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A Nonconvex Nonorientable Crossing Number Sequence

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Abstract It is well known that finding the crossing number of a graph on nonplanar surfaces is very difficult. In this paper we study the crossing number of the circular graph $C(10,4)$ on the projective plane and determine the nonorientable crossing number sequence of $C(10,4)$. On the basis of the result, we show that the nonorientable crossing number sequence of $C(10,4)$ is not convex.

Keywords crossing number; embedding; non-orientable surface.

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1. Introduction

Here we consider simple connected graphs with the vertex set V and the edge set E. For terms and notations, we refer to [1, 2]. A drawing of G in the plane $\mathcal{R}^2$ is an immersion $\phi : G \to \mathcal{R}^2$ such that

(1) $\phi(v) \cap \phi(x) = \emptyset$ for each $v \in V(G)$ and $x \in (V(G) \cup E(G)) - \{v\}$, and
(2) $\phi(e) \cap \phi(f)$ is finite for each pair $\{e, f\}$ of edges of G.

A drawing is good, if for all $\phi(E)$, no one crosses itself, no two cross more than once, and no more than two cross at a point in the plane. A crossing in a good drawing is a point of intersection of two elements in $\phi(E)$. A good drawing is said to be optimal if it minimizes the number of crossings. The crossing number $cr_0(G)$ of a graph G is the number of crossings in any optimal drawing of G in the plane. Similarly, we can define the crossing number of a graph G drawn on the nonorientable surfaces by $\widetilde{cr}_i(G)$, where i is the genus of the nonorientable surface in which graph G is drawn. Here, $\widetilde{cr}_0(G)$ is the crossing number of G on the plane. Archdeacon, Bonnington and Širáň [3] considered the crossing number sequence of graphs. A sequence $C_0, C_1, \ldots, C_n, \ldots$ is called the orientable crossing number of G if each C_k is the crossing number of G in the orientable surface S_k, the sphere with k handles. We may define the concept nonorientable crossing number sequence of graphs similarly. Širáň [4] introduced the concept

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of convex crossing number sequence of graphs. A sequence $c_1, c_2, \ldots, c_n, \ldots$ is called convex if $c_i - c_{i+1} \leq c_{i-1} - c_i$ for each i. He conjectured that the orientable crossing number sequence is convex. This idea is based on the following reasons [3]:

If adding the second handle saves more edges than adding the first handle, why not add the second handle first? Later, Archdeacon et al, constructed a counter-example of this conjecture. But what about nonorientable crossing number sequence? (That is, is every nonorientable crossing number sequence of graphs convex?). Here, in this paper we consider the case of nonorientable crossing number sequence.

Before stating our main results, we have to give some definitions for graph embeddings. An embedding (or 2-cell embedding) of a graph G in a surface S is a drawing of G in S such that no edge-crossing is permitted and edges meet only at their common vertices. Each component of S-G is an open disc. By Euler's equation, one may see that for every surface, there are graphs which cannot be embedded in the surface. A circular graph $C(m, n)$ is an m-cycle $C = (1, 2, \ldots, m)$ together with the chords such as (i, j) with $|i - j| \equiv 0 (\mathrm{mod}\, n)$. It is clear that all circular graphs are 4-regular except $C(2m, m)$. There are many papers written for the crossing number of circular graphs on the plane [5–8], but little is known for the crossing numbers of them on the surfaces. Here we study the crossing number sequence of $C(10, 4)$ on the nonorientable surfaces and show that its crossing number sequence on nonorientable surface is $5, 3, 0, 0, \ldots$. This shows that there exists a graph whose nonorientable crossing number sequence is not convex.

2. The main result

Archdeacon, Glover, Huneke et al proved the following result:

Lemma 1 ([9, 10]) *There are 103 minimal forbidden graphs for the projective plane.*

If we are allowed to delete both vertices and edges, and also to contract edges, then the resulting graph is called a minor. If a graph G has a minor isomorphic to one of those 103 forbidden graphs for the projective plane, then G cannot be embedded on the projective plane.

