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On the Least Eigenvalue of Graphs with Cut Vertices

Yi WANG, Yun QIAO, Yi Zheng FAN*
School of Mathematical Sciences, Anhui University, Anhui 230039*, P. R. China*

Abstract Let $\mathscr{S}$ be a certain set of graphs. A graph is called a minimizing graph in the set $\mathscr{S}$ if its least eigenvalue attains the minimum among all graphs in $\mathscr{S}$. In this paper, we determine the unique minimizing graph in $\mathscr{G}_n$, where $\mathscr{G}_n$ denotes the set of connected graphs of order n with cut vertices.

Keywords adjacency matrix; least eigenvalue; minimizing graph; cut vertex.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The adjacency matrix of G is defined to be a $(0, 1)$ matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i is adjacent to v_j, $a_{ij} = 0$ otherwise. The zeros of the characteristic polynomial $P(G, \lambda) = \det(\lambda I - A(G))$ of $A(G)$ are called the eigenvalues of G. Since $A(G)$ is symmetric, its eigenvalues are real and can be arranged as follows:

$$\lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_n(G).$$

One can find that $\lambda_n(G)$, denoted by $\rho(G)$, is exactly the spectral radius of $A(G)$. If, in addition, G is connected, then $A(G)$ is irreducible; and by Perron-Frobenius Theorem, the eigenvalue $\rho(G)$ is simple and there exists a unique (up to a multiple) corresponding positive eigenvector, usually referred to as the Perron vector of $A(G)$. There are many results in literatures concerning the spectral radius of the adjacency matrix of a graph, which involve the work in two directions: one for the bounds of spectral radius, e.g. [1–3], and one for the structure of graphs with extreme spectral radius subject to one or more given parameters, such as order and size [4], maximal degree [5], diameter [6–8], matching number [9], chromatic number [10], domination number [11], number of cut vertices [12], number of cut edges [13], number of pendant vertices

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* Corresponding author
E-mail address: wangy@ahu.edu.cn (Y. WANG); fanyz@ahu.edu.cn (Y. Z. FAN)

[14]. One can also refer to [15–17] for basis results on the spectral radius of the adjacency matrix of a graph.

However, much less is known about the least eigenvalue $\lambda_1(G)$ of the graph G, now denoted by $\lambda_{\min}(G)$. An eigenvector corresponding to $\lambda_{\min}(G)$ of $A(G)$ is called the least vector of G. For a graph with at least one edge, the least eigenvalue is negative, and is less than or equal to -1 with equality if and only if each component of the graph is complete; in addition, the least vectors contain both positive and negative entries, which may be a real reason why the least eigenvalue is not taken more attention than the spectral radius.

In the past, the main work on the least eigenvalue of a graph is about its bounds; one can refer to [1], [18]–[20]. Recently, two papers of Bell et.al [21, 22] and one paper of ours [23] appear in the same issue of the journal Linear Algebra and Its Applications. Bell et.al. studied the graph whose least eigenvalue is minimal among all connected graphs of given order and size. We determined the unique graph with the minimal least eigenvalue among all connected unicyclic graphs of fixed order and fixed girth. We think there will be more work on the least eigenvalue of a graph. In this paper, we continue this work, and characterize the unique graph with the minimal least eigenvalue among all connected graphs of fixed order which contain cut vertices.

2. Preliminaries

Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, and let G be a graph on vertices $v_1, v_2, \ldots, v_n$. Then x can be considered as a function defined on the vertex set of G, that is, for any vertex v_i, we map it to $x_i = x(v_i)$. We often say x_i is a value of vertex v_i given by x. One can find that

$$x^{\mathrm{T}} A(G) x = 2 \sum_{uv \in E(G)} x(u)x(v), \tag{2.1}$$

and λ is an eigenvector of G corresponding to an eigenvector x if and only if $x \neq 0$ and

$$\lambda x(v) = \sum_{u \in N_G(v)} x(u), \quad \text{for each } v \in V(G), \tag{2.2}$$

where $N_G(v)$ denotes the neighborhood of v in G. The equation (2.2) is also called a (λ, x)-eigenequation of G. In addition, for an arbitrary unit vector $x \in \mathbb{R}^n$,

$$\lambda_{\min}(G) \leq x^{\mathrm{T}} A(G) x \tag{2.3}$$

with equality if and only if x is a least vector of G.

