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On X-s-Permutable Subgroups of a Finite Group

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Abstract Let X be a nonempty subset of a group G. A subgroup H of G is said to be X-s-permutable in G if there exists an element $x \in X$ such that $HP^x = P^xH$ for every Sylow subgroup P of G. In this paper, some new results are given under the assumption that some suited subgroups of G are X-s-permutable in G.

Keywords finite group; X-s-permutable subgroup; the generalized Fitting subgroup; formation.

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1. Introduction and statements of the results

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [1]. G always denotes a finite group, $\pi(G)$ is the set of the primes which divide the order of G, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$, and $M \lessdot G$ means that M is a maximal subgroup of G.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation [1, p.713, Satz 8.6].

From the permutability property with other subgroups, the concept of normal subgroup was generalized. Two subgroups H and K of G are said to permute if $HK = KH$. It is easily seen that two subgroups of G, H and K, permute, if and only if the set of HK is a subgroup of G. A subgroup H of G is said to be permutable in G if it permutes with every subgroup of G; H is called s-permutable (or S-quasinormal) in G if it permutes with every Sylow subgroup of G. Recently, s-permutable subgroup was generalized as s-conditionally permutable subgroup [2]: a subgroup H of G is an s-conditionally permutable subgroup of G if there exists an element $x \in G$ such that $HP^x = P^xH$ for every Sylow subgroup P of G. More recently, the following

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new concept was introduced in [3].

Definition *Let X be a nonempty subset of a group G. A subgroup H of G is said to be X-s-permutable in G if there exists an element $x \in X$ such that $HP^x = P^xH$ for every Sylow subgroup P of G.*

We know that the Fitting subgroup $F(G)$ is a useful conception in the study of the solvable groups. In [4], the subgroup $\tilde{F}(G)$ of G was introduced, where $\tilde{F}(G)$ satisfies $\tilde{F}(G)/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$. It is easy to see that $\tilde{F}(G)$ is a generalization of $F(G)$. We now define a sequence of subgroups $\{\tilde{F}_i(G)\}$ of G by the rules

$$\tilde{F}_1(G) = \tilde{F}(G), \tilde{F}_i(G)/\tilde{F}_{i-1}(G) = \tilde{F}(G/\tilde{F}_{i-1}(G)) \quad \text{for} \quad i > 1.$$

Obviously, $\tilde{F}_n(G)$ is a generalization of $F_n(G)$, the Fitting subgroup of degree n (see [1]). Since G is finite and $\tilde{F}_i(G) > \tilde{F}_{i-1}(G)$, there exists an integer m such that $\tilde{F}_m(G) = G$.

On the other hand, the Fitting subgroup $F(G)$ of G was generalized as $F^*(G)$, the unique maximal normal quasinilpotent subgroup of G (see [5]), which has played an important role in the proof of the theorem of the classification of finite simple groups [6]. Its definition and important properties can be found in [5, Chapter X, § 13]. Similarly, we can also define a sequence of subgroups $F_i^*(G)$ of G by the rules

$$F_1^*(G) = F^*(G), F_i^*(G)/F_{i-1}^*(G) = F^*(G/F_{i-1}^*(G)) \quad \text{for} \quad i > 1.$$

And similarly, $F_i^*(G) > F_{i-1}^*(G)$ and there exists an integer m such that $F_m^*(G) = G$.

Remark 1.1 The following example shows that $F^*(G)$ is not equal to $\tilde{F}(G)$ usually.

Example 1.2 Suppose that G is a non-split extension $(Z_2)^3L_3(2)$ of an elementary abelian subgroup $(Z_2)^3$ of order 2^3 by $L_3(2)$. G is a maximal subgroup of $G_2(3)$ (ref. ATLAS, page 61). Then $\tilde{F}(G) = G$, but $F^*(G) = F(G) = (Z_2)^3$.

In [3], there are following interesting theorems which are the generalizations of some recent results in the literature.

