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# Virtual and Immediate Basins of Newton's Method for a Class of Entire Functions

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**Abstract** In this paper, we consider Newton's method for a class of entire functions with infinite order. By using theory of dynamics of functions meromorphic outside a small set, we find there are some series of virtual immediate basins in which the dynamics converges to infinity and a series of immediate basins with finite area in the Fatou sets of Newton's method.

**Keywords** Newton's method; Baker domain; virtual immediate Basin; Fatou set; Julia set.

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## 1. Introduction

Newton's method is a classical way to approximate roots of differentiable functions by an iterative procedure. We can investigate the procedure in view of complex dynamical systems. (See [1] for general references on this subject.)

Newton's method for a complex polynomial $P(z)$ is the iteration of a rational function $N_P = z-\frac{P(z)}{P'(z)}$ on the Riemann sphere. Such dynamical systems have been extensively studied in recent years. Przytycki [2] has shown that all immediate basins (Definition 3.1) are simply connected and unbounded. Shishikura [3] has shown more generally that if a rational map has a multiply connected Fatou component, then it must have two weakly repelling fixed points. Tan [4] gave a complete classification of the Newton maps of cubic polynomials.

If $f(z)$ is a transcendental entire function, then the associated Newton map $N_f$ will generally be transcendental meromorphic, except in the special case $f(z) = p(z)e^{q(z)}$ with polynomials $p(z)$ and $q(z)$ which was studied by Haruta [5]. Bergweiler [6] proved a no-wandering-domains theorem for transcendental Newton maps that satisfy several finiteness assumptions. Mayer and Schleicher [7] have shown that immediate basins for the Newton maps of entire functions are simply connected and unbounded, extending a result of Przytycki [2] in the polynomial case. They have also shown that the Newton maps of transcendental functions may exhibit a type of Fatou component that does not appear for the Newton maps of polynomials, so called virtual immediate basins (Definition 3.2) in which the dynamics converges to infinity. The thesis

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[8] investigated the Newton map of the transcendental function $f(z) = ze^{e^z}$ and proved that it exhibits virtual immediate basins. While immediate basins (Definition 3.1) of roots are by definition related to zeroes of $f$, a virtual immediate basin often contains an asymptotic path of an asymptotic value at 0 for $f$ (see [9]).

In this paper, we investigate the Newton maps $N_f(z)$ for a class of entire functions $f(z) = (e^z - 1)e^{e^{-z}P(e^z)}$, where $P(z)$ is a real coefficient polynomial with $\deg(P) \geq 2$ and $P(0) \neq 0$. In the Fatou set $F(N_f)$ of $N_f(z)$, we find that there are some series of simply connected invariant Baker domains, which are virtual immediate basin; and that there are a series of supper-attracting immediate basins. We also show each supper-attracting immediate basin has finite area while each is unbounded. Moreover, in case of $\deg(P) = 1$ and $P(0) = 0$, we find the immediate basin may have infinite area (see Remark 1).

Throughout this paper, $\ln z$ denotes the principal branch of the logarithm function $\log z = \ln|z| + \arg z + 2k\pi i$, $k \in \mathbb{Z}$.

## 2. Dynamics of functions meromorphic outside a small set

To investigate the dynamics of the meromorphic function $N_f(z)$, we need to analyse the dynamics of function in the following class $M$.

$M = \{f$ : there is a compact totally disconnected set $E = E(f)$ such that $f$ is meromorphic in $E^c$ and $C(f, E^c, z_0) = \hat{\mathbb{C}}$ for all $z_0 \in E$. If $E = \emptyset$ we make the further assumption that $f$ is neither constant nor univalent in $\hat{\mathbb{C}}\}$, where the cluster set $C(f, E^c, z_0) = \{w : w = \lim_{n\to\infty} f(z_n)$ for some $z_n \in E^c$ with $z_n \to z_0\}$.