Let G_1, G_2 be two disjoint connected graphs, and let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. We obtain a graph G from $(G_1 - v_1) \cup (G_2 - v_2)$ by adding a new vertex u and together with edges joining u to the vertices of $N_{G_1}(v_1) \cup N_{G_2}(v_2)$. The graph G is called a *coalescence* of G_1 and G_2 at vertices v_1, v_2 (see [16]), denoted by $G_1(v_1) \cdot G_2(v_2)$. Intuitively, $G_1(v_1) \cdot G_2(v_2)$ is obtained from G_1, G_2 by identifying v_1 with v_2 and forming a new vertex u. The graph $G_1(v_1) \cdot G_2(v_2)$ is also written as $G_1(u) \cdot G_2(u)$.

Lemma 2.1 ([23]) *Let G_1 and G_2 be two disjoint nontrivial connected graphs, and let $\{v_1, v_2\} \subseteq V(G_1), u \in V(G_2)$. Let $G = G_1(v_2) \cdot G_2(u)$ and let $\widetilde{G} = G_1(v_1) \cdot G_2(u)$. If there exists the least* *vector x of G such that $|x(v_1)| \geq |x(v_2)|$, then*

$$\lambda_{\min}(\widetilde{G}) \leq \lambda_{\min}(G) \tag{2.4}$$

with equality if and only if x is the least vector of $\widetilde{G}$, $x(v_1) = x(v_2)$ and $\sum_{w \in N_{G_2}(u)} x(w) = 0$.

Lemma 2.2 *Let G be a graph with two nonadjacent vertices p, q, and let $\widetilde{G}$ be obtained from the graph G by adding the edge pq. Let x be the least vector of G. Then*

(1) $\lambda_{\min}(\widetilde{G}) < \lambda_{\min}(G)$ if $x(p)x(q) < 0$;

(2) $\lambda_{\min}(\widetilde{G}) \leq \lambda_{\min}(G)$ if $x(p) = 0$ or $x(q) = 0$. In this case, the equality holds if and only if x is the least vector of $\widetilde{G}$ and $x(p) = x(q) = 0$.

Proof Assuming that x has unit length, by (2.3) we have

$$\begin{aligned}\lambda_{\min}(\widetilde{G}) &\leq x^{\mathrm{T}} A(\widetilde{G}) x = 2 \sum_{uv \in E(\widetilde{G})} x(u)x(v) = 2\Big(\sum_{uv \in E(G)} x(u)x(v) + x(p)x(q) \Big) \\ &= \lambda_{\min}(G) + 2x(p)x(q).\end{aligned}$$

If $x(p)x(q) < 0$, surely $\lambda_{\min}(\widetilde{G}) < \lambda_{\min}(G)$. If $x(p) = 0$ or $x(q) = 0$, then $\lambda_{\min}(\widetilde{G}) \leq \lambda_{\min}(G)$. In this case, if the equality holds, then x is also the least vector of $\widetilde{G}$. Denoting $\beta := \lambda_{\min}(G) = \lambda_{\min}(\widetilde{G})$, and comparing the (β, x)-eigenequation of G and $\widetilde{G}$ on the vertex p or q, we have $x(p) = x(q) = 0$. The sufficiency is easily verified by above inequalities. $\square$

Let G_1, G_2 be two disjoint connected graphs, and let $G_1 \vee G_2$ denote the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. Denote by O_n an empty graph of order n (without edges). Thus $O_p \vee O_q$ is a complete bipartite graph. Denote by $G(p, q)$ $(1 \leq p \leq q)$ a graph of order $(p + q + 1)$ obtained from $O_p \vee O_q$ by adding a new vertex together with an edge joining this vertex to some vertex of O_p; see Figure 1.1.

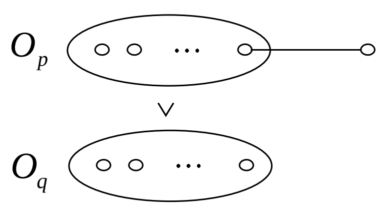

Figure 1.1 The graph $G(p, q)$ where $1 \leq p \leq q$

We need the following lemma for calculating the least eigenvalue of $G(p, q)$.

Lemma 2.3 ([16]) *Let G be a graph containing a vertex u, and let $\mathscr{C}(u)$ be the set of all cycles of G containing u. Then*

$$P(G, \lambda) = \lambda P(G - u, \lambda) - \sum_{v \in N_G(u)} P(G - u - v, \lambda) - 2 \sum_{Z \in \mathscr{C}(u)} P(G - V(Z), \lambda).$$

By Lemma 2.3, we have

$$P(G(p, q), \lambda) = \lambda^{n-4}[\lambda^4 - (pq + 1)\lambda^2 + (p - 1)q]$$

so that

$$\lambda_{\min}(G(p,q)) = -\sqrt{\frac{pq+1+\sqrt{(pq+1)^2-4(p-1)q}}{2}}.$$

Given n such that $p+q=n-1$, $\lambda_{\min}(G(p,q))$ attains a minimum when p takes uniquely at $p=\lfloor\frac{n-1}{2}\rfloor$ and hence q takes uniquely at $q=\lceil\frac{n-1}{2}\rceil$.

