Journal of Mathematical Research & Exposition
Nov., 2010, Vol. 30, No. 6, pp. 992–996
DOI:10.3770/j.issn:1000-341X.2010.06.006
Http://jmre.dlut.edu.cn

On a Kind of Summation Formula Using Howard's Generalized Stirling Numbers

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Abstract It is shown that Howard's degenerate weighted Stirling numbers can be used to construct a fruitful summation formula for a class of formal power series involving generalized factorials. This is achieved with the aid of a series-transformation formula due to He, Hsu and Shiue, and several identities involving generalized Stirling numbers and Bell numbers are given to illustrate the application of the formula obtained.

Keywords generalized factorial; Howard's GSN; series-transformation formula; formal series summation.

Document code A
MR(2000) Subject Classification 05A10; 05A15; 05A19
Chinese Library Classification O158; O173.1

1. Introduction

For a given positive integer p and any real or complex number θ we define

$$(t|\theta)_p := \prod_{j=0}^{p-1}(t - j\,\theta), \tag{1}$$

and call it the generalized falling factorial of order p with increment θ. In particular we define $(t|\theta)_0 = 1$, and we may write $(t|0)_p = t^p$ and $(t|1)_p = (t)_p$ with $(t)_0 = 1$.

It is known that Howard's generalized Stirling numbers (GSN) $S(p, j, \lambda|\theta)$ $(0 \le j \le p)$, also called degenerate weighted Stirling numbers with real or complex parameters λ and θ, may be defined by the expansion

$$(t + \lambda|\theta)_p = \sum_{j=0}^{p} S(p, j, \lambda|\theta)(t)_j. \tag{2}$$

Consequently one may note $S(p, j, \lambda|\theta) = 0$ $(j > p)$, and we see that $S(p, j, \lambda|\theta)$ $(p \ge j \ge 0)$ satisfy the recurrence relations [3, 4]

$$S(p + 1, j, \lambda|\theta) = S(p, j - 1, \lambda|\theta) + (j - p\,\theta + \lambda)\, S(p, j, \lambda|\theta) \tag{3}$$

with $S(0, 0, \lambda|\theta) = S(p, p, \lambda|\theta) = 1$ and $S(p, 0, \lambda|\theta) = (\lambda|\theta)_p$.

Received January 15, 2009; Accepted June 30, 2009
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Howard's GSN may also be defined by the generating function [3]

$$(1+\theta\, t)^{\lambda/\theta}\left((1+\theta\, t)^{1/\theta}-1\right)^j = j\,!\sum_{p=j}^{\infty} S(p,j,\lambda|\theta)\frac{t^p}{p\,!} \tag{4}$$

which is actually equivalent to the relation (2).

Note that Carlitz's GSN $S(p,j\,|\theta)$ and the ordinary Stirling numbers $S(p,j)$ of the second kind are just special cases given in [1, 3]

$$S(p,j\,|\theta) = S(p,j,0\,|\theta), \quad S(p,j) = S(p,j,0\,|\,0),$$

respectively. Frequently $S(p,j)$ is also written as $\left\{{p \atop j}\right\}$.

Using the difference operator Δ defined by $\Delta f(t) = f(t+1) - f(t)$ and $\Delta^j = \Delta\Delta^{j-1}$ $(j \geq 2)$, and applying the Newton interpolation formula to the LHS (left-hand side) of (2), we see that $j\,!\,S(p,\lambda,\lambda\,|\,\theta)$ can be written in the form

$$j\,!S(p,j,\lambda\,|\,\theta) = \left[\Delta^j(t+\lambda\,|\,\theta)_p\right]_{t=0} = \sum_{r=0}^{j}(-1)^{j-r}\binom{j}{r}(r+\lambda\,|\,\theta)_p. \tag{5}$$

In particular, $\Delta^0 \equiv 1$ denotes an identity operator so that $\Delta^0 f(t) = f(t)$.