The class $M$ was studied in [10]–[15]. In [13] and [14], where the basic concepts such as the Fatou set and the Julia set and the basic properties of dynamics of functions in $M$ were established. It was proved in [13] that the class $M$ is closed under composition and if $f, g \in M$, then $E(f \circ g) = E(g) \bigcup g^{-1}(E(f))$. For $f \in M$, we define $f^0$ to be the identity function with $E_0 = \emptyset$, $f^n = f \circ f^{n-1}$, then $f^n \in M$, $n \in \mathbb{N}$, and $E_n = E(f^n) = \bigcup_{j=0}^{n-1} f^{-j}(E) = \{$ singularities of $f^{-n}\}$. Let $J_1(f) = \overline{\bigcup_{n=0}^{+\infty} E_n}$ and $F_1(f) = \hat{\mathbb{C}} \setminus J_1(f)$. Then $F_1(f)$ is the largest open set in which all $f^n$ are defined and $f(F_1(f)) \subset F_1(f)$. As in [13], for $f \in M$, we define the Fatou set of $f$, denoted by $F(f)$, to be the largest open set in which (i) all composition $f^n$ are meromorphic and (ii) the family $\{f^n\}$ is a normal family; and the Julia set of $f$, denoted by $J(f)$, to be the complement of $F(f)$. If the set $J_1(f)$ is either empty or contains one point or two points, then $f$ is conjugate to a rational map or entire function or an analytic map of the punctured plane $\mathbb{C}^*$, respectively. In these cases the condition (i) is trivial and the Fatou sets are determined by (ii). In all other cases, by Montel's theorem, $F(f) = F_1(f)$ and $J(f) = J_1(f)$. It is clear that for $f \in M$, $F(f)$ is open and completely invariant. Let $U$ be a connected component of $F(f)$. Then $f^n(U)$ is contained in a component $U_n$ of $F(f)$. If for any pair of $m \neq n$, $U_m \neq U_n$, then $U$ is called a wandering domain of $f$. Otherwise, $U$ is said to be preperiodic. If for some $n \in N$, $U_n = U$, namely, $f^n(U) \subset U$, then $U$ is said to be periodic, and the smallest positive $n \in \mathbb{N}$ is called the period of $U$. For a periodic component of $F(f)$ we have the following classification theorem:

**Therorem A** ([13]) *Let $U$ be a periodic component of the Fatou set of period $p$. Then precisely one of the following is true:*

*(i) $U$ is a (super)attracting domain of a (super)attracting periodic point $a$ of $f$ of period $p$ such that $f^{np}|_U \to a$ as $n \to +\infty$ and $a \in U$.*

*(ii) $U$ is a parabolic domain of a rational neutral periodic point $a$ of $f$ of period $p$ such that $f^{np}|_U \to a$ as $n \to +\infty$ and $a \in \partial U$.*

*(iii) $U$ is a Siegel disk of period $p$ such that there exists an analytic homeomorphism $\varphi : U \to \triangle$, where $\triangle = \{z : |z| < 1\}$, satisfying $\varphi(f^p(\varphi^{-1}(z))) = e^{2\pi\alpha i}z$ for some irrational number $\alpha$ and $\varphi^{-1}(0) \in U$ is an irrational neutral periodic point of $f$ of period $p$.*

*(iv) $U$ is a Herman ring of period $p$ such that there exists an analytic homeomorphism $\varphi : U \to A$, where $A = \{z : 1 < |z| < r\}$, satisfying $\varphi(f^p(\varphi^{-1}(z))) = e^{2\pi\alpha i}z$ for some irrational number $\alpha$.*

*(v) $U$ is a Baker domain of period $p$ such that $f^{np}|_U \to a \in J(f)$ as $n \to +\infty$ but $f^p$ is not meromorphic at $a$. If $p = 1$, then $a \in E(f)$.*

As to the local structure of rationally indifferent periodic point, with similar discussion as that of §6.5 in [1], or §3.1.6 in [16], we have the following Theorem B and C.

**Theorem B** *Suppose that the map $f \in M$ has the Taylor expansion*

$$f(z) = z - z^{p+1} + O(z^{2p+1})$$

*at the origin. Then for sufficiently small $t$, $f$ has $p$ petals*

$$\Pi_k(t) = \{re^{i\theta} : r^p < t(1 + \cos(p\theta)); |\frac{2k\pi}{p} - \theta| < \frac{\pi}{p}\}, \quad k = 0, 1, \ldots, p-1$$

*lying in distinct parabolic domains at the origin, such that:*

*(i) $f$ maps each petal $\Pi_k(t)$ into itself, and $f : \Pi_k(t) \mapsto \Pi_k(t)$ is conjugate to $T(z) = z + 1$;*

*(ii) $f^n(z) \mapsto 0$ uniformly on each petal as $n \mapsto \infty$;*

*(iii) $\arg(f^n(z)) \mapsto \frac{2k\pi}{p}$ locally uniformly on $\Pi_k$ as $n \mapsto \infty$;*

*(iv) $|f(z)| < |z|$ on a neighborhood of the axis of each petal.*

**Theorem C** *Suppose that the map $f \in M$ has the Taylor expansion*

$$f(z) = z + az^{p+1} + O(z^{p+2})$$

*at the origin with $a \neq 0$. Then there is a function $F(z) = z - z^{p+1} + O(z^{2p+1})$ and a polynomial $\varphi(z) = e^{\frac{\ln a}{p}}z + \beta z^2 + \cdots + \gamma z^{p!}$, such that $F \circ \varphi = \varphi \circ f$.*

As for the relation between the dynamics of two commutable functions in $M$, we have:

**Theorem D** ([16, Theorem 3.1.14]) *Let $f, g \in M$, $\varphi$ be a meromorphic function and $\varphi(f(z)) = g(\varphi(z))$. If $J(f) = J_1(f)$ and either $\infty \in E(f)$ or $f(\infty) \neq \infty$, then $J(f) = \varphi^{-1}(J(g))$ and $F(f) = \varphi^{-1}(F(g))$.*

**Theorem E** ([16, Theorem 3.1.17]) *Let $f, g \in M$, and $\exp f(z) = g(e^z)$. If $\infty \in E(f)$ or*

$f(\infty) \neq \infty$, *then* $\exp J(f) = J(g) \setminus \{0\}$ *and* $\exp F(f) = F(g) \setminus \{0\}$.