Theorem 1.3 ([3, Theorem 3.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group and X a solvable normal subgroup of G. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and every maximal subgroup of every Sylow subgroup of H is X-s-permutable in G.*

Theorem 1.4 ([3, Theorem 3.2]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group and X a solvable normal subgroup of G. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and every maximal subgroup of every Sylow subgroup of $\tilde{F}(H)$ is X-s-permutable in G.*

In this paper, we first unify Theorems 1.3 and 1.4 as follows:

Theorem 1.5 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group and X a solvable normal subgroup of G. Then $G \in \mathcal{F}$ if and only if there exist a normal subgroup H of G*

such that $G/H \in \mathcal{F}$, and a positive integer n such that every maximal subgroup of every Sylow subgroup of $\tilde{F}_n(H)$ is X-s-permutable in G.

Then we get a parallel result by replacing $\tilde{F}(G)$ by $F^*(G)$.

Theorem 1.6 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group and X a solvable normal subgroup of G. Then $G \in \mathcal{F}$ if and only if there exist a normal subgroup H such that $G/H \in \mathcal{F}$, and a positive integer n such that every maximal subgroup of every Sylow subgroup of $F_n^*(H)$ is X-s-permutable in G.*

From Theorem 1.6, we have following corollaries immediately.

Corollary 1.7 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group and X a solvable normal subgroup of G. Then $G \in \mathcal{F}$ if and only if there exist a solvable normal subgroup H such that $G/H \in \mathcal{F}$, and a positive integer n such that every maximal subgroup of every Sylow subgroup of $F_n(H)$ is X-s-permutable in G.*

Corollary 1.8 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group and X a solvable normal subgroup of G. Then $G \in \mathcal{F}$ if and only if there exist a normal subgroup H such that $G/H \in \mathcal{F}$, and a positive integer n such that every maximal subgroup of every Sylow subgroup of $F_n^*(H)$ is s-permutable in G.*

Remark 1.9 Corollary 1.8 is a generalization of results in [7].

Corollary 1.10 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group and X a solvable normal subgroup of G. Then $G \in \mathcal{F}$ if and only if there exist a solvable normal subgroup H such that $G/H \in \mathcal{F}$, and a positive integer n such that every maximal subgroup of every Sylow subgroup of $F_n(H)$ is s-permutable in G.*

Remark 1.11 The following example illustrates that the condition "X is solvable" is necessary for Theorems 1.5 and 1.6. Hence the assumption of solvability (p-solvability) of the groups is also necessary in the results of [2, 3].

Example 1.12 Suppose that $G = A_5$ and $X = A_5$. Since the only Sylow subgroups of G which are not of prime order are Sylow 2-subgroups and A_5 contains D_{10} and S_3 (ref. ATLAS, page 2), every maximal subgroup of every Sylow subgroup of G is X-s-permutable in G. But G is not supersolvable.

2. Preliminaries

Lemma 2.1 *Let X be a nonempty subset of G. Suppose that $N \trianglelefteq G$ and $H \leq G$. Then*

(1) If H is X-s-permutable in G and $H \leq N$, then H is X-s-permutable in N;

(2) If H is X-s-permutable in G, then HN/N is XN/N-s-permutable in G/N;

(3) Suppose that T is a subgroup of G containing N and every maximal subgroup of any Sylow subgroup of T is X-s-permutable in G. Then every maximal subgroup of any Sylow subgroup of T/N is XN/N-s-permutable in G/N.

Proof For (1), (2), please see [3, Lemma 2.1]. Now we prove (3).

Let PN/N be a Sylow subgroup of T/N, where P is a Sylow subgroup of T. Let M/N be a maximal subgroup of PN/N. Then $M = N(P \cap M)$. Suppose that P_1 is a maximal subgroup of P containing $P \cap M$, then $M \leq P_1 N$. Since $P \cap M = P_1 \cap M$, $M = N(P \cap M) = N(P_1 \cap M)$, we have $P_1 \cap N = P \cap N$ by calculating the orders of $N(P \cap M)$ and $N(P_1 \cap M)$. So $P_1 N$ is maximal in PN by the orders of $P_1 N$ and PN. Therefore $M = P_1 N$. By the hypotheses, P_1 is X-s-permutable in G, hence $M/N = P_1 N/N$ is XN/N-s-permutable in G/N by (2). □

Lemma 2.2 *Let M be a subgroup of G. Then*

(1) $F^*(G) = F(G)E(G)$ and $[F(G), E(G)] = 1$, where $E(G)$ is the layer of G;

(2) If M is normal in G, then $F^*(M) \leq F^*(G)$;

(3) $F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G))$;

(4) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$;

(5) Let $N = Z(E(G))\Phi(F(G))$. Then $F^*(G/N) = F^*(G)/N$;

(6) Suppose that P is a normal subgroup of G contained in $O_p(G)$, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$;

(7) If K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.