Both (2) and (5) suggest that Howard's GSN can be used to sum some power series involving $(k+\lambda\,|\,\theta)_p$ $(k = 0, 1, 2, \ldots)$ as coefficients. Actually, Hsu-Shiue's paper [4] provided a pair of summation formulas as follows

$$\sum_{k=0}^{\infty}(k+\lambda\,|\,\theta)_p\, t^k = \sum_{j=0}^{p}\frac{j\,!\,S(p,j,\lambda\,|\,\theta)\,t^j}{(1-t)^{j+1}}, \quad |t| < 1, \tag{6}$$

$$\sum_{k=0}^{n}(k+\lambda\,|\,\theta)_p\, t^k = \sum_{j=0}^{p} j\,!\,S(p,j,\lambda\,|\,\theta)\phi(t,n,j), \tag{7}$$

where (7) holds for any given t, in which $\phi(1,n,j) = \binom{n+1}{j+1}$, and for $t \neq 1$, $\phi(t,n,j)$ is given by

$$\phi(t,n,j) = \frac{1}{1-t}\left\{\left(\frac{t}{1-t}\right)^j - t^{n+1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{t}{1-t}\right)^r\right\}, \tag{8}$$

As may be observed, for $|t| < 1$ we have

$$\lim_{n\to\infty}\phi\,(t,n,j) = t^j/(1-t)^{j+1},$$

so that (6) is implied by (7) with $|t| < 1$. Also, it is evident that Euler's formula for $\sum_{k=0}^{\infty} k^p\, t^k$ and Stirling's formula for $\sum_{k=1}^{n} k^p$ are consequences of (6) and (7) with $\lambda = \theta = 0$, respectively.

2. A summation formula and its consequences

What we want to investigate is a general summation formula contained in the following theorem.

Theorem *Let $\alpha(t) = \Sigma_{k=0}^{\infty}\alpha_k t^k$ be a formal power series with real or compex coefficients. Then there is a formal series identity of the form*

$$\sum_{k=0}^{\infty}(k+\lambda\,|\,\theta)_p\,\alpha_k\, t^k = \sum_{j=0}^{p} S(p,j,\lambda\,|\,\theta)\,\alpha^{(j)}(t)t^j, \tag{9}$$

where $\alpha^{(j)}(t)$ is the jth formal derivative of $\alpha(t)$. Moreover, (9) becomes a sum formula of convergent power series for $|t| < 1$ whenever $\alpha(t)$ is an analytic function for $|t| < 1$; and for $t = 1$ whenever $\alpha_k = O(k^{-p-d})$ $(d > 1, k \to \infty)$.

Proof Actually, (9) may be obtained as a useful specialization of a formal series-transformation formula provided by He, Hsu and Shiue [2], namely

$$\sum_{k=0}^{\infty} f(k)\, g^{(k)}(0)\, \frac{t^k}{k\,!} = \sum_{k=0}^{\infty} \Delta^k f(0)\, g^{(k)}(t)\, \frac{t^k}{k\,!}, \tag{10}$$

where $\{f(k)\}$ is any given sequence of real or complex numbers, and $g^{(k)}(t)$ denotes the jth formal derivative of a formal power series $g(t)$.

To deduce (9) from (10), it suffices to take $f(t) = (t + \lambda|\theta)_p$, $g(t) = \alpha(t)$ in (9), and to make use of (5), so that

$$\Delta^k f(0) = \left[\Delta^k (t + \lambda \,|\, \theta)_p\right]_{t=0} = k\,!\, S(p, k, \lambda \,|\, \theta), \quad g^{(k)}(t) = \alpha^{(k)}(t).$$

Note that $S(p, k, \lambda \,|\, \theta) = 0$ for $k > p$. Hence the right-hand side of (9) follows from that of (10).

The other part of the conclusion in the theorem is obvious, since the analyticity of $\alpha(t)$ ($|t| < 1$) implies that of the power series in (9) for $|t| < 1$; and the order condition $\alpha_k = O(k^{-p-d})$ ($d > 1$) sufficiently ensures the absolute convergence of the infinite series in (9) with $t = 1$. □

It is easy to observe that Hsu-Shiue's formulas (6) and (7) are included in (9) as particular consequences when $\alpha(t)$ is taken to be

$$\alpha_1(t) = \sum_{k=0}^{\infty} t^k = 1/(1-t), \quad |t| < 1$$

and

$$\alpha_2(t) = \sum_{k=0}^{n} t^k = (1 - t^{n+1})/(1-t), \quad |t| < 1$$

respectively. Indeed, we have the jth derivatives

$$\alpha_1^{(j)}(t) = j\,!\,(1-t)^{-j-1}, \qquad \alpha_2^{(j)}(t) = j\,!\,\phi(t, n, j)/t^{j^2},$$

where $\phi(t, n, j)$ as given by (8). More consequences will be presented in the next section.