## 3. Immediate basins and virtual Immediate basins

Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an entire function. Newton's method of finding the zero of $f$ consists of iterating the meromorphic function $N_f(z)$ defined by $N_f(z) = z - \frac{f(z)}{f'(z)}$. In fact, zeroes of $f$ are attracting fixed points of $N_f(z)$, and vice versa. The simple zeroes of $f$ are super-attracting fixed points of $N_f(z)$.

**Definition 3.1** *Let $\xi$ be an attracting fixed point of $N_f(z)$. The basin of attraction of $\xi$ is the open set of all points $z$ such that $(N_f^m(z))$ converges to $\xi$ as $m \to \infty$. The connected component containing $\xi$ of the basin is called the immediate basin of $\xi$.*

**Definition 3.2** *An unbounded domain $U \subset \mathbb{C}$ is called virtual immediate basin of $N_f(z)$ if it is maximal (among domains in $\mathbb{C}$) with respect to the following properties:*

*(i)* $\lim_{n\to\infty} N_f^{\circ n}(z) = \infty$ *for all* $z \in U$*;*

*(ii) There is a connected and simply connected subdomain $S_0 \subset U$ such that $N_f(\overline{S_0}) \subset S_0$ and for all $z \in U$ there is an $m \in N$ such that $N_f^{\circ n}(z) \in S_0$. We call the domain $S_0$ an absorbing set for $U$.*

If $f$ is a transcendental entire function, then the associated Newton map $N_f$ will generally be transcendental meromorphic, except in the special case $f(z) = p(z)e^{q(z)}$ with polynomials $p(z)$ and $q(z)$. Mayer and Schleicher [7] have shown that the Newton maps of transcendental functions may exhibit virtual immediate basin that does not appear for the Newton maps of polynomials.

Now let entire function $f(z) = (e^z - 1)e^{e^{-z}P(e^z)}$, and $N_f(z)$ be the corresponding Newton map, where $P(z)$ is a real coefficient polynomial with $\deg(P) = d \geq 2$ and $P(0) \neq 0$. Then $N_f(z) = z + R(e^z)$, where $R(z) = -\frac{z(z-1)}{z^2+(z-1)[zP'(z)-P(z)]}$. Let $g(z) = ze^{R(z)}$. Then $e^{N_f(z)} = g(e^z)$. According to the nature of logarithmic function and $e^{N_f(z)} = g(e^z)$, Theorem E implies that the dynamics of $N_f$ in horizontal strip regions $\{z : (2m-1)\pi < \operatorname{Im} z < (2m+1)\pi\}$ are the same for different $m \in \mathbb{Z}$. So, we just need to consider dynamics of $N_f$ in the horizontal strip region

$$\Xi = \{z : -\pi < \operatorname{Im} z < \pi\}.$$

Without loss of generality, let $P(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$ with $a_d a_0 \neq 0$. Then

$$N_f(z) = \frac{w - w^2}{(d-1)a_d w^{d+1} + ((d-2)a_{d-1} - (d-1)a_d)w^d + \cdots - a_0 w + a_0} \circ e^z + z,$$

and

$$g(z) = z \exp \frac{z - z^2}{(d-1)a_d z^{d+1} + ((d-2)a_{d-1} - (d-1)a_d)z^d + \cdots - a_0 z + a_0}.$$

**Theorem 3.3** *In Fatou set of $g(z)$, there are one super-attracting component $V_1$ containing 1, one parabolic domain $V_0$ such that $g^n|_{V_0} \to 0$ as $n \to +\infty$ and $d-1$ invariant parabolic domains $V_\infty^k$ $(k = 0, 1, \ldots, d-2)$ such that $g^n|_{V_\infty^k} \to 0$ as $n \to +\infty$.*

**Proof** It is easy to see $g(z) \in M$. In view of $\deg(P) = d \geq 2$, $R(z) \to 0$ as $z \to \infty$, $g(z)$ only has the pole at infinity. $g(z)$ has two fixed points 0 and 1, moreover $g'(0) = 1$ and $g'(1) = 0$. So by Theorem A, in the Fatou set $F(g)$ of $g(z)$, there is an invariant immediate super-attracting basin $V_1$ containing 1.