Proof (1)–(4) please see [5, Chapter X, § 13], (5) is [6, Proposition 4.10], (6) and (7) are the corollaries of (5). □

The following Lemma 2.3 is a corollary of Theorem 1.4.

Lemma 2.3 *Suppose that H is a solvable normal subgroup of G such that G/H is supersolvable and X is a normal solvable subgroup of G. If all maximal subgroups of any Sylow subgroup of $F(H)$ are X-s-permutable in G, then G is supersolvable.*

Lemma 2.4 ([7, Lemma 2.6]) *Let H be a normal subgroup of G. If $H \cap \Phi(G) = 1$ and $F(H) \neq 1$, then $F(H)$ is the direct product of minimal normal subgroups of G which are contained in $F(H)$. In particular, if $\Phi(G) = 1$ and $F(G) \neq 1$, then $F(G)$ is the direct product of minimal normal subgroups of G which are contained in $F(G)$.*

Lemma 2.5 ([7, Theorem 3.1]) *Suppose that G is a group and every maximal subgroup of any Sylow subgroup of $F^*(G)$ is s-permutable in G, then G is supersolvable.*

3. The proofs

The Proof of Theorem 1.5 We only need to prove the sufficiency. Suppose that the theorem is false and let G be a counterexample of minimal order.

If $n = 1$, then $G \in \mathcal{F}$ by Theorem 1.4, a contradiction. Thus suppose that $n \geq 2$. Denote $N = \tilde{F}_{n-1}(H)$ and consider factor groups G/N and H/N.

By Lemma 2.1(3), we know that every Sylow subgroup of $\tilde{F}(H/N) = \tilde{F}_n(H)/N$ is XN/N-s-permutable in G/N. Applying Theorem 1.4 for G/N and H/N, we can get that $G/N \in \mathcal{F}$. Hence $G/\tilde{F}_n(H) \cong (G/N)/(\tilde{F}_n(H)/N) \in \mathcal{F}$. Now by the hypotheses we know that every maximal subgroup of any Sylow subgroup of $\tilde{F}_n(H)$ is X-s-permutable in G, thus $G \in \mathcal{F}$ by applying Theorem 1.3 for G and $\tilde{F}_n(H)$, a contradiction.

These complete the proof of Theorem 1.5. □

To prove Theorem 1.6, we give some preliminary results. The following is a generalization of Lemma 2.3.

Theorem 3.1 *Suppose that G is a group and X is a normal solvable subgroup of G and H is a normal subgroup of G such that G/H is supersolvable. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are X-s-permutable in G, then G is supersolvable.*

Proof Suppose that the theorem is false and let G be a counterexample of minimal order. Then

(1) Every proper normal subgroup of G containing $F^*(H)$ is supersolvable.

If N is a proper normal subgroup of G containing $F^*(H)$, we have $F^*(H) = F^*(F^*(H)) \leq F^*(N) \leq F^*(H)$ by Lemma 2.2 (2). So $F^*(H) = F^*(N)$. Consequently, N satisfies the hypotheses of the theorem by Lemma 2.1 (1), the minimal choice of G implies that N is supersolvable.

(2) $H = G$ and $F^*(G) = F(G) < G$.

If $H < G$, then H is supersolvable by (1). Thus G is supersolvable by Lemma 2.3. a contradiction. Hence $H = G$.

If $F^*(G) = G$, then G is supersolvable by applying Theorem 1.1 for the special case $\mathcal{F} = \mathcal{U}$, a contradiction. Hence $F^*(G) < G$. Thus $F^*(G) = F(G)$ by Lemma 2.2(4).