Lemma 2 ([11]) *If $n = 2l + 2$ with $l \geq 3$, then $\bar{\gamma}(C(n, l)) = 2$, where $\bar{\gamma}$ is the nonorientable genus of circular graph $C(n, l)$.*

Lemma 3 ([12]) *The crossing number of circular graph $C(2m + 2, m)$ $(m \geq 3)$ is $m + 1$.*

Let G be a nonplanar graph. Then G contains a subdivision of K_5 or $K_{3,3}$ by Kuratowski's Theorem [13]. It is clear that removing edges from G will decrease the possibility of existence of such Kuratowski's subgraphs. The removal number of a graph G is the least number of edges of G such that after removing them, the resulting graph is planar. Similarly, we may define the removal number of a graph with respect to a fixed surface. The following result shows the relation between the crossing number and the removal number.

Lemma 4 *Let G be a graph and S be a surface. Then the crossing number of G on S is no less than the removal number of G on S.*

Theorem A *The crossing number of $C(10, 4)$ on N_1 is 3, i.e., $\widetilde{cr}_1(C(10, 4)) = 3$.*

By Lemma 4, our proof proceeds as follows.

Step 1. We shall prove that the removal number of $C(10,4)$ on N_1 is at least 3;

Step 2. We shall provide an optimal drawing of $C(10,4)$ on N_1 which implies that the crossing number of $C(10,4)$ on N_1 is 3.

Lemma 5 *The removal number of* $C(10,4)$ *on* N_1 *is at least 3 (i.e., deleting any two edges from* $C(10,4)$ *will not result in a projective planar graph).*

Proof Let

$$\begin{aligned}
E_1 &= \{(v_2,v_8),(v_3,v_7),(v_5,v_9),(v_6,v_{10})\},\\
E_2 &= \{(v_1,v_{10}),(v_3,v_9),(v_5,v_6),(v_7,v_8)\},\\
E_3 &= \{(v_1,v_7),(v_2,v_6),(v_6,v_{10})\},\\
E_4 &= \{(v_1,v_{10}),(v_4,v_{10}),(v_5,v_6)\},\\
E_5 &= \{(v_1,v_{10}),(v_9,v_{10}),(v_5,v_6)\}.
\end{aligned}$$

We can find two graphs, called $F_4(10,16)$ and $E_{18}(8,15)$ in the Appendix A of [2] in those 103 graphs which are minimal forbidden for the projective plane. It is easy to see that $F_4(10,16)$ and $E_{18}(8,15)$ are minors of $C(10,4)$ by deleting or contracting some edges of $C(10,4)$ (as shown in Figures 1–5).

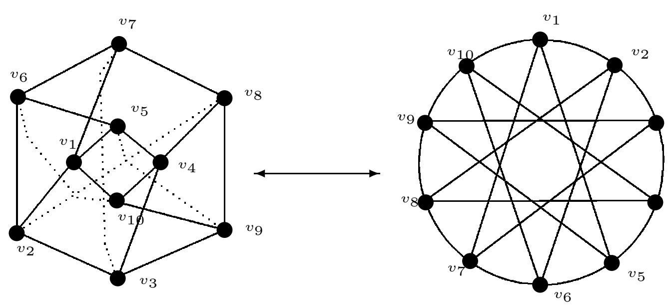

Figure 1 $F_4(10,16)$ and its minor $C(10,4)$

where $F_4(10,16)$ is obtained from C(10,4) by deleting $(v_2,v_8),(v_3,v_7),(v_5,v_9),(v_{10},v_6)$.

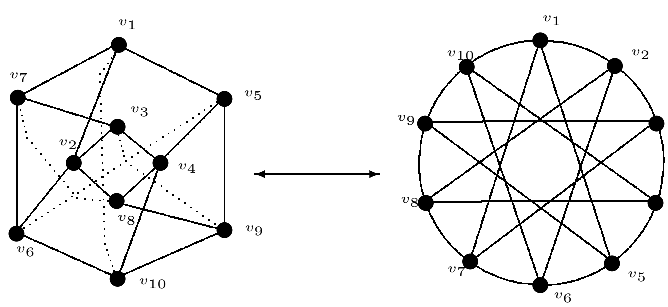

Figure 2 $F_4(10,16)$ and its minor $C(10,4)$

where $F_4(10,16)$ is obtained from C(10,4) by deleting $(v_1,v_{10}),(v_3,v_9),(v_5,v_6),(v_7,v_8)$.

