of a group G and P is a Sylow p-subgroup of G. Let $X = O_{p'p}(G)$. If every cyclic subgroup of P is X-semipermutable in G, then G is p-nilpotent.*

Proof Assume that the assertion is false and let G be a counterexample of minimal order. Then

(1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Then it is easy to see that $G/O_{p'}(G)$ satisfies the hypothesis. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction.

(2) $O_p(G) \neq 1$.

Assume that $O_p(G) = 1$. Then $X = 1$ by (1). Let A be a minimal subgroup of P and T a supplement of A in G such that A is X-permutable with every subgroup of T. If $A \cap T = 1$, then $|G : T| = p$ and so T is normal in G (see [16, II, Proposition 4.6]). It is easy to see that T satisfies the hypothesis and hence T is p-nilpotent by the choice of G. If $O_{p'}(T) \neq 1$, then $O_{p'}(G) \neq 1$ because $O_{p'}(T)$ char $T \trianglelefteq G$. If $O_{p'}(T) = 1$, then G is a p-group. This contradiction implies that $A \cap T = A$. Thus $G = T$ and so A is a permutable subgroup of G. By [14, (13.2.2)], A is subnormal in G and consequently $A \leq O_p(G)$ by [1, A, Lemma 8.6]. This contradiction shows that $O_p(G) \neq 1$.

(3) $\Phi(G) = 1$.

By (1), we have $F(G) = O_p(G)$. If $\Phi(G) \neq 1$, then $\Phi(G) \leq O_p(G)$. Obviously, $G/\Phi(G)$ satisfies the hypothesis. Hence $G/\Phi(G)$ is p-nilpotent by the choice of G. It follows that G is p-nilpotent, a contradiction.

(4) $O_p(G)$ is a minimal normal subgroup of G.

By Lemma 2.2 and (3), $O_p(G)$ is a direct product of some minimal normal subgroup of G. If $O_p(G)$ is not a minimal normal subgroup of G, then obviously G/N satisfies the hypothesis for any minimal normal subgroup N of G contained in $O_p(G)$. Hence G/N is p-nilpotent by choice of G. Since the class of all p-nilpotent groups is a saturated formation, $O_p(G)$ must be a minimal normal subgroup of G.

(5) The final contradiction.

If P is normal in G, then $G/O_p(G)$ is a p'-subgroup and G is soluble. If $P \neq O_p(G)$, then obviously $G/O_p(G)$ satisfies the hypothesis. Hence $G/O_p(G)$ is p-nilpotent. Let $K/O_p(G)$ be the normal p-complement of $P/O_p(G)$ in $G/O_p(G)$. Since p is the smallest prime dividing the order of G, $K/O_p(G)$ is soluble by the well known Feit-Thompson theorem. It follows that K is soluble and hence G is soluble. Then by (4) and Lemma 2.4, $O_p(G)$ is cyclic of order p. Let g be an element of G such that $(|g|, p) = 1$ and $E = O_p(G)\langle g\rangle$. Then $\langle g\rangle$ is normal in E by [14, (10.1.9)] and so $\langle g\rangle \leq C_G(O_p(G)) = O_p(G)$. The final contradiction completes the proof. □

Remark 3.7 The condition that "every cyclic subgroup of P is X-semipermutable in G" in Theorem 3.6 cannot be replaced by "every minimal subgroup of P is X-semipermutable in G". For example, let $G = [\langle a, b\rangle]\langle\alpha\rangle$, where $a^4 = 1$, $a^2 = b^2 = [a, b]$ and $a^\alpha = b$, $b^\alpha = ab$. Then G is a counterexample.

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