Now, let P be the Sylow p-subgroup of $F(G)$ for an arbitrary prime $p \in \pi(F(G))$.

(3) $\Phi(P) = 1$, i.e., P is an elementary abelian p-group.

If $\Phi(P) \neq 1$, then we consider the factor group $G/\Phi(P)$. By Lemma 2.2(6), $F^*(G/\Phi(P)) = F^*(G)/\Phi(P) = F(G)/\Phi(P)$. Then we have all maximal subgroups of any Sylow subgroup of $F^*(G/\Phi(P))$ are $X\Phi(P)/\Phi(P)$-s-permutable subgroups of $G/\Phi(P)$ by Lemma 2.1(3). By the minimality of G, $G/\Phi(P)$ is supersolvable. Since $\Phi(P) \leq \Phi(G)$, G is supersolvable, a contradiction.

(4) There is no subgroup of order p which is normal in G. In particular, $Z(G) = 1$.

If not, let P_0 be such a subgroup of G. Then $P_0 \leq P$. Since $P_0 \leq Z(P) \leq Z(F(G))$, $F(G) \leq C_G(P_0) \leq G$. Note that $C_G(P_0)$ is normal in G, $F^*(C_G(P_0)) = F^*(G) = F(G)$. If further $C_G(P_0) < G$, then $C_G(P_0)$ is supersolvable by (1). Since $G/C_G(P_0)$ is cyclic, G is supersolvable by Lemma 2.3, a contradiction. If $C_G(P_0) = G$, then $P_0 \leq Z(G)$. By Lemma 2.2(7), $F^*(G/P_0) = F^*(G)/P_0$. Now we have all maximal subgroups of any Sylow subgroup of $F^*(G/P_0)$ are XP_0/P_0-s-permutable subgroups of G/P_0 by Lemma 2.1(3). The minimal choice of G implies that G/P_0 is supersolvable and so G is supersolvable, a contradiction.

(5) $P \cap \Phi(G) \neq 1$.

If $P \cap \Phi(G) = 1$, then we have $P = L_1 \times L_2 \times \cdots \times L_r$ by Lemma 2.4, where L_i $(i = 1, 2, \ldots, r)$ are minimal normal subgroups of G contained in P. Now pick a maximal subgroup P_0 of P such that P_0 is normal in G_p. Then P_0 is X-s-permutable in G by the hypotheses. So there exists

$x \in X$ such that $P_0G_q^x \leq G$ for any $q \in \pi(G)$, where $q \neq p$. Hence

$$[P_0, G_q^x] \leq P \cap P_0G_q^x = P_0,$$

which means that P_0 is normalized by G_q^x. Therefore P_0 is normal in G. Then P/P_0 is a G-composite factor of P of order p. On the other hand, we know that $1 \trianglelefteq L_1 \trianglelefteq L_1L_2 \trianglelefteq L_1L_2 \cdots L_r = P$ is a G-composite series of P. According to Jordan-Hölder theorem, P/P_0 is isomorphic to some L_t. Thus $|L_t| = p$, contrary to (4).

Now let p be a fixed prime in $\pi(F(G))$. Pick a minimal normal subgroup L of G contained in $P \cap \Phi(G)$.

(6) $F(G) = P$.

Let $Q \neq 1$ be the Sylow q-subgroup of $F(G)$ and Q_0 a minimal normal subgroup of G contained in Q, where $q \neq p$. By Lemma 2.2 (1), $F^*(G/L) = F(G/L) \cdot E(G/L) = F(G)/L \cdot E/L$, where $E(G/L)$ is the Layer of G/L. Since $[F(G/L), E(G/L)] = 1$ by Lemma 2.2 (1), we suppose $[Q_0, E] \leq L \cap Q_0 = 1$, i.e., $E \leq C_G(Q_0)$. We have $C_G(Q_0) < G$ by step (4). Clearly, $F^*(G) = F(G) \leq C_G(Q_0)$, we have $C_G(Q_0)$ is supersolvable by step (1). Then $E = L$. This implies that $F^*(G/L) = F(G)/L = F^*(G)/L$. By Lemma 2.1 (3) and the minimality of G, we have G/L is supersolvable. Consequently, G is supersolvable, a contradiction. Hence $F(G)$ is a p-group.