By assumption $a_0 = P(0) \neq 0$, $g(z)$ has the Taylor expansion $g(z) = z + \frac{1}{a_0}z^2 + O(z^3)$ at the origin. Theorem B and A imply that there is an invariant parabolic domain $V_0$ such that $g^n|_{V_0} \to 0$ as $n \to +\infty$.

Let $\sigma(z) = \frac{1}{z}$, and $h(z) = z \exp \frac{(1-z)z^{d-1}}{(d-1)a_d+((d-2)a_{d-1}-(d-1)a_d)z+\cdots-a_0z^d+a_0z^{d+1}}$. Then $\sigma \circ g(z) = h \circ \sigma(z)$. $h(z)$ has the Taylor expansion $h(z) = z + \frac{1}{(d-1)a_d}z^d + O(z^{d+1})$ at the origin. By Theorem B and A, there are $d-1$ invariant parabolic domains $B^k$ $(k = 0, 1, \ldots, d-2)$ such that $h^n|_{B^k} \to 0$ as $n \to +\infty$. So there are $d-1$ invariant parabolic domains $V_\infty^k = \sigma(B^k)$ $(k = 0, 1, \ldots, d-2)$ such that $g^n|_{V_\infty^k} \to 0$ as $n \to +\infty$. The proof of Theorem 3.3 is completed. □

**Theorem 3.4** *In the Fatou set of $N_f(z)$, there are one simply connected invariant super-attracting basin and $d$ invariant Baker domains in $\Xi$.*

**Proof** Since $e^{N_f(z)} = g(e^z)$ and $N_f(z)$, $g(z) \in M$, based on Theorem 3.3 and E, there is an invariant immediate super-attracting basin $U_1 = \ln(V_1)$ in the Fatou set $F(N_f)$ of $N_f(z)$, and the corresponding super-attracting fixed point is 0. According to Theorem 2.7 in [7], $U_1$ is simply connected.

On the other hand, by Theorem 3.3 and B, $g(z)$ has a parabolic domain $V_0$ such that $g^n|_{V_0} \to 0$ as $n \to +\infty$, and for positive number $t$ small enough, $V_0$ contains an absorbing petal with an absorbing axis $l = \{re^{\theta i} : \theta = \pi + \arg(a_0), 0 < r < t\}$. Consequently, Theorem E implies $N_f(z)$ has a component $U_0 = \ln(V_0)$ such that $N_f^n|_{U_0} \to \infty$ as $n \to +\infty$. Considering that $P(z)$ is a real coefficient polynomial, $R(z) = -\frac{z(z-1)}{z^2+(z-1)[zP'(z)-P(z)]}$ is a real coefficient rational function, then the Newton map $N_f(z) = z + R(e^z)$ maps $L_k = \{x + iy : -\infty < x < \ln t, y = \pi + \arg(a_0) + 2k\pi\}$ $(k \in \mathbb{Z})$ to itself, where $L_k$ is the image of $l$ of a branch of the logarithmic function $\log z$, and $L_0 = \ln l$ lies in $U_0$. So $U_0$ is not a wandering domain but a Baker domain.

Proceeding with similar discussion, we can show $N_f(z)$ has other $d-1$ Baker domains $U_\infty^k = \ln(V_\infty^k)$ $(k = 0, 1, \ldots, d-2)$. The proof of Theorem 3.4 is completed. □

**Theorem 3.5** *In the Fatou set of $N_f(z)$, each Baker domain is virtual immediate basin.*

**Proof** From the proof of Theorem 3.4, each Baker domain in the Fatou set $F(N_f(z))$ comes from parabolic domain of $g(z)$.

On the other hand, from the proof of Theorem 3.3, in a neighborhood of the origin, $g(z) = z + \frac{1}{a_0}z^2 + O(z^3)$, $h(z) = z + \frac{1}{(d-1)a_d}z^d + O(z^{d+1})$. Theorem C implies $g(z)$ is conjugate to a function $F_1(z) = z - z^2 + O(z^3)$ via $\varphi(z) = \frac{-1}{a_0}z$ and $h(z)$ is conjugate to a function $F_2(z) = z - z^d + O(z^{2d-1})$ via a polynomial $\psi(z) = \lambda z + \beta z^2 + \cdots + \gamma z^{(d-1)!}$ near the origin, where $\lambda = ((d-1)\,|a_d|)^{-\frac{1}{d-1}} e^{-\frac{\arg(a_d)}{d-1}i}$.