3. Some identities involving Stirling and Bell numbers

Note that Howard's generalized Bell numbers $W(p, \lambda \,|\, \theta)$ can be defined by the expression

$$W(p, \lambda \,|\, \theta) := \sum_{j=0}^{p} S(p, j, \lambda \,|\, \theta). \tag{11}$$

The classical Bell number is of course given by $W(p) = W(p, 0 \,|\, 0)$. Let us rewrite (4) in the form

$$(1 + \theta\, t)^{\lambda/\theta} \big((1 + \theta\, t)^{1/\theta} - 1\big)^j / j\,! = \sum_{p=0}^{\infty} S(p, j, \lambda \,|\, \theta) \frac{t^p}{p\,!} \tag{4 bis}$$

and sum the both sides of (4 bis) for all j $(0 \le j \le \infty)$. Then we obtain [1, 3]

$$(1 + \theta\, t)^{\lambda/\theta} exp[(1 + \theta\, t)^{1/\theta} - 1] = \sum_{p=0}^{\infty} W(p, \lambda \,|\, \theta) \frac{t^p}{p\,!}. \tag{12}$$

This is the exponential generating function for the number sequence $\{W(p,\lambda\,|\,\theta)\}$.

According to the theorem in Section 2 and taking

$$\alpha(t) = f(t) = \sum_{k=0}^{\infty} f^{(k)}(0)\, t^k/k\,!$$

with $\alpha_k = f^{(k)}(0)/k\,! = O(k^{-p-d})$ $(d>1,\ k\to\infty)$, we see that (9) yields an exact identity for $t=1$:

$$\sum_{k=0}^{\infty} \frac{(k+\lambda\,|\,\theta)_p}{k\,!} f^{(k)}(0) = \sum_{j=0}^{p} S(p,j,\lambda\,|\,\theta) f^{(j)}(1). \tag{13}$$

This formula may be used to produce various identities of some interest.

1° Taking $f(t)=e^t$ so that $f^{(k)}(0)=1$ and $f^{(k)}(1)=e$, we find that (13) gives an identity of the form

$$\frac{1}{e}\sum_{k=0}^{\infty} \frac{(k+\lambda\,|\,\theta)_p}{k\,!} = W(p,\lambda\,|\,\theta). \tag{14}$$

This may be called the Dobinski-type identity for Howard's GSN. Clearly (12) and (14) with $\lambda=0$ and $\theta\to 0$ reduce to the classical formulas for the ordinary Bell numbers

$$e^{e^t-1} = \sum_{p=0}^{\infty} W(p)\, t^p/p\,!, \quad \frac{1}{e}\sum_{k=0}^{\infty} k^p/k\,! = W(p).$$

2° Here we would like to adopt Knuth's notations for the ordinary Stirling numbers of the first and second kinds, namely

$$\genfrac{[}{]}{0pt}{}{n}{k} := \frac{1}{k\,!}\big[D^k (t)_n\big]_{t=0}, \quad \genfrac{\{}{\}}{0pt}{}{n}{k} := \frac{1}{k\,!}\big[\Delta^k t^n\big]_{t=0} = S(n,k),$$

where $D\equiv \mathrm{d}/\mathrm{d}t$ and $D^k \equiv (\mathrm{d}/\mathrm{d}t)^k$. To use (13), let us take $f(t)=(t)_n$, so that $f^{(k)}(0) = k\,!\genfrac{[}{]}{0pt}{}{n}{k}$. We may compute $f^{(k)}(1)$ as follows

$$\begin{aligned} f^{(k)}(1) &= \big[D^k(t)_n\big]_{t=1} = \big[D^k(t+1)_n\big]_{t=0} = \big[(t=1)D^k(t)_{n-1}\big]_{t=0} + \binom{k}{1}\big[D^{k-1}(t)_{n-1}\big]_{t=0} \\ &= k\,!\Big(\genfrac{[}{]}{0pt}{}{n-1}{k} + \genfrac{[}{]}{0pt}{}{n-1}{k-1}\Big). \end{aligned}$$