(7) $F^*(G/L) = G/L$.

If $F^*(G/L) < G/L$, we have $F^*(G/L) = F(G/L) \cdot E(G/L) = F(G)/L \cdot E/L$, where $E(G/L)$ is the Layer of G/L by Lemma 2.2 (1). Since $F^*(G) = F(G) \leq F(G)E < G$, $F(G)E$ is supersolvable by (1), $F(G)E/L = F^*(G/L)$ is supersolvable. Thus $F^*(G/L) = F(G/L) = F(G)/L = F^*(G)/L$. By Lemma 2.1 (3) and the minimality of G, we have G/L is supersolvable. Consequently, G is supersolvable, a contradiction.

(8) G/P is a non abelian simple group.

By (6), (7) and Lemma 2.2 (3), $G/P \cong (G/L)/(P/L) = F^*(G/L)/F(G/L) = Soc(F(G/L) \cdot C_{G/L}(F(G/L))/F(G/L))$. Let $\mathrm{Soc}(F(G/L) \cdot C_{G/L}(F(G/L))/F(G/L)) = (N_1/L)/F(G/L) \times (N_2/L)/F(G/L) \times \cdots \times (N_s/L)/F(G/L)$, where $(N_i/L)/F(G/L)$ is a minimal normal subgroup of $(G/L)/F(G/L)$. If $s > 1$, then for each i, N_i is a proper normal subgroup of G containing $F^*(G)$. Then N_i is supersolvable by (1). Consequently, G is solvable, a contradiction. Thus $s = 1$ and G/P is a non abelian simple group.

(9) The final contradiction.

Since XP/P is a solvable normal subgroup of G/P, $XP/P = 1$ by (8). It means that $X \leq P$. For any maximal subgroup P_1 of P. By hypotheses, P_1 is X-s-permutable in G, then there exists some $x \in X$ such that $P_1G_q^x = G_q^xP_1$ for any $q \in \pi(G)$ and $G_q \in \mathrm{Syl}_q(G)$. Then $P_1G_q = G_qP_1$, i.e., P_1 is s-permutable in G. By Lemma 2.5, we know that G is supersolvable, the final contradiction.

These complete the proof of the theorem. □

Theorem 3.2 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and suppose that G is a group and X is a normal solvable subgroup of G. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup*

H of G such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(H)$ are X-s-permutable in G.

Proof We only need prove the sufficiency. By hypotheses and Lemma 2.1 (1), all maximal subgroups of all *Sylow* subgroups of $F^*(H)$ are X-s-permutable in H. We have H is supersolvable by Theorem 3.1. Hence $F^*(H) = F(H)$. By Theorem 1.4, $G \in \mathcal{F}$. □

The Proof of Theorem 1.6 Replacing $\tilde{F}(H)$ by $F^*(H)$ in the proof of Theorem 1.5 gives the proof of Theorem 1.6. □

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References

[1] HUPPERT B. *Endliche Gruppen (I)* [M]. Springer-Verlag, Berlin-New York, 1967.
[2] HUANG Jianhong, GUO Wenbin. *s-conditionally permutable subgroups of finite groups* [J]. Chinese Ann. Math. Ser. A, 2007, **28**(1): 17–26.
[3] SHI Lei, GUO Wenbin, YI Xiaolan. *X-s-permutable subgroups* [J]. J. Math. Res. Exposition, 2008, **28**(2): 257–265.
[4] GUO Wenbin. *The Theory of Class of Groups* [M]. Kluwer Academic Publishers Group, Dordrecht; Science Press, Beijing, 2000.
[5] HUPPERT B, BLACKBURN B. *Finite Groups (III)* [M]. Springer-Verlag, Berlin-New York, 1982.
[6] GORENSTEIN D, LYONS R, SOLOMON R. *The Classification of the Finite Simple Groups* [M]. American Mathematical Society, Providence, RI, 1994.
[7] LI Yangming, WANG Yanming, WEI Huaquan. *The influence of π-quasinormality of some subgroups of a finite group* [J]. Arch. Math. (Basel), 2003, **81**(3): 245–252.