Using Theorem B, at the origin, for sufficiently small positive numbers $t_1$, $t_2$, $s_1$ and $s_2$, $F_1(z)$ has a petal

$$\Pi(t_1) = \{re^{i\theta} : r < t_1(1 + \cos\theta); |\theta| < \pi\}$$

with repelling axis $L = \{re^{i\theta} : 0 < r < s_1, \theta = \pi\}$, and $F_2(z)$ has $d-1$ petals

$$\Pi_k(t_2) = \{re^{i\theta} : r^{d-1} < t_2(1+\cos(d-1)\theta); |\frac{2k\pi}{d-1} - \theta| < \frac{\pi}{d-1}\}$$

with repelling axis $L^k = \{re^{i\theta} : 0 < r < s_2, \theta = \frac{2k+1}{d-1}\pi\}$ $(k = 0, 1, \ldots, d-2)$. Consequently, the Baker domain $U_0 = \ln(V_0)$ has an absorbing set $\ln\circ\varphi^{-1}(\Pi(t))$, and the Baker domain $U_\infty^k = \ln(V_\infty^k)$ has an absorbing set $\ln\circ\sigma\circ\psi^{-1}(\Pi_k(t))$ $(k = 0, 1, \ldots, d-2)$. So each Baker domain is a virtual immediate basin. The proof of Theorem 3.5 is completed. $\square$

**Theorem 3.6** *In $\Xi$, complement of the union of all virtual immediate basins of $N_f(z)$ has finite area.*

**Proof** Theorem 3.5 implies each Baker domain of $N_f(z)$ has an absorbing set, therefore, the complement of the union of all virtual immediate basins of $N_f(z)$ is a subset of the complement of union of these absorbing sets. To complete this proof, we need only to show: in $\Xi$, the complement of union of these absorbing sets has finite area.

Following the Proof of Theorem 3.5.

Let $0 < t < \min\{t_1, t_2\}$, $\frac{3}{4}\pi < \theta_1 < \pi$, $\frac{3}{4(d-1)}\pi < \theta_2 < \frac{1}{d-1}\pi$,

$$\gamma_1 = \{re^{i\theta} : r = t(1+\cos\theta); \theta_1 < \theta < \pi\},$$

$$\gamma_2 = \{re^{i\theta} : r = t(1+\cos\theta); -\theta_1 > \theta > -\pi\},$$

$$\gamma_1^k = \{re^{i\theta} : r^{d-1} = t(1+\cos(d-1)\theta), \frac{2k\pi}{d-1} + \theta_2 < \theta < \frac{2k\pi}{d-1} + \frac{\pi}{d-1}\},$$

$$\gamma_2^k = \{re^{i\theta} : r^{d-1} = t(1+\cos(d-1)\theta), \frac{2k\pi}{d-1} - \theta_2 > \theta > \frac{2k\pi}{d-1} - \frac{\pi}{d-1}\}$$

$(k = 0, 1, \ldots, d-2)$. Then $\gamma_1$ and $\gamma_2$ are two simple curves in $\Pi(t_1)$ and $\gamma_1^k$ and $\gamma_2^k$ are two simple curves in $\Pi_k(t_2)$. Accordingly, $\Gamma_1 = \varphi^{-1}(\gamma_1)$ and $\Gamma_2 = \varphi^{-1}(\gamma_2)$ are two simple curves in the parabolic domain in the Fatou set $F(g(z))$. Choose $\psi^{-1}$ the branch of the inverse function of $\psi$ which fixes 0, namely, $\psi^{-1}(z) = \frac{1}{\lambda}z + \alpha_1 z^2 + \alpha_2 z^3 + \cdots$. Then $\Gamma_1^k = \psi^{-1}(\gamma_1^k)$ and $\Gamma_2^k = \psi^{-1}(\gamma_2^k)$ are two simple curves in the parabolic domain in the Fatou set $F(h(z))$.

Since $e^{N_f(z)} = g(e^z)$ and $\sigma\circ g(z) = h\circ\sigma(z)$, $\tilde{\Gamma}_1 = \ln\circ\varphi^{-1}(\gamma_1)$, $\tilde{\Gamma}_2 = \ln\circ\varphi^{-1}(\gamma_2)$, $\tilde{\Gamma}_1^k = \ln\circ\sigma\circ\psi^{-1}(\gamma_1^k)$ and $\tilde{\Gamma}_2^k = \ln\circ\sigma\circ\psi^{-1}(\gamma_2^k)$ are simple curves in above-mentioned Baker domains in the Fatou set $F(N_f(z))$.

For $\varphi^{-1}(z) = |a_0|\, e^{i\arg(-a_0)}z$, we have

$$\tilde{\Gamma}_1 = \left\{ X(\theta) + iY(\theta) : \begin{array}{r} X(\theta) = \ln(|a_0|\, t(1+\cos\theta)), Y(\theta) = \theta + \arg(-a_0), \\ \theta_1 < \theta < \pi \end{array} \right\},$$

$$\tilde{\Gamma}_2 = \left\{ X(\theta) + iY(\theta) : \begin{array}{r} X(\theta) = \ln(|a_0|\, t(1+\cos\theta)), Y(\theta) = \theta + \arg(-a_0), \\ -\theta_1 > \theta > -\pi \end{array} \right\}.$$

Furthermore, the curve $\tilde{\Gamma}_1$ is monotonously decreasing, and has an asymptote $Y = \pi + \arg(-a_0)$ as $\theta \to \pi$, $\tilde{\Gamma}_2$ is monotonously increasing, and has an asymptote $Y = -\pi + \arg(-a_0)$ as $\theta \to -\pi$.