By the symmetry of circular graphs, after changing the order of vertices of $C = (v_1, \ldots, v_{10})$ into an anticlockwise order, deleting $(v_1, v_{10}), (v_3, v_9)$ is equivalent to deleting $(v_1, v_{10}), (v_2, v_8)$ on the right side of Figure 2, where we only consider the position of edges but not labels of vertices.

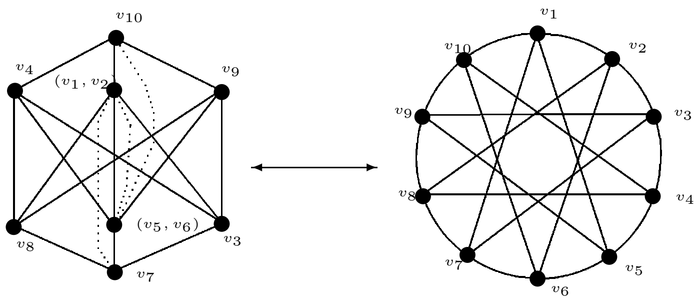

Figure 3 $E_{18}(8, 15)$ and its minor $C(10, 4)$

where $E_{18}(8, 15)$ is obtained from C(10,4) by deleting $(v_1, v_7), (v_2, v_6), (v_{10}, v_6)$ and contracting $(v_1, v_2), (v_5, v_6)$.

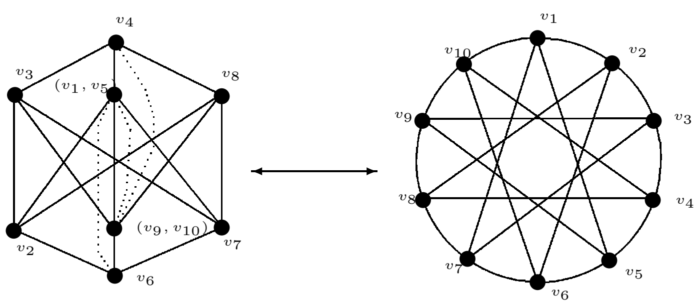

Figure 4 $E_{18}(8, 15)$ and its minor $C(10, 4)$

where $E_{18}(8, 15)$ is obtained from C(10,4) by deleting $(v_1, v_{10}), (v_4, v_{10}), (v_5, v_6)$ and contracting $(v_1, v_5), (v_9, v_{10})$.

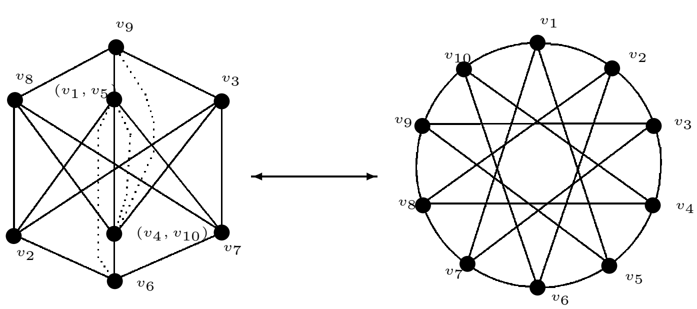

Figure 5 $E_{18}(8, 15)$ and its minor $C(10, 4)$

where $E_{18}(8, 15)$ is obtained from C(10,4) by deleting $(v_1, v_{10}), (v_9, v_{10}), (v_5, v_6)$ and contracting $(v_1, v_5), (v_4, v_{10})$.

By the symmetry of circular graphs, the deletion of (v_1, v_{10}) is equivalent to deletion of (v_4, v_5) in Figure 5. Then by Lemma 1 and Figures 1–5, we have the following.