3. Main results

Let $\mathscr{S}$ be a certain set of graphs. A graph is called a minimizing graph in the set $\mathscr{S}$ if its least eigenvalue attains the minimum among all graphs in $\mathscr{S}$. Recall that a cut vertex in a connected graph is one whose deletion yields the resulting graph into two (or more) components. For convenience, denote the set of all connected graphs of order n with cut vertices by $\mathscr{G}_n$, and denote by α_n the minimum of the least eigenvalues among all graphs in $\mathscr{G}_n$.

Lemma 3.1 *α_n is strictly decreasing in n.*

Proof Let G be a minimizing graph in $\mathscr{G}_n$, and let x be the least vector of G of unit length. We assert that there exists at least one block B of G such that B contains two vertices p,q satisfying $x(p)+x(q)\neq 0$. Otherwise, each block of G contains exactly two vertices (that is, G is a tree) and the sum of their values given by x is zero. Discussing the (α_n,x)-eigenequation of G on any pendent vertex of G, we have $\alpha_n=-1$, a contradiction.

Let $\widetilde{G}$ be obtained from G by adding a new vertex w and joining w to both p and q, and let $\widetilde{x}\in\mathbb{R}^{n+1}$ such that $\widetilde{x}(w)=0$ and $\widetilde{x}(v)=x(v)$ for any vertex v of G. We have

$$\begin{aligned}\lambda_{\min}(\widetilde{G}) &\le \widetilde{x}^{\mathrm{T}}A(\widetilde{G})\widetilde{x} = 2\sum_{uv\in E(\widetilde{G})}\widetilde{x}(u)\widetilde{x}(v) = 2\Big(\sum_{uv\in E(G)}\widetilde{x}(u)\widetilde{x}(v)+\widetilde{x}(w)[\widetilde{x}(p)+\widetilde{x}(q)]\Big)\\ &= 2\sum_{uv\in E(G)}x(u)x(v)=\lambda_{\min}(G).\end{aligned}$$

If the equality holds, then $\widetilde{x}$ is the least vector of $\widetilde{G}$ corresponding to the eigenvalue $\lambda_{\min}(\widetilde{G})$. By considering the $(\lambda_{\min}(\widetilde{G}),\widetilde{x})$-eigenequation of $\widetilde{G}$ on vertex w, we have $0=\lambda_{\min}(\widetilde{G})\widetilde{x}(w)=\widetilde{x}(p)+\widetilde{x}(q)=x(p)+x(q)$, a contradiction. Obviously, $\widetilde{G}\in\mathscr{G}_{n+1}$, and then

$$\alpha_{n+1}\le\lambda_{\min}(\widetilde{G})<\lambda_{\min}(G)=\alpha_n.$$

Lemma 3.2 *Let G be a minimizing graph in $\mathscr{G}_n$, and let x be the least vector of G. Then x contains no zero entries.*

Proof Assume to the contrary, G contains a vertex u such that $x(u)=0$. If u is a cut-vertex of G, then G can be considered as a coalescence of two subgraphs, and written as $G=G_1(u)\cdot G_2(u)$. Note that one graph among G_1 and G_2, say G_1, contains a vertex $x(w)\neq 0$. Now let $\widetilde{G}=G_1(w)\cdot G_2(u)$. Surely $\widetilde{G}\in\mathscr{G}_n$, and by Lemma 2.1, $\lambda_{\min}(\widetilde{G})<\lambda_{\min}(G)$ as $x(u)\neq x(w)$, a contradiction. If u is not a cut vertex of G. Then $G-u\in\mathscr{G}_{n-1}$. Let $\widetilde{x}$ be subvector of x only by deleting the entry corresponding to u. Assume that x has unit length, then

$$\alpha_{n-1}\le\lambda_{\min}(G-u)\le\widetilde{x}^{\mathrm{T}}A(G-u)\widetilde{x}=x^{\mathrm{T}}A(G)x=\lambda_{\min}(G)=\alpha_n,$$

a contradiction to Lemma 3.1. □

Lemma 3.3 *Each block of a minimizing graph in $\mathscr{G}_n$ is a complete bipartite graph.*

Proof Let G be a minimizing graph in $\mathscr{G}_n$, and let x be a unit least vector of G. By Lemma 3.2, x contains no zero entries. Let B be any block of G. Denote by $V_B^+ = \{v \in B : x(v) > 0\}$, $V_B^- = \{v \in B : x(v) < 0\}$. By Lemma 2.2(1), every pair of vertices of B with opposite signs are adjacent. So there exists an edge between each vertex of V_B^+ and each vertex of V_B^-.