Write $\psi^{-1}(z)=r_v e^{i\theta_v}z$, where $r_v=|\frac{1}{\lambda}+\alpha_1 z+\alpha_2 z^2+\cdots|$ and $\theta_v=\arg(\frac{1}{\lambda}+\alpha_1 z+\alpha_2 z^2+\cdots)$ are continuous functions. Then

$$\tilde\Gamma_1^k=\left\{X(\theta)+iY(\theta):\begin{array}{r}X(\theta)=-\ln(r_v(t+t\cos(d-1)\theta)^{\frac{1}{d-1}}),Y(\theta)=-\theta-\theta_v,\\ \dfrac{2k}{d-1}\pi+\theta_2<\theta<\dfrac{2k+1}{d-1}\pi\end{array}\right\},$$

$$\tilde\Gamma_2^k=\left\{X(\theta)+iY(\theta):\begin{array}{r}X(\theta)=-\ln(r_v(t+t\cos(d-1)\theta)^{\frac{1}{d-1}}),Y(\theta)=-\theta-\theta_v,\\ \dfrac{2k}{d-1}\pi-\theta_2>\theta>\dfrac{2k-1}{d-1}\pi\end{array}\right\}.$$

In view of $r_v(z)\to\left|\frac{1}{\lambda}\right|$ and $\theta_v(z)\to\arg\frac{1}{\lambda}$ as $z\to 0$, the curve $\tilde\Gamma_1^k$ is monotonously decreasing, and has an asymptote $Y=-\frac{2k+1}{d-1}\pi-\arg\frac{1}{\lambda}$ as $\theta\to\frac{2k+1}{d-1}\pi$, while the curve $\tilde\Gamma_2^k$ is monotonously increasing, and has an asymptote $Y=-\frac{2k-1}{d-1}\pi-\arg\frac{1}{\lambda}$ as $\theta\to\frac{2k-1}{d-1}\pi$.

In the same way, repelling axis $L=\{re^{i\theta}:0<r<s_1,\theta=\pi\}$ and $L^k=\{re^{i\theta}:0<r<s_2,\theta=\frac{2k+1}{d-1}\pi\}$ respectively produce repelling axis of $N_f(z)$ as follows:

$$\ln\circ\varphi^{-1}(L)=\left\{X(r)+iY(r):\begin{array}{l}X(r)=\ln(|a_0|\,r),\\ Y(r)=\pi+\arg(-a_0),\end{array}\ 0<r<s_1\right\},$$

$$\ln\circ\sigma\circ\psi^{-1}(L^k)=\left\{X(r)+iY(r):\begin{array}{l}X(r)=-\ln(r_v r),\\ Y(r)=-\dfrac{2k+1}{d-1}\pi-\arg\dfrac{1}{\lambda},\end{array}\ 0<r<s_2\right\}.$$

It is easy to see that the asymptote of $\tilde\Gamma_1$ or $\tilde\Gamma_2$ is the horizontal line in which $\ln\circ\varphi^{-1}(L)$ lies and the asymptote of $\tilde\Gamma_1^k$ or $\tilde\Gamma_2^k$ is the horizontal line in which $\ln\circ\sigma\circ\psi^{-1}(L^k)$ lies.

Next we show that the area of each unbounded wedge sharped region between above curve and the corresponding asymptote is finite.

The area of unbounded wedge sharped region $W_1$ between $\tilde\Gamma_1$ and the corresponding asymptote $Y=\pi+\arg(-a_0)$ is the following integration:

$$\int_\pi^{\theta_1}(\pi+\arg(-a_0)-Y(\theta))\mathrm{d}X(\theta)=\int_\pi^{\theta_1}(\pi-\theta)d(\ln(|a_0|\,t(1+\cos\theta))$$
$$=\int_\pi^{\theta_1}\frac{-(\pi-\theta)\sin\theta}{1+\cos\theta}\mathrm{d}\theta=\int_0^{\pi-\theta_1}\frac{\theta\sin\theta}{1-\cos\theta}\mathrm{d}\theta=4\int_0^{\frac{\pi-\theta_1}{2}}\frac{\theta}{\tan\theta}\mathrm{d}\theta<2(\pi-\theta_1),$$

where $X(\theta)=\ln(|a_0|\,t(1+\cos\theta))$, $Y(\theta)=\theta+\arg(-a_0)$. So the area of $W_1$ is finite.