Thus (13) implies the identity

$$\sum_{k=0}^{n} (k+\lambda\,|\,\theta)_p \genfrac{[}{]}{0pt}{}{n}{k} = \sum_{j=0}^{p} S(p,j,\lambda\,|\,\theta)\, j\,!\Big(\genfrac{[}{]}{0pt}{}{n-1}{j-1} + \genfrac{[}{]}{0pt}{}{n-1}{j}\Big). \tag{15}$$

This may be compared with the known identity [4]:

$$\sum_{k=0}^{n} (k+\lambda\,|\,\theta)_p \binom{n}{k} = \sum_{j=0}^{p} S(p,j,\lambda\,|\,\theta)\, j\,!\, 2^{n-j}\binom{n}{j}. \tag{16}$$

Certainly, (15) and (16) involve the following pair of combinatorial identities for $n\ge p\ge 1$:

$$\sum_{k=1}^{n} k^p \genfrac{[}{]}{0pt}{}{n}{k} = \sum_{j=1}^{p} j\,!\genfrac{\{}{\}}{0pt}{}{p}{j}\Big(\genfrac{[}{]}{0pt}{}{n-1}{j-1} + \genfrac{[}{]}{0pt}{}{n-1}{j}\Big) \tag{17}$$

$$\sum_{k=1}^{n} k^p \binom{n}{k} = \sum_{j=1}^{p} j\,!\, 2^{n-j}\genfrac{\{}{\}}{0pt}{}{p}{j}\binom{n}{j}. \tag{18}$$

As summation formulas, it is clear that (15)–(18) may be practically available when n is much bigger than p. Here we also guess that there does not seem to exist simple sum formulas (with numbers of terms depending only on p) for the following finite series

$$\sum_{k=1}^{n} k^p \left\{ {n \atop k} \right\} \quad \text{and} \quad \sum_{k=1}^{n} (k+\lambda\,|\,\theta)_p \left\{ {n \atop k} \right\}.$$

3° Recall that Carlitz's degenerate Stirling numbers are given by $S(p,j\,|\,\theta) = S(p,j,0\,|\,\theta)$, so that the case $\lambda = 0$ of (13) is of some interest, viz.

$$\sum_{k=0}^{\infty} \frac{(k\,|\,\theta)_p}{k\,!} f^{(k)}(0) = \sum_{j=0}^{p} S(p,j\,|\,\theta)\, f^{(j)}(1). \tag{19}$$

Note that for $\theta = 0$ and $\theta = -1$ we have

$$S(p,j\,|\,0) = S(p,j) = \left\{ {p \atop j} \right\}$$

$$\begin{aligned} S(p,j\,|\,-1) &= \frac{1}{j\,!}\left[\Delta^j (t\,|\,-1)_p\right]_{t=0} = \frac{p\,!}{j\,!}\left[\Delta^j \binom{t+p-1}{p}\right]_{t=0} \\ &= \frac{p\,!}{j\,!}\binom{p-1}{p-j} = \frac{p\,!}{j\,!}\binom{p-1}{j-1} = |L(p,j)|, \end{aligned}$$

where $|L(p,j)|$ are known as Lah numbers, with $|L(0,0)| = 1$ and $|L(p,j)| = 0$ $(p < j)$. Consequently (19) implies the following identities

$$\sum_{k=0}^{\infty} \frac{(k+p-1)_p}{k\,!} f^{(k)}(0) = \sum_{j=0}^{p} |L(p,j)|\, f^{(j)}(1), \tag{20}$$

$$\frac{1}{e}\sum_{k=0}^{\infty} \frac{(k+p-1)_p}{k\,!} = \sum_{j=0}^{p} |L(p,j)|. \tag{21}$$

In particular, taking $f(t) = (t)_n$ we find that (20) yields the identity

$$\sum_{k=1}^{n} \binom{k+p-1}{p} \left[{n \atop k} \right] = \sum_{j=1}^{p} \binom{p-1}{p-j} \left(\left[{n-1 \atop j-1} \right] + \left[{n-1 \atop j} \right] \right). \tag{22}$$

Acknowledgment The authors are grateful to Professors HE Tianxiao, HSU L C and SHIUE P J-S for providing reference materials [2] and [4].

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