Furthermore, there exist no edges within V_B^+ or V_B^-; otherwise, let uv be such an edge. If uv is not a cut edge, the graph $G - uv$ is connected and also belongs to $\mathscr{G}_n$. However, $x^{\mathrm{T}} A(G-uv)x < x^{\mathrm{T}} A(G)x$, a contradiction. If uv is a cut edge, then $G - uv$ contains exactly two components, say G_1 and G_2. Let $\widetilde{x}$ be obtained from x by replacing $x(v)$ by $-x(v)$ for each vertex $v \in V(G_1)$ and preserving the values of other vertices. We have

$$\lambda_{\min}(G) \leq \widetilde{x}^{\mathrm{T}} A(G)\widetilde{x} < x^{\mathrm{T}} A(G)x = \lambda_{\min}(G),$$

a contradiction. The result follows. □

Theorem 3.4 *The graph $G(\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil)$ of Figure 1.1 is the unique minimizing graph in $\mathscr{G}_n$.*

Proof Let G be a minimizing graph in $\mathscr{G}_n$, and let x be the least vector of G of unit length. We first assert that G has exactly two blocks. Otherwise, let $G_1, G_2, \ldots, G_k$ $(k \geq 3)$ be all blocks of G and G_1 have exactly one vertex belonging to other blocks. By Lemmas 3.2 and 3.3, x contains no zero entries, and each block of G is bipartite complete with opposite signs on bipartition of the vertex set. So we can get a new graph $\widetilde{G} \in \mathscr{G}_n$ from G by joining the vertices of $G_2 \cup G_3 \cup \cdots \cup G_k$ with opposite signs. By Lemma 2.2(1), $\lambda_{\min}(\widetilde{G}) < \lambda_{\min}(G)$, a contradiction.

Now assume G_1, G_2 are the all blocks of G, which share a common vertex u. We will prove that one of G_1, G_2 contains only two vertices. Denote by $V_i^+ = \{v \in G_i : x(v) > 0\}$, $V_i^- = \{v \in G_i : x(v) < 0\}$ for $i = 1, 2$. By Lemma 3.3, G_1 and G_2 are both complete bipartite with bipartitions (V_1^+, V_1^-) and (V_2^+, V_2^-), respectively. Without loss of generality, we assume that $x(u) < 0$ and there exist vertices $u_1 \in V_1^+$, $u_2 \in V_2^+$ such that $x(u_1) \geq x(u_2)$; see Figure 3.1.

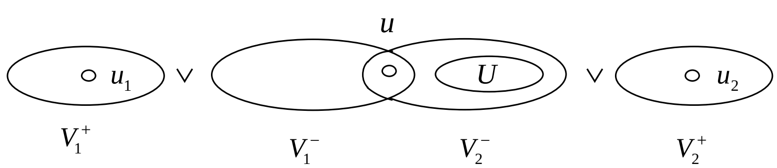

Figure 3.1 Illustration of the Proof of Theorem 3.4

If for some i $(1 \leq i \leq 2)$, V_i^+ or V_i^- contains only one vertex, then G_i contains exactly two vertices since it is a block and also bipartite with (V_i^+, V_i^-) as the bipartition. The result follows in this case. Now assume that each of V_1^+, V_1^-, V_2^+, V_2^- contains two or more vertices. Let $U := V_2^- - \{u\} \neq \emptyset$. Deleting all edges between U and u_2, and adding all possible edges between U and V_1^+, we obtain a graph $\widetilde{G} \in \mathscr{G}_n$. Observe that

$$x^{\mathrm{T}} A(\widetilde{G})x - x^{\mathrm{T}} A(G)x = \sum_{v \in V_1^+, w \in U} x(v)x(w) - \sum_{v = u_2, w \in U} x(v)x(w)$$

$$< \sum_{v=u_1,w\in U} x(v)x(w) - \sum_{v=u_2,w\in U} x(v)x(w)$$
$$= [x(u_1) - x(u_2)] \sum_{w\in U} x(w) \le 0.$$

So this case cannot occur.

By above discussion, G is of structure of the graph $G(p,q)$ of Figure 1.1 for some p or q. From the discussion after Lemma 2.3, we find $G = G(\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil)$, and the result follows. $\square$

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