To analyse the area of unbounded wedge sharped region $W_1^k$ between $\tilde\Gamma_1^k$ and the corresponding asymptote $Y=-\frac{2k+1}{d-1}\pi-\arg\frac{1}{\lambda}$, we construct another unbounded wedge sharped region $\bar W_1^k$. For positive numbers $\delta_1$ and $\delta_2$, we define $\bar W_1^k$ to be the region between curve

$$\bar\Gamma_1^k=\ln\circ\sigma\circ\bar\psi(\gamma_1^k)=\left\{X(\theta)+iY(\theta):\begin{array}{r}X(\theta)=-\ln(\dfrac{\delta_1}{|\lambda|}(t(1+\cos(d-1)\theta))^{\frac{1}{d-1}}),\\ Y(\theta)=-\theta-(\arg\dfrac{1}{\lambda}-\delta_2(\dfrac{2k+1}{d-1}\pi-\theta)),\\ \dfrac{2k}{d-1}\pi+\theta_2<\theta<\dfrac{2k+1}{d-1}\pi\end{array}\right\}$$

and its asymptote $Y=-\frac{2k+1}{d-1}\pi-\arg\frac{1}{\lambda}$, where $\bar\psi(z)=z\frac{\delta_1}{|\lambda|}e^{i(\arg\frac{1}{\lambda}-\delta_2(\frac{2k+1}{d-1}\pi-\theta))}$. For some appropriate small positive numbers $\delta_1$ and $\delta_2$, and $z\in\gamma_1^k$, the Euclidian distance from the point

$\ln\circ\sigma\circ\bar{\psi}(z)$ to the line $Y=-\frac{2k+1}{d-1}\pi-\arg\frac{1}{\lambda}$ is greater than that from the point $\ln(\sigma\circ\psi^{-1}(z))$ to the same line, namely, $\bar{\Gamma}_1^k$ lies above $\tilde{\Gamma}_1^k$. Moreover, the difference between areas of $\bar{W}_1^k$ and $W_1^k$ is finite. The area of $\bar{W}_1^k$ is the following integration:

$$\int_{\frac{2k}{d-1}\pi+\theta_2}^{\frac{2k+1}{d-1}\pi}(Y(\theta)+\frac{2k+1}{d-1}\pi+\arg\frac{1}{\lambda})\mathrm{d}X(\theta)=\int_{\frac{2k\pi}{d-1}+\theta_2}^{\frac{2k+1}{d-1}\pi}\frac{(\delta_2+1)(\frac{2k+1}{d-1}\pi-\theta)\sin(d-1)\theta}{1+\cos(d-1)\theta}\mathrm{d}\theta$$
$$=\int_0^{\frac{\pi}{d-1}-\theta_2}\frac{(\delta_2+1)\theta\sin(d-1)\theta}{1-\cos(d-1)\theta}\mathrm{d}\theta<\frac{2(\delta_2+1)(\pi-(d-1)\theta_2)}{(d-1)^2},$$

where $X(\theta)=-\ln(\frac{\delta_1}{|\lambda|}(t(1+\cos(d-1)\theta))^{\frac{1}{d-1}})$, $Y(\theta)=-\theta-(\arg\frac{1}{\lambda}-\delta_2(\frac{2k+1}{d-1}\pi-\theta))$. So $\bar{W}_1^k$ and then $W_1^k$ has finite area.

The symmetry implies that the area of the wedge sharped region $W_2$ between $\tilde{\Gamma}_2$ and the corresponding asymptote $Y=\pi+\arg(-a_0)$ takes the same value as the area of $W_1$, and the area of wedge sharped region $W_2^k$ between $\tilde{\Gamma}_2^k$ and the corresponding asymptote $Y=-\frac{2k+1}{d-1}\pi-\arg\frac{1}{\lambda}$ takes the same value as the area of $W_1^k$.

Denote union of these wedge sharped regions by $W$ and $\ln\circ\varphi^{-1}(\Pi(t))\cup(\bigcup_{k=0}^{d-2}(\ln\circ\sigma\circ\psi^{-1}(\Pi_k(t))))$ by $\Pi$. In view of that $\ln\circ\varphi^{-1}(\Pi(t))$ and $\ln\circ\sigma\circ\psi^{-1}(\Pi_k(t))$ are also absorbing sets of those Baker domains respectively, and that those asymptotes alternately exist with alternation as $2\pi$ and $\frac{2\pi}{d-1}$ respectively, $\Xi\setminus(W\cup\Pi)$ is bounded domain, and $\Xi\setminus\Pi=W\cup(\Xi\setminus W\cup\Pi)$ has finite area. So the complement of the union of all virtual immediate basins of $N_f(z)$, a subset of $\Xi\setminus\Pi$, has finite area. The proof is completed. □