Claim 6 For any two edges $e_1, e_2 \in E_i$ $(1 \le i \le 5)$, $C(10,4) - e_1 - e_2$ cannot be embedded on N_1.

Now, by the symmetry of $C(10,4)$, we only consider the case of deleting edges $e_1 = (v_{10}, v_1), e_2 = (v_1, v_5)$ from $C(10,4)$. Suppose that $C(10,4) - e_1 - e_2$ may be embedded on N_1. Then by Euler's formula, we have

Claim 7 If $C(10,4) - e_1 - e_2$ can be embedded on N_1, then all of such embeddings are quadrangular (i.e., each of 9 faces is a 4-gon.)

Jordan Curve Theorem states that any simple closed curve (cycle) C on the plane devides the plane into two arcwise connected components. Similarly, we have the following generalized version of Jordan Curve Theorem:

Claim 8 Any simple closed curve (cycle) C on a surface S which is contractible devides S into two connected components such that they have C as their common boundary.

Here a curve (cycle) C on a surface S is contractible, if one of two components of $S - C$ is an open disc. Otherwise, C is noncontractible. If C is contractible, one of interior or outer of C has genus zero, say the interior of C and we denote it by $\mathrm{int}(C)$. Now we concentrate on the 4-cycles passing through the vertex v_1. Since every edge is contained in exact two faces (in an embedded graph), we see that there are exact three 4-cycles passing through (v_7, v_1) and (v_1, v_2), $C_1 = (v_1, v_2, v_8, v_7)$, $C_2 = (v_1, v_2, v_6, v_7)$ and $C_3 = (v_1, v_2, v_3, v_7)$ such that two of them are contractible.

Case 1 Two 4-cycles of C_1, C_2, C_3 are contractible and one is noncontractible on N_1.

Subcase 1.1 $C_1 = (v_1, v_2, v_8, v_7)$ is noncontractible and C_2, C_3 are contractible on N_1.

Let us consider $C_2 = (v_1, v_2, v_6, v_7)$. Since C_2 is contractible, $\mathrm{int}(C_2)$ is an open disc. If there exists a vertex $x \in \mathrm{int}(C_2)$ (other than v_1, v_2, v_6, v_7), then $C(10,4) - (v_1, v_{10}) - (v_1, v_5) - \{v_2, v_6, v_7\}$ will have at least three distinct components, a contrary to the fact that this subgraph has exact two components. So, C_2 is a facial cycle (i.e., a 4-gon). Similarly, C_3 is also a facial cycle. So, we may suppose that the local rotation of edges around v_2 and v_7 are, respectively,

$$\rho(v_2) : (v_2, v_8) \to (v_2, v_3) \to (v_2, v_1) \to (v_2, v_6),$$
$$\rho(v_7) : (v_7, v_8) \to (v_7, v_6) \to (v_7, v_1) \to (v_7, v_3).$$

Since every face is a 4-gon (Claim 6), there exists a vertex $x \in N(v_6) \cap N(v_8) - \{v_2\} - \{v_7\}$ (i.e., (x, v_8, v_2, v_6) bounds a 4-gon). This is impossible since $|N(v_6) \cap N(v_8)| = 2$.

Subcase 1.2 $C_2 = (v_1, v_2, v_6, v_7)$ is noncontractible and C_1, C_3 are contractible on N_1.

After a similar discussion as we did in Subcase 1.1, we conclude that both C_1 and C_3 are facial cycles and this allows us to suppose that the local rotation of edges around v_2 and v_7 are, respectively:

$$\rho(v_2) : (v_2, v_1) \to (v_2, v_8) \to (v_2, v_6) \to (v_2, v_3),$$
$$\rho(v_7) : (v_7, v_6) \to (v_7, v_8) \to (v_7, v_1) \to (v_7, v_3).$$

As we have reasoned in Subcase 1.1, there exists a vertex $x \in N(v_6) \cap N(v_8) - \{v_2\} - \{v_7\}$, also a contradiction.

Subcase 1.3 $C_3 = (v_1, v_2, v_3, v_7)$ is noncontractible and C_1, C_2 are contractible on N_1.