**Corollary 3.7** *Each immediate basin of $N_f(z)$ has finite area.*

**Remark 3.8** In the case $\deg(P)=1$, i.e., $P(z)=a_1z+a_0$ with $(a_1\neq 0)$, $f(z)=(e^z-1)e^{a_1+a_0e^{-z}}$, $N_f(z)=z-\frac{e^{2z}-e^z}{e^{2z}-a_0e^z+a_0}$ and $g(z)=ze^{\frac{z-z^2}{z^2-a_0z+a_0}}$. Then 1 is super attracting fixed point of $g(z)$, and zero is either rational indifferent fixed point or essential singularity of $g(z)$. For example, $P(z)=a_1z$, then $g(z)=ze^{\frac{1-z}{z}}$, zero is essential singularity of $g(z)$, and the attracting basin of 1 contains region $\Theta=\{re^{i\theta}:1<r,|\theta|<\frac{\pi}{3}\}$.

In fact, if $z\in\Theta$, $g(z)=z(1+\sum_{m=1}^{+\infty}\frac{1}{m!}(\frac{1-z}{z})^m)$, and $\left|\frac{1-z}{z}\right|=(\frac{1-2r\cos\theta+r^2}{r^2})^{\frac{1}{2}}<1$. Therefore,

$$|\frac{g(z)-1}{z-1}|=|\frac{z-1+z\sum_{m=1}^{+\infty}\frac{1}{m!}(\frac{1-z}{z})^m}{z-1}|=|\sum_{m=2}^{+\infty}\frac{1}{m!}(\frac{1-z}{z})^{m-1}|$$
$$\leq\sum_{m=2}^{+\infty}\frac{1}{m!}|\frac{1-z}{z}|^{m-1}<\sum_{m=2}^{+\infty}\frac{1}{m!}<1.$$

This implies that the attracting basin of 1 contains region $\Theta$, hence, in $\Xi$, the immediate basin $U_1$ of $N_f(z)$ has infinite area.

**Remark 3.9** Let $M_f(z)=N_f(z)+2\pi i$. Then $U_0$, $U_1$ and $U_\infty^k$ $(k=0,1,\ldots,n-2)$ are wandering domains of $M_f(z)$.

## References

[1] BEARDON A F. *Iteration of Rational Fuctions* [M]. Springer-Verlag, New York, 1991.
[2] PRZYTYCKI F. *Remarks on the Simple Connectedness of Basins of Sinks for Iterations of Rational Maps* [J]. Banach Center Publ., 23, PWN, Warsaw, 1989.
[3] SHISHIKURA M. *The connectivity of the Julia set and fixed points* [J]. Wellesley, MA, 2009.
[4] TAN Lei. *Branched coverings and cubic Newton maps* [J]. Fund. Math., 1997, **154**(3): 207–260.
[5] HARUTA M E. *Newton's method on the complex exponential function* [J]. Trans. Amer. Math. Soc., 1999, **351**(6): 2499–2513
[6] BERGWEILER W. *Newton's method and a class of meromorphic functions without wandering domains* [J]. Ergodic Theory Dynam. Systems, 1993, **13**(2): 231–247.
[7] MAYER S, SCHLEICHER D. *Immediate and virtual basins of Newton's method for entire functions* [J]. Ann. Inst. Fourier (Grenoble), 2006, **56**(2): 325–336.
[8] MAYER S. *Newton's method for transcendental functions* [D]. Diplomarbeit, TU Munchen, 2002.
[9] BUFF X, RÜCKERT J. *Virtual immediate basins of Newton maps and asymptotic values* [J]. Int. Math. Res. Not. 2006, Art. ID 65498, 18 pp.
[10] BOLSCH A. *Repulsive periodic points of meromorphic functions* [J]. Complex Variables Theory Appl., 1996, **31**(1): 75–79.
[11] BOLSCH A. *Iteration of meromorphic functions with countably many singularities* [D]. Dissertation, Technische Universitat, Berlin, 1997.
[12] HERRING M E. *An extension of the Julia-Fatou theory of iteration* [D]. PhD Thesis, Imperial College, 1994.
[13] BAKER I N, DOMINGUEZ P, HERRING M E. *Dynamics of functions meromorphic outside a small set* [J]. Ergodic Theory Dynam. Systems, 2001, **21**(3): 647–672.
[14] BAKER I N, DOMINGUEZ P, HERRING M E. *Functions meromorphic outside a small set: completely invariant domains* [J]. Complex Var. Theory Appl., 2004, **49**(2): 95–100.
[15] ZHENG Jianhua. *Iteration of functions meromorphic outside a small set* [J]. Tohoku Math. J., 2005, **57** (1): 23–43.
[16] ZHENG Jianhua. *Dynamics of Meromorphic Functions* [M]. Beijing: Tsinghua University, 2006.