Let us consider $C_1 = (v_1, v_2, v_8.v_7)$ and $C_2 = (v_1, v_2, v_6, v_7)$. Since both of C_1 and C_2 are contractible and $C(10,4) - (v_1, v_{10}) - (v_1, v_5) - \{v_2, v_7\}$ has exact two components, both of C_1 and C_2 are facial cycles. So, we may further suppose that the local rotation of edges around v_2 and v_7 are, respectively,

$$\rho(v_2) : (v_2, v_1) \to (v_2, v_6) \to (v_2, v_3) \to (v_2, v_8),$$
$$\rho(v_7) : (v_7, v_3) \to (v_7, v_6) \to (v_7, v_1) \to (v_7, v_8).$$

Since each face is a 4-gon, there exists a vertex $x \in N(v_6) \cap N(v_3) - \{v_2\} - \{v_7\}$ such that (v_2, v_6, x, v_3) is a 4-gon. This contradicts the fact that $N(v_3) \cap N(v_6) = \{v_2, v_7\}$.

Case 2 C_1, C_2, C_3 are all contractible on N_1.

Subcase 2.1 Both C_1 and C_2 are facial cycles.
Then C_3 must be nonfacial (since otherwise (v_1, v_2) should be contained in the boundaries of three distinct facial cycles). So, we may assume that the local rotation of edges around v_2 and v_7 are, respectively,

$$\rho(v_2) : (v_2, v_1) \to (v_2, v_8) \to (v_2, v_3) \to (v_2, v_6),$$
$$\rho(v_7) : (v_7, v_3) \to (v_7, v_8) \to (v_7, v_1) \to (v_7, v_6).$$

Thus, we have

Claim 9 $\{v_6, v_8\} \cap \mathrm{int}(C_3) \neq \varnothing$, where int (C_3) denotes the open disc bounded by C_3.
This implies that $C(10,4) - (v_1, v_{10}) - (v_1, v_5) - \{v_2, v_7\}$ has at least three distinct components, a contradiction as desired.

Subcase 2.2 Both C_2 and C_3 are facial.
Then C_1 must be nonfacial. This allows us to assume that the local rotation of edges around v_2 and v_7 are, respectively,

$$\rho(v_2) : (v_2, v_1) \to (v_2, v_6) \to (v_2, v_8) \to (v_2, v_3),$$
$$\rho(v_7) : (v_7, v_8) \to (v_7, v_6) \to (v_7, v_1) \to (v_7, v_3).$$

Thus, $\{v_3, v_6\} \cap \mathrm{int}(C_1) \neq \varnothing$. This implies that $C(10,4) - (v_1, v_{10}) - (v_1, v_5) - \{v_2, v_7\}$ has at least three distinct components. A contradiction.

Subcase 2.3 Both C_1 and C_3 are facial.
Now C_2 is nonfacial. As we have discussed in Subcase 2.1, we have that $\{v_3, v_8\} \cap \mathrm{int}(C_2) \neq \varnothing$, which implies that $C(10,4) - (v_1, v_{10}) - (v_1, v_5) - \{v_2, v_7\}$ has three components. Also a contradiction to the fact that this subgraph has exact two components. This completes the proof of Lemma 5. □

Now, Theorem A follows from Lemma 5 and the optimal drawing of $C(10,4)$ shown in Figure

6.

Theorem B *The nonorientable crossing number sequence of* $C(10,4)$ *is* $5,3,0,0,\ldots$.

Proof By Lemmas 2, 3 and Theorem A, we can complete the proof. □

Remark Theorem B shows that the nonorientable crossing number sequence is not always convex.

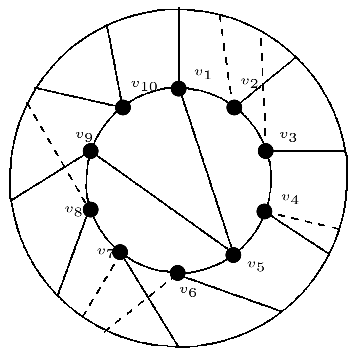

Figure 6 A drawing of $C(10,4)$ on